Statistical Stability in Financial Modeling

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Abstract

In this paper, a theory of different stability concepts is developed including α-stability, max-stability and pot-stability. Special cases of stable distributions are introduced with some detail. The properties of heavy-tailedness and asymmetry together with the real line support encouraged in trusting stable distributions and considering them an attractive reliable environment in financial modeling. Modeling procedures are applied on real data of size 600 from the DAX portfolio[3] restricted to the financial factors such as profitability, as a ratio of the net income to total assets and net income to sales, leverage , obtained by dividing the equity to total assets, and turnover expressed as a ratio of the sales to total assets. Risk measures such as value at risk and expected shortfall are also modeled via the POT-stable distributions.

Key words *: α-stability, Max-stability, Pot-stability, Gaussian modeling, Gaussian mixture, Extreme value distributions, Generalized Pareto distributions, Financial modeling, risk measures.*

1. Introduction

The present work introduces and investigates the concept of statistical stability and its application in financial modeling *via* three types of stable distributions, namely, α-stable, max-stable and pot-stable distributions. The α-stable family is completely defined by its characteristic function with its four parameters that depict the skewness, fatness and heavy-tailedness. The max-stable and pot-stable families are determined by three parameters, namely, the location parameter, the scale parameter and the shape parameter and represented by density functions. Each of the families is used to model some financial factors, for more details about financial modeling we refer to [1], [3] and [9]. Student's t-distribution and Gaussian distribution are introduced as special cases of α-stable distributions and used to

[3] Refer to Comdirect Bank AG, www.comdirect.de

model financial factors such as profitability and asset turnover. The extreme value distributions, being special cases of the max-stable family, are considered appropriate for modeling the largest observations collected from large samples of identically distributed observation, the family is used, in the present work, for modeling the leverage factor. Different graphical aids, e.g., histogram, kernel density and cumulative distribution, are utilized. The concept of heavy-tailedness, which is related to the nonexistence of higher order moments, is an elegant property of generalized Pareto distribution that makes it most relevant for modeling risk measures such as the Value at Risk, VaR, and the Expected Shortfall, ES, [8].

2. α-stable Modeling

The family of α-stable distributions is named due to the concept of stability of the tail index parameter α , which states that the linear combination of α-stable distributions is also α-stable with same tail index. The family is generally described by four parameters, namely, the tail parameter α, 0 < α ≤ 2, the skewness parameter β, -1 < β < 1, the location parameter, $\mu \in \mathbb{R}$, and the scale parameter $\sigma \in \mathbb{R}^{+}$.

2.1 Theoretical Background

To explain the concept of α-stability, let

$$F^{m,c}(x) = Pr\left\{\sum_{i \leq m} X_i \leq x\right\}, X_i \sim F$$

be the $m - th$ convolution of a distribution function F, and c for the notation of convolution. We say that F is α-stable if

$$F^{m,c}(a_m\, x + b) = F(x)$$

for certain constants $a_m = m^{\frac{1}{\alpha}} > 0,\ 0 < \alpha \leq 2$. [8]. The asymmetry of the distribution is determined by the value and sign of the skewness parameter α, the distribution is skewed to right for positive values of β and skewed to the left for negative value and symmetric for zero value whereas smaller values of α indicate heavier tails.

Unlike other families of distributions, the α-stable family has no formal theoretical probabilistic model, but can be completely determined by its characteristic function $\phi_X(t)$,

$$\phi_X(t) = \begin{cases} exp\left\{-|t|^{\alpha}\left(1 + i\beta tan\left(\frac{\alpha\,\pi}{2}\right) sign\,(t)\,(|t|^{1-\alpha})\right)\right\}, & \alpha \in (1,2)\backslash\{1\} \\ exp\left\{-|t|\left(1 + i\beta\left(\frac{\pi}{2}\right) sign\,(t)\, ln(|t|)\right)\right\}, & \alpha \in \{1\} \end{cases}$$

where

$$sign\ (t) = \begin{cases} -1, & t < 0 \\ 0, & t = 0 \\ +1, & t > 0 \end{cases}$$

For more details we refer to [8].

2.2 Special α-stable Models

It is of interest that both the Gaussian distribution and the student's t-distribution are special cases of the α-stable family. If, in particular, one puts $\beta = 0$ in the above α-stable characteristic function, then gets the characteristic function $\phi_X(t) = exp\{|t|^\alpha\}$ to represent symmetric sub-family including Student's t-distribution which can be completely characterized by its location parameter μ, scale parameter $\sigma > 0$ and tail index ν determined by the number of degrees of freedom to control the tail fairly heaviness. Furthermore, if we put $\alpha = 2$ then we get the characteristic function of Gauss distribution, namely, $\phi_X(t) = exp\{|t|^2\}$ with the property of thin tails.

Due to the α-stability property, the Gaussian mixture with c-components is also stable.

2.3 Financial Modeling *via* α- stability

The properties of heavy-tailedness and asymmetry together with the real line support property play an important role in trusting the family and making it an attractive reliable environment in financial modeling. As an example, the technical limitations and skewness property of profitability and growth factors make the family a good choice for modeling these factors see Fig 1.

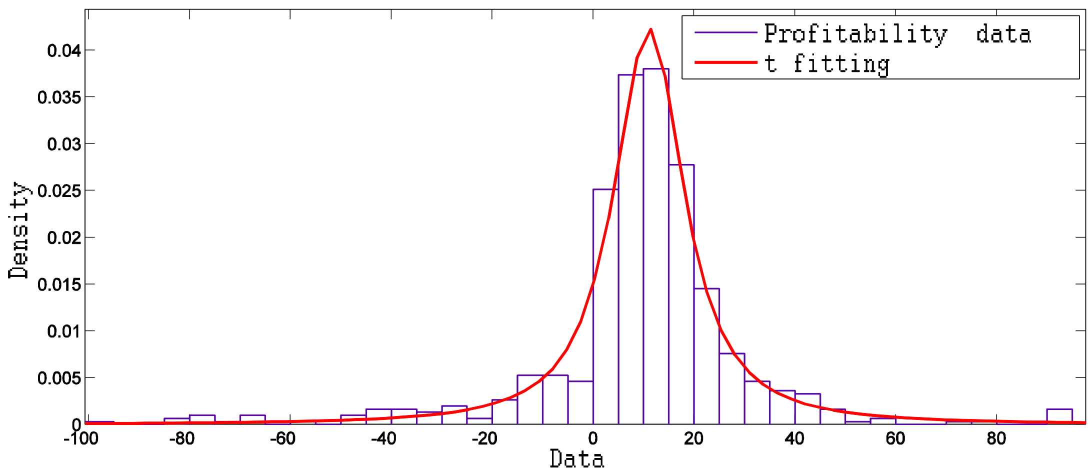

Figure 1. Modeling profitability factor using t-distribution

Figure 1 shows the density fitting of profitability factor using t-distribution with parameter estimates $\hat{\mu} = 11$ (s.e $= 0.44$), $\hat{\sigma} = 8$ (s.e $= 0.50$), $\hat{\nu} = 2$ (s.e $= 0.12$).

Moreover, the Gaussian mixture can be used as a theoretical model for the total asset turnover , [1], see Fig 2.

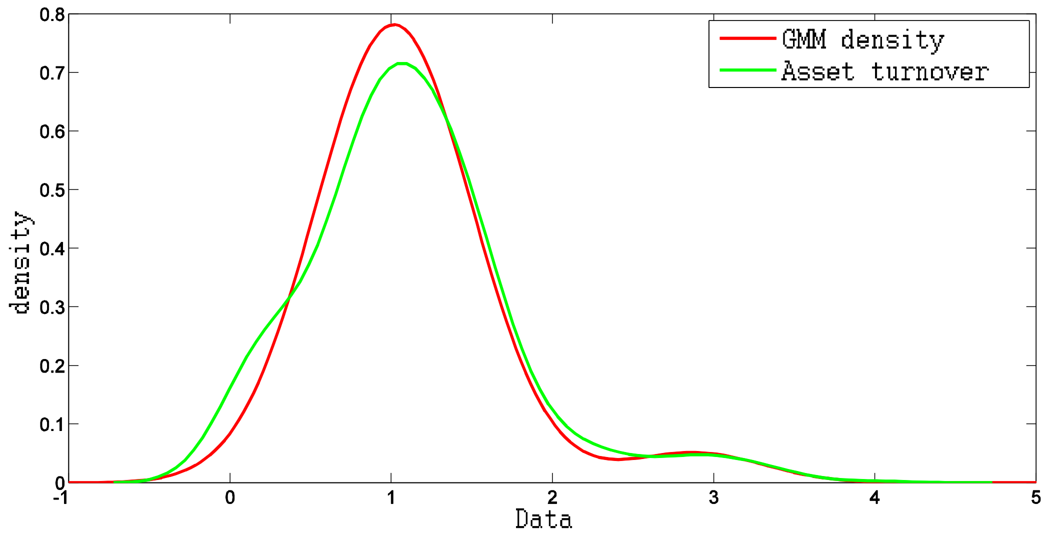

Figure 2. Modeling of asset turnover with using a Gaussian mixture

Figure (2) shows the modeling of asset turnover with a Gaussian mixture model, GMM, of $0.95\ \mathrm{N}(1{,}0.23)$ and a $0.05\ \mathrm{N}(2.88, 0.18)$.

3. Max-stable Modeling

The max-stable family is concerned to the block maxima and determined by three parameters, namely, the location parameter $\mu \in \mathbb{R}$, the scale parameter $\sigma > 0$ and the shape parameter $\gamma \in \mathbb{R}$. The family is represented by its unified density function as given below.

3.1 Theoretical Background

To explain the concept of max-stability, let $X_i \in F, i = 1,2,3, \dots, n$. be identically distributed random variables and let

$$F^n(x) = Pr\left\{\max_{i \le n} X_i \le x\right\} = Pr\{X_1 \le x, X_2 \le x, \dots, X_n \le x\}$$

be the distribution function of the block maxima. The distribution function is said to be max-stable if

$$F^n(a_n x + b_n) = F(x)$$

for proper choice of $a_n > 0$ and $b_n \in \mathbb{R}$.

3.2 Special Max-stable Models

A special case of the max-stable distributions is the family of extreme value distributions determined by its location parameter $\mu \in \mathbb{R}$, scale parameter $\sigma > 0$ and shape parameter $\gamma \in \mathbb{R}$. According to von Mises, 1954, model of the family is given by

$$G_{\gamma,\mu,\sigma}(x) = exp\left\{-\left[1+\gamma\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\gamma}}\right\}$$

For $\gamma > 0$, the distribution is called *Frechet* distribution with heavy upper tail and right endpoint at $\frac{1}{|\gamma|}$ where as for , $\gamma < 0$ the distribution is called *Weibull* distribution with left endpoint at $-\frac{1}{\gamma}$, and called *Gumbel* distribution for $\gamma \longrightarrow 0$.

A distribution function F is said to be in the max-domain of attraction of a non-degenerate function G, $F \in D(G)$, if it satisfies the following weak convergence property

$$\lim_{n\uparrow\infty} |F^n(a_n\, x + \, b_n) - G(x)| = 0$$

where G is an extreme value distribution function.

3.3 Financial Modeling *via* Max-stability

The nonnegative support line of the *Frechet* makes it good for modeling both the leverage factors, see Fig 3.

Block maxima models, although less useful than the forthcoming threshold models, they models are not without practical relevance and could be used to provide estimates of stress losses. For example, if we record daily or hourly losses and profits from trading a particular instrument or group of instruments, the block maxima method provides a model which may be appropriate for the quarterly or annual maximum of such values. We see a possible role for this method in the definition and analysis of stress losses, [7].

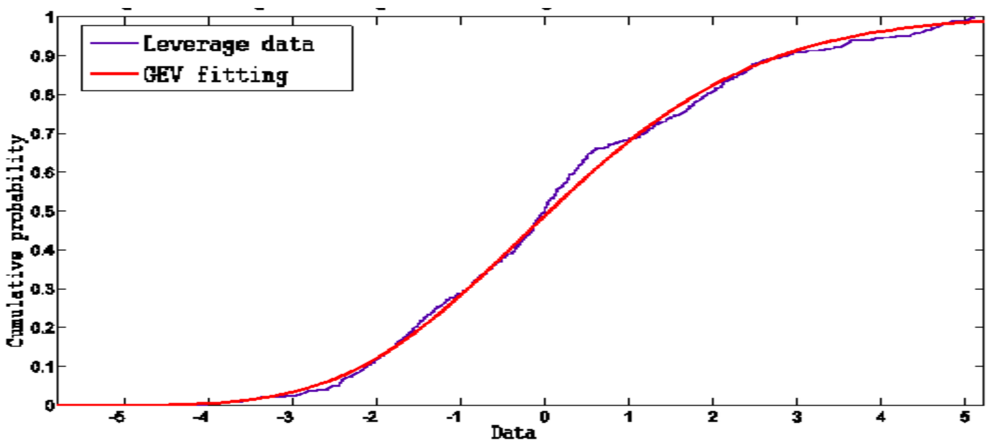

Figure 3. Modeling of leverasge factor with generalized extreme value distribution

Figure (3) shows the fitting of leverasge factor using cumulative distribution of generalized extreme value distribution with parameter estimates μ ̂= -0.56 (s.e=0.08),σ ̂=1.8 (s.e=0.06),γ= -0.016 (s.e=0.03).

4. Pot-stable Modeling

The POT (Peak-Over-Threshold) notation is concerned with the exceedances over a certain threshold u in order to represent the upper tail of the underlying distribution. Similar to the max-stable family, the POT-stable family is defined by three parameters, namely, the location parameter $\mu \in \mathbb{R}$, the scale parameter $\sigma > 0$ and the shape parameter $\gamma \in \mathbb{R}$, and represented by a unified density function.

4.1 Theoretical Background

To explain the concept of POT-stability, let

$$F^{[u]}(x) = Pr\{X \leq x \mid x > u\}$$

be the exceedance distribution function, with u be a high threshold pertaining to F.

The distribution function F is said to be POT-stable if it satisfies

$$F^{[u]}(a_u x + b_u) = F(x)$$

for a proper choice of $a_u > 0$ and $b_u \in \mathbb{R}$, [8].

4.2 Special Pot-stable Models

A special case of the POT-stable distributions is the generalized Pareto distribution, a right-skewed distribution determined by its location (threshold) parameter $\mu \in \mathbb{R}$, scale parameter $\sigma > 0$ and shape (tail index) parameter $\gamma \in \mathbb{R}$. and defined by the combined formula given by

$$W_{\gamma,\mu,\sigma}(x) = \begin{cases} 1 - \left[1 + \gamma\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\gamma}}, \gamma \neq 0 \\ 1 - exp\left\{-\left(\frac{x-\mu}{\sigma}\right)\right\}, \gamma \longrightarrow 0 \end{cases}$$

For $\gamma > 0$, the distribution is called Pareto distribution with heavy upper tail, whereas for $\gamma < 0$ the distribution is called beta distribution with finite upper limit of the tail, and called exponential distribution for $\gamma \longrightarrow 0$, with domain restriction

$$D\left(W_{\gamma,\mu,\sigma}(x)\right) = \begin{cases} x \in [\mu, \infty) \quad , \ \gamma > 0 \\ x \in \left[\mu, \mu + \frac{\sigma}{|\gamma|}\right], \ \gamma < 0 \end{cases}$$

Fig 4 shows the density function for location at zero, unit scale parameter and different values of tail index.

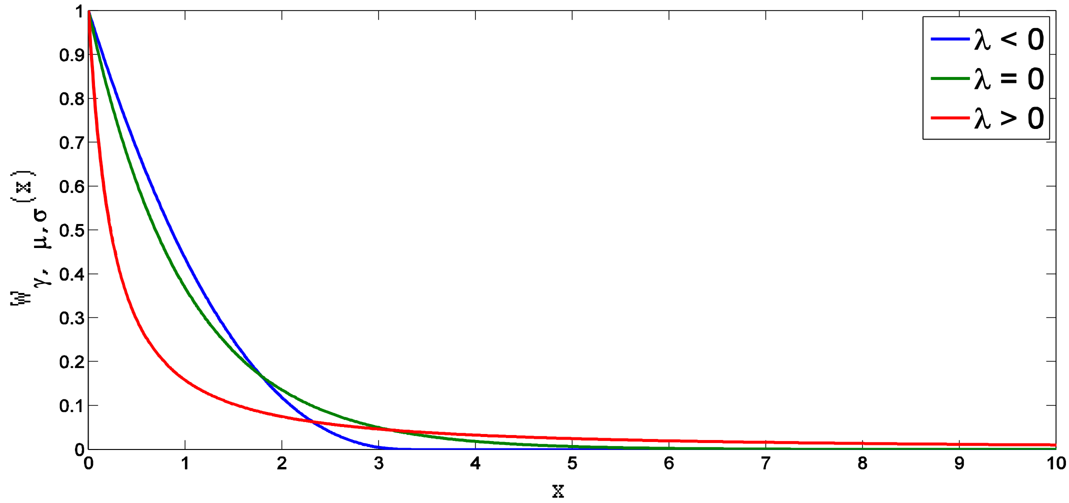

Figure 4. Generalized Pareto distributions

Figure (4) shows the density function of generalized Pareto distributions for negative, zero and positive values of shape parameter.

A distribution function F is said to be in the domain of attraction of generalized Pareto function W, $F \in D(W)$, if it satisfies the following weak convergence property

$$\lim_{u \uparrow x_w} sup_{0<x<x_w-u}\left|F^{[u]}(x) - W_{\gamma,u,\sigma_u}(x)\right| = 0$$

, $x_w = \sup\{x \in \mathbb{R} : F < 1\}$ is the right endpoint of F, [1].

In accordance with the POT-stability property, the generalized Pareto distribution, $W^{[u]}$, truncated at a high threshold u, still remains a generalized Pareto with same tail index γ, where

$$W^{[u]}_{\gamma,\mu,\sigma} = W_{\gamma,u,\sigma(u-\mu)}$$

with location shifted to u and scale shifted to $\sigma(u-\mu)$. The truncated generalized distribution can also fit the upper tail, $X|X > u$, of the original distribution function, F, for a certain choice of γ and σ. To verify this, let $X \in F$, $(X-u) \sim F^{[u]}$, then

$$Pr\{X-u > y \,|X-u > 0\} = Pr\{X > y+u \,|X > u\}$$

$$= \frac{1-F(u+y)}{1-F(u)}$$

$$\approx \frac{\left[1+\gamma\left(\frac{u+y-\mu}{\sigma}\right)\right]^{-\frac{\gamma}{2}}}{\left[1+\gamma\left(\frac{u-\mu}{\sigma}\right)\right]^{-\frac{\gamma}{2}}}$$

$$\approx \left[1+\gamma\left(\frac{u}{\sigma_u}\right)\right]^{-\frac{\gamma}{2}}$$

Hence,

$$Pr\{X-u<y\,|X-u>0\}\approx 1-\left[1+\gamma\left(\frac{u}{\sigma_u}\right)\right]^{-\frac{\gamma}{2}}=W_{\gamma,u,\sigma(u-\mu)}=W^{[u]}_{\gamma,\mu,\sigma}$$

4.3 Financial Modeling *via* Pot-stability

In addition to the general concept of upper tail modeling, risk measures such as value at risk, expected shortfall and others are examples of measures that can be modeled *via* the generalized Pareto distribution.

4.3.1 Upper tail

Regarding the generalized Pareto distribution parameter estimates, the tail estimator of underlying distribution $F(x)$

$$\hat{F}^{[u]}(x)=1-\frac{N_u}{n}\left(1+\hat{\gamma}\frac{x-u}{\hat{\sigma}}\right)^{-\frac{1}{\hat{\gamma}}},x>u$$

, where n is the number of observations, $N_u=\sum 1_{(x>u)}$, a counter of the number of exceedances over the threshold u , [7].

Reiss and Thomas used the generalized Pareto estimates of the lower and upper tail indices of the daily returns to the Standard & Poors index. Besides they suggested the exponential modeling of the upper tail in case of Swiss exchange rate returns, see [8]. For our data, we suggest modeling the upper tail of 100 points of negative profitability using generalized Pareto. exceeding a threshold $u=0.21$, distribution with tail index and scale estimates, $\hat{\gamma}=0.28\ (s.e=0.17), \hat{\sigma}=9.7\ (s.e=1.9)$ respectively, see Fig 5 and Fig 6.

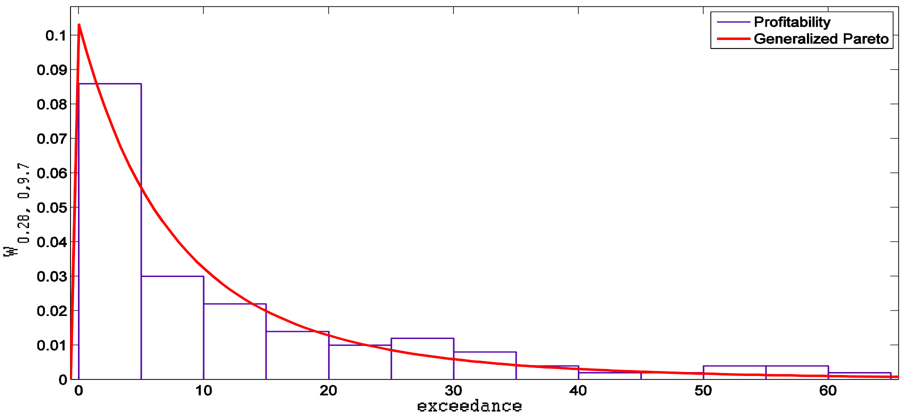

Figure 5. Fitting Generalized Pareto density function to profitability.

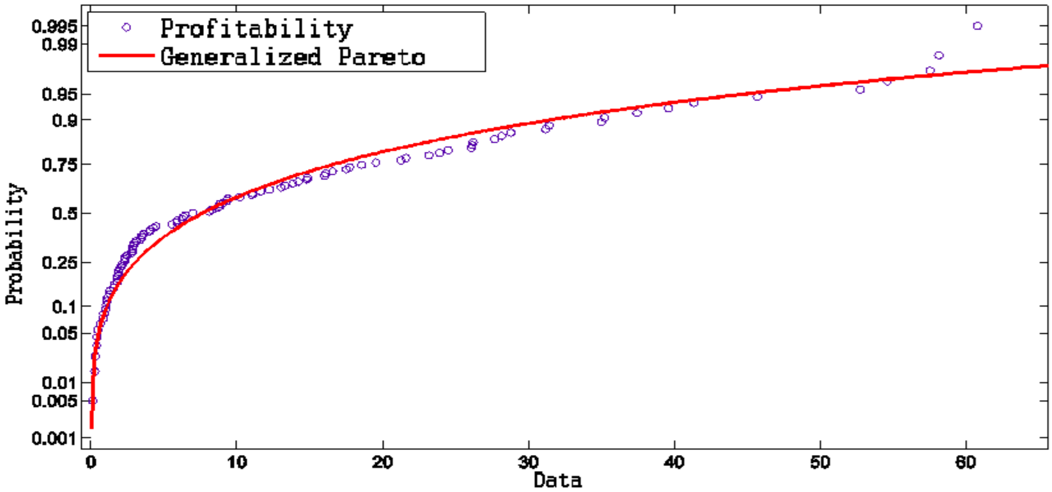

Figure 6. A probability plot of generalized Pareto distribution.

4.3.2 Value at Risk

In the field of risk management, Value at Risk, VaR, is a measure of the maximal amount of a financial portfolio that may be lost over a certain period of time at a certain confidence level $(1-\alpha)\%$. Technically, VaR is needed to fulfil the requirement of capital adequacy, in such a case a higher confidence level,say 99% or 99.9% is advisable. Statistically, VaR is not more that than the $(1-\alpha)th$ quantile of the underlying distribution, say, the loss distribution F, that is

$$VaR_{1-\alpha} = F^{-1}(1-\alpha), 0 < \alpha < 0.05$$

The most relevant model for VaR is by using the Pareto distribution where the heavy tail index $\gamma > 0$ indicates a higher than usual probability of portfolio loss. An estimate is given by

$$\widehat{VaR}_{1-\alpha} = u + \frac{\hat{\sigma}}{\hat{\gamma}}\left(\left(\frac{\alpha \times n}{N_u}\right)^{-\hat{\gamma}} - 1\right)$$

4.3.3 Expected shortfall

Even though VaR is not a coherent measure but is utilized the coherent Expected Shortfall, ES. The measures has been the cornerstone of banking risk management for more than two decades. The Expected Shortfall is the tail conditional expectation

$$ES_{1-\alpha} = E(X|X > VaR_{1-\alpha}) = \int_{VaR_{1-\alpha}}^{\infty} x \, dF^{[VaR_{1-\alpha}]}(x)$$

$$= \frac{1}{\alpha}\int_{1-\alpha}^{\infty} VaR(x)dx$$

with estimator given by

$$\widehat{ES}_{1-\alpha} = \frac{\widehat{VaR}_{1-\alpha} + \hat{\sigma} - \hat{\gamma}\, u}{1 - \hat{\gamma}}$$

For implementation to ISE 100 Index, with different levels of confidence, we refer to [2] and [8].

For implementation to profitability data for different levels of significance, see Table 1.

Table 1: Estimated Value at Risk and Expected shortfall for the profitability factor

Risk Measures	$1-\alpha$		
	90%	95%	99%
$\widehat{VaR}_{1-\alpha}$	18.96	29.54	34.17
$\widehat{ES}_{1-\alpha}$	26.24	35.45	37.89

Table 1 shows the modeling of Value at Risk, VaR and Expected Shortfall, ES using maximum likelihood estimates of generalized Pareto distribution parameters for $\alpha = 0.10, 0.5, 0.01$.

5. Discussion and Conclusion

The literature and previous studies showed that the characteristics of symmetry and thin tail alone are not enough for modeling purposes of financial data. Instead, due to its amazing modeling characteristics ranging from symmetry to asymmetry and from thin-tailedness to heavy-tailedness, the α −stable family can be considered as rich environment for modeling financial data, for example, the t-distribution, being a member of the this family, may be used to model the symmetry and heavy-tailed phenomena of financial factors such as profitability whereas the asymmetry property of asset turnover may allow using the Gaussian mixture modeling. Another modeling environment of block maxima is the generalized extreme value theory which may be used to catch the skewness of the leverage factor.

An environment for modeling risks represented by the tail heaviness is the generalized Pareto distribution which provides the best estimate for the upper quantiles of underlying financial distribution and hence used for modeling the potential risks such as the value at risk and expected shortfall.

We conclude either from the present work and on the basis of the underlying data, that the concept of stability and its characteristics, as it has been introduced in the present work, is a rich environment for modeling financial factors and can be recommended for more investigations among financial modeling.

6. References

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