The Distortion in Conformal Mappingin Hyperbolic Geometry

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Abstract

Nehari-type inequalities for normalized univalent functions are combined with elementary monotonicity arguments to give quick and simple proofs for numerous sharp two-point distortion theorems for conformal maps. These inequalities are used to prove two theorems of sharp two-point distortion theorems for conformal maps from the unit disk into the unit disk equipped with hyperbolic geometry.

Keywords: Distortion, Conformal maps, Unit disc, Sharp two-point, and hyperbolic geometry.

1. Introduction

The study of distortion under the conformal maps started at first since 1994, where the first paper was published by S. Kim and D. Minda [5] under the title “ two-point distortion theorems for univalent function”. After three years and so, again D. Minda and W. Ma [6] they published another paper. Also, in 1999 and 2000 W. Ma and D. Minda [7, 8] published more papers about the same kind of theorems. In [5] through [14] the publishers used different approaches in their proofs, like differential geometry, general coefficient theory, and control theory to prove these kinds of theorems.

Our aim of this research to prove same kind of theorems concerning the distortion in conformal mapping for univalent function. The first theorem says that: If $f:D\to D$ is univalent function, then for any $z_1, z_2 \in D$ and $p \le 0$, where D is the unit disk, then we have the inequality (3). The second theorem says that: If: $f:D\to D$ is univalent, then for any two distinct points $z_1, z_2 \in D$, and any $p \le 0$, then we have the inequalities (8).

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In our research we used different approaches to prove these kind of theorems, that is; we used the Nehari-type inequalities with elementary monotonicity arguments to give quick and simple proofs. The distortions in conformal mapping in hyperbolic geometry was measured by the quantity of differential operator $|D_D\ f(z)|$ which defined below. The hyperbolic distance with metric space d_D defined by

$$d_D(\xi, z) = \{|\xi - z| > a, a \neq 0 \text{ for all } \xi, z \text{ in } D\}$$

We consider the class of conformal maps related to hyperbolic geometry:
Conformal maps from the unit disk D with the hyperbolic distance d_D into the unit disc D with the hyperbolic distance d_D.

If f is a conformal map from the metric space (D, d_D), then the local length distortion of f at a point $z \in D$ is measured by the quantity

$$|D_D\ f(z)| = \lim_{\xi \to z} \frac{d_D(f(\xi), f(z))}{d_D(\xi, z)}.$$

A two-point distortion theorem for conformal maps $f:(D, d_D) \to (D, d_D)$ provide sharp upper and lower bounds for the global length distortion $d_D(f(z_1), f(z_2))$ for two points $z_1, z_2 \in D$ in terms of the local length distortion $|D_D\ f(z_1)|$ and $|D_D\ f(z_2)|$ at these two points as well at the hyperbolic distance $d_D(z_1, z_2)$ between z_1 and z_2.

Recently, a multitude of two-point distortion theorems for univalent function has been obtained using :

(a) Differential geometric methods [1-5],

(b) The general coefficient theorem [3],

(c) Control theory [13], [14].

Here, we show how Nehari-type inequalities for univalent functions can effectively and systematically be combined with elementary monotonicity argument to establish new distortion theorem for conformal maps. As a byproduct, we also obtain quick proofs of a distortion estimate due to Jenkins [3], [4] and Ma and Minda [7].

If $f: D \to D$ is a (bounded) conformal map, then it is natural to consider f as a map

$f:(D,d_D)\to(D,d_D)$. Now, the local length distortion of f at a point $z\in D$ is given by [14]:

$$\left|D_D f(z)\right| = \lim_{\xi\to z}\frac{d_D(f(\xi),f(z))}{d_D(\xi,z)} = \frac{1-|z|^2}{1-|f(z)|^2}\left|f'(z)\right|$$

which might be called (hyperbolic-hyperbolic) derivative of f at $z\in D$. A two-point distortion theorems for bounded univalent functions $f:D\to D$ aim at giving upper and lower bounds for the global length distortion of $f:(D,d_D)\to(D,d_D)$ at two given points z_1 and z_2 in terms of the local length distortion at these points.

There are almost natural quantities to measure the local length distortion of a conformal map $f:(D,d_D)\to(D,d_D)$ at two points z_1 and z_2.

Ma and Minda [6] (see [13]) employed the expression

$$\left(\left(\frac{\left|D_D f(z_1)\right|}{1-\left|D_D f(z_1)\right|}\right)^p + \left(\frac{\left|D_D f(z_2)\right|}{1-\left|D_D f(z_2)\right|}\right)^p\right)^{\frac{1}{p}},\quad p\in\mathbb{R} \tag{1}$$

whereas Jenkins [4] used the quantity

$$\left(\left|D_D f(z_1)\right|^p + \left|D_D f(z_2)\right|^p\right)^{\frac{1}{p}},\quad p\in\mathbb{R}, \tag{2}$$

to state sharp upper and lower bounds for the local length distortion of f.

We would like to emphasize that in some case formula (1) gives better results than formula (2), whereas in other cases (1) is more advantageous (see [14]) for a discussion of this matter. Here, we focus on the quantity (2) and establish the following sharp upper and lower bounds for the locale length distortion of f in terms of (2) for negative parameters p.

2. Main Results

Theorem 2.1

If $f: D \to D$ is univalent, then for any $z_1, z_2 \in D$ and any $p \le 0$, we have

$$\tanh\big(d_D(f(z_1), f(z_2))\big) \ge \Big(|D_D f(z_1)|^p + |D_D f(z_2)|^p\Big)^{1/p} \cdot \frac{\tanh(d_D(z_1, z_2))}{2^{1/p}}. \quad (3)$$

Note that equality holds for two distinct points z_1 and z_2 if and only if f maps D onto D slit along two hyperbolic rays on the hyperbolic geodesic γ which is perpendicular to the hyperbolic geodesic joining $f(z_1)$ and $f(z_2)$ and such that $f(z_1)$ and $f(z_2)$ are symmetric with respect to γ. Conversely, if $f: D \to D$ is a non-constant analytic function satisfying (3), then f is univalent.

2.2 Proof

We first recall the Nehari inequalities for bounded univalent functions. Let $f: D \to D$ be a univalent function, $z_1, z_2, \ldots, z_n$ points in D and $\lambda_1, \lambda_2, \ldots, \lambda_n$ complex numbers, then, see [10]

$$\operatorname{Re}\left(\sum_{j,k=1}^{n} \lambda_j \lambda_k \log \frac{\big(f(z_k) - f(z_j)\big)}{z_k - z_j}\right) \le -\sum_{j,k=1}^{n} \lambda_j \overline{\lambda_k} \log\left(\frac{1 - f(z_j)\overline{f(z_k)}}{1 - z_j \overline{z_k}}\right), \quad (4)$$

With equality possible only if f maps D onto D slit along a system of arcs $\omega = \omega(t)$ satisfying

$$\operatorname{Re}\left(\sum_{k=1}^{n} \lambda_k \log(\omega - f(z_k)) - \overline{\lambda_k} \log(1 - \overline{f(z_k)}\,\omega)\right) = 0 .$$

Now, we begin with the f-part of the equality statement. Note that for fixed $\mu \in (0,1]$ and fixed $c \in [-2,2]$ the equation

$$K_c\big(P_{\mu,c}(z)\big) = \mu\, K_c(z) \quad , \; z \in D, \quad (5)$$

Where $K_c(z) = \dfrac{z}{(1 + z + z^2)}$, and c is constant complex number.

define a conformal map $P_{\mu,c}: D \to D$, which maps D onto D slit along two (possibly degenerate) segments $(-1, l_{\mu,c}]$ and $[r_{\mu,c}, 1)$ on the real axis. By construction, the points $l_{\mu,c} \in [-1,0)$ and $r_{\mu,c} \in (0,1]$ can be arbitrarily prescribed by varying μ and c. The hyperbolic derivative of $P_{\mu,c}$ is given by

$$|D_D P_{\mu,c}(z)| = \mu \frac{1-|z|^2}{1-|P_{\mu,c}(z)|^2} \frac{|K'_c(z)|}{|K'_c(P_{\mu,c}(z))|}. \tag{6}$$

Subtracting (5) for $\bar{z}$ from (5) for z, using $P_{\mu,c}(\bar{z}) = \overline{P_{\mu,c}(z)}$ and

$$|K_c(z) - K_c(\bar{z})| = \left(1-|z|^2\right)|K'_c(z)| \frac{|z-\bar{z}|}{|1-z^2|},$$

yield

$$\frac{|1-z^2|}{|1-P^2_{\mu,c}(z)|}\left|\frac{P_{\mu,c}(z)-\overline{P_{\mu,c}(z)}}{z-\bar{z}}\right| = \mu \frac{1-|z|^2}{1-|P_{\mu,c}(z)|^2} \cdot \frac{|K'_c(z)|}{|K'_c(P_{\mu,c}(z))|}.$$

Combining this and (6) gives

$$|D_D P_{\mu,c}(z)| = |D_D P_{\mu,c}(\bar{z})| = \left|\frac{P_{\mu,c}(z)-\overline{P_{\mu,c}(z)}}{z-\bar{z}}\right| \frac{|1-z^2|}{|1-P^2_{\mu,c}(z)|}$$

$$= \frac{\tanh\left(d_D\left(P_{\mu,c}(z), \overline{P_{\mu,c}(z)}\right)\right)}{\tanh(d_D(z,\bar{z}))}. \tag{7}$$

Consequently, the equality holds in (3) for $f = P_{\mu,c}$ and $z = \bar{z}_2$. Now if f maps D conformally onto D slit along two hyperbolic rays on the hyperbolic geodesic γ which is perpendicular to the hyperbolic geodesic joining $f(z_1)$ and $f(z_2)$ such that $f(z_1)$ and $f(z_2)$ are symmetric with respect to γ, then it is easy to see that $f = S \circ P_{\mu,c} \circ T$ for some $\mu \in (0,1]$ and $c \in [-2,2]$, where S and T are conformal automorphisms of D such that $T(z_1)=\overline{T(z_2)}$. This proves the f-part of the equality statement of theorem 2.1. The distortion estimate (3) for $p=0$ follows immediately from Nehari's inequality (4) for $n=2$, $\lambda_1=1$, and $\lambda_2=-1$, this is equivalent to

$$\left|\frac{f(z_1)-f(z_2)}{1-\overline{f(z_1)}\,f(z_2)}\right|^2 \geq \frac{1-|z_1|^2}{1-|f(z_1)|^2}|f'(z_1)|\frac{1-|z_2|^2}{1-|f(z_2)|^2}\cdot\left|\frac{z_1-z_2}{1-\bar{z}_1 z_2}\right|^2 ,$$

that is

$$\tanh\left(d_D(f(z_1),f(z_2))\right)\geq \tanh\left(d_D(z_1,z_2)\right)\sqrt{|D_D f(z_1)\|D_D f(z_2)|} \; .$$

Equality is only possible if f maps D onto D slit along two hyperbolic rays on the hyperbolic geodesic γ which is perpendicular to the hyperbolic geodesic joining $f(z_1)$ and $f(z_2)$, such that $f(z_1)$ and $f(z_2)$ are symmetric with respect to γ .

Since

$$\left(|D_D f(z_1)|^p+|D_D f(z_2)|^p\right)^{1/p}/2^{1/p},$$

is an increasing function of $p\leq 0$, we deduce that only-if part of the equality statement and that (3) holds for every $p\leq 0$. Conversely, if $f:D\to D$ is a non-constant analytic function satisfying (3) for some $p\leq 0$, then the Kim-Minda argument [5] shows that f is univalent.

Theorem 2.3

If $f:D\to D$ is univalent, then for any two distinct points $z_1,z_2\in D$ and any $p\leq 0$

$$\left(|D_D f(z_1)|^p+|D_D f(z_2)|^p\right)^{1/p}\geq\left(2\cosh(2p(\rho'-\rho))\right)^{1/p}\cdot\frac{\sinh(2\rho')}{\sinh(2\rho)} \; . \quad (8)$$

Where ρ is the hyperbolic distance between z_1 and z_2, and ρ' is the hyperbolic distance between $f(z_1)$ and $f(z_2)$. For $p<0$ the equality holds for two distinct points z_1 and z_2 if and only if f maps D onto D slit along a hyperbolic ray on the hyperbolic geodesic through $f(z_1)$ and $f(z_2)$.

Lemma 2.4 [9]

If f is a conformal map of D, then for any $z_1, z_2 \in D$, the expression

$$\frac{\left(|D_D f(z_1)|^p + |D_D f(z_2)|^p\right)^{1/p}}{\left(2\cosh(2p\, d_D(z_1, z_2))\right)^{1/p}},$$

is a decreasing function for $p \le 0$.

2.5 Proof

By applying (4) with $n = 2$, $\lambda_1 = i$, and $\lambda_2 = -i$ we get

$$\frac{\left|1-\overline{f(z_1)}\,f(z_2)\right|^2 |f(z_1)-f(z_2)|^2}{\left(1-|f(z_1)|^2\right)\left(1-|f(z_2)|^2\right)} \le |f'(z_1)||f'(z_2)| \cdot \frac{|1-z_1 z_2|^2 |z_1-z_2|^2}{\left(1-|z_1|^2\right)\left(1-|z_2|^2\right)}.$$

or

$$\sinh(2\rho') \le \sqrt{|D_D f(z_1)||D_D f(z_2)|}\,\sinh(2\rho), \tag{9}$$

with $\rho' = d_D(f(z_1), f(z_2))$ and $\rho = d_D(z_1, z_2)$.

Equality is possible only if f maps D onto D slit along one or two rays on the hyperbolic geodesic through $f(z_1)$ and $f(z_2)$. We observe that

$$\left(\frac{|D_D f(z_1)|^p + |D_D f(z_2)|^p}{2\cosh(2p(\rho'-\rho)}\right)^{1/p},$$

is a decreasing function for $p \le 0$. This follows from lemma 2.4. Now, we have to use the fact that:

$$|D_D f(z_2)| \le |D_D f(z_1)|\, e^{-4\rho'} e^{\rho 4} \tag{10}$$

For bounded univalent functions.

Consequently, the equation (4) holds for any $p \le 0$ and equality is only possible if f maps D onto D slit along one or two rays on the hyperbolic geodesic through $f(z_1)$ and $f(z_2)$. If f is such a conformal map, then by replacing f by $S \circ f \circ T$ with conformal automorphisms S , T of D, we may assume $z_2 = 0$, $z_1 = r \in (0,1)$ and $f(z) = P_{\mu,c}(z)$ for some $c \in [-2,2]$. The calculation is a straightforward and for $p < 0$ the expression

$$c \cdot \left(\frac{\left| D_D P_{\mu,c}(z_2) \right|^p + \left| D_D P_{\mu,c}(z_1) \right|^p}{2\cosh(2p\, d_D(P_{\mu,c}(z_2), P_{\mu,c}(z_1))} \right)^{1/p} \cdot \left(\frac{1}{\sinh d_D(P_{\mu,c}(z_2), P_{\mu,c}(z_1))} \right),$$

attains its minimal value in the interval [-2,2] only for $c = 2$ or $c = -2$. Thus $f(z) = P_{\mu,-2}(z)$ or $f(z) = P_{\mu,2}(z)$, so f maps D onto D slit along a single ray on the real axis. Conversely, if $f(z) = P_{\mu,-2}(z)$ or $f(z) = P_{\mu,2}(z)$, then the equality holds in equation (8) for $z_2 = 0$ and $z_1 \in (0,1)$. This proves the equality statement of theorem 2.3.

2. Remarks

(a) A quick proof of (10) runs as follows:

We may assume $z_1 = 0$, $z_2 = z$, $f(0) = 0$ and $f(z) > 0$. Then

$$g(z) = \frac{1}{f'(0)} \cdot \frac{f(z)}{(1 - f(z))^2},$$

belongs to the set of functions

$$g(z) = z + a_2 z^2 + a_3 z^3 + \dots\dots\dots,$$

so $\qquad |g'(z)| \le (1 + |z|)/(1 - |z|)^3$

which is equivalent to

$$\left| f'(z) \right| \leq \left| f'(0) \right| \frac{1+|z|}{(1-|z|)^3} \left| \frac{(1-f(z))^3}{1+f(z)} \right|$$

$$= \left| f'(0) \right| \frac{1+|z|}{(1-|z|)^3} \frac{(1-|f(z)|)^3}{1+|f(z)|} .$$

Thus

$$\left| D_D f(z) \right| = (1-|z|^2) \frac{|f'(z)|}{1-|f(z)|^2} \leq \left| f'(0) \right| \left(\frac{1-|f(z)|}{1+|f(z)|} \right)^2 \left(\frac{1+|z|}{1-|z|} \right)^2$$

$$= \left| D_D f(0) \right| \exp(-4 d_D(f(z), f(0))) \cdot \exp(4 d_D(z,0))$$

which proves (10).

(b) We notice that Jenkins [4], gave an estimate of the form

$$\left(\left| D_D f(z_1) \right|^p + \left| D_D f(z_2) \right|^p \right)^{1/p} \geq \frac{e^{2\rho'}+1}{e^{2\rho}+1} \left(\frac{e^{\rho'}+1}{e^{\rho}+1} \right)^2 \left(\frac{\cosh(\rho'/2)}{\cosh(\rho/2)} \right)^2$$

for any conformal map $f : D \to D$ and any $p > 0$. Unfortunately, this formula is not quite correct [12]. It has to be replaced by

$$\sinh\left(2 d_D(f(z_1), f(z_2))\right) \leq \left(\left| D_D f(z_1) \right|^p + \left| D_D f(z_2) \right|^p \right)^{1/p} \times \frac{\sinh\,(2 d_D(z_1, z_2))}{2^{1/p}} . \qquad (11)$$

Equality occurs for fixed $p > 0$ if and only if f maps D onto D slit along symmetric rays on the hyperbolic geodesic through $f(z_1)$ and $f(z_2)$. The distortion estimate (11) for $p > 0$ follows immediately from (9) by monotonicity. The discussion of the case of equality is similar to the Euclidean case (see [2], preprint) and will be omitted.

(c) As in the Euclidean case (see [2]) the one-parameter family (8) of distortion estimates can be deduced from the inequalities (8) for $p = 0$ or $p = -\infty$

combined with a monotonicity argument. For this case, as indicated in the above proof of theorem 2.3, it remains to show that the inequality (10) can be derived from (8) for $p=-\infty$.

If we set $z_1 = z, z_2 = 0$ and assume $f(0)=0$, then (8) for $p=-\infty$ takes the form

$$\min\left\{\frac{1-|z|^2}{1-|f(z)|^2}\,|f'(z)|,\ |f'(0)|\right\} \geq \frac{|f(z)|}{|z|}\left(\frac{1-|z|}{1-|f(z)|}\right)^2 .$$

This is a well-known estimate for normalized bounded univalent functions due to Robinson [12]. Now, from

$$|f'(0)| \geq \frac{|f(z)|}{|z|}\left(\frac{1-|z|}{1-|f(z)|}\right)^2,$$

So, we immediately obtain for

$$g(z) = a_1 z + a_2 z^2 + a_3 z^3 + ...$$

Pick's coefficient inequality [10]. i.e.;

$$|a_2| \leq 2|a_1|\left(1-|a_1|\right) .$$

If $f: D \to D$ is not necessarily a normalized bounded function, then

$$|g'(0)| \leq 4|g'(0)|\left(1-|g'(0)|\right)$$

for

$$g(\xi) = \frac{f\left(\dfrac{\xi+z}{1+\bar{z}\xi}\right) - f(z)}{1-\overline{f(z)}\,f\left(\dfrac{\xi+z}{1+\bar{z}\xi}\right)},$$

where z is a fixed point in D

this is equivalent to

$$\left|\frac{f'(z)\left(1-|z|^2\right)}{1-|f(z)|^2} - \frac{2\bar{z}f'(z)}{1-|f(z)|^2} + 2\,\overline{f(z)}\,\frac{[f'(z)]^2\left(1-|z|^2\right)}{\left(1-|f(z)|^2\right)^2}\right|$$

$$\le \frac{4|f'(z)|}{1-|f(z)|^2}\left(1-\frac{|f(z)|\left(1-|z|^2\right)}{1-|f(z)|^2}\right).$$

By integrating the right hand side of the last inequality from 0 to $|z|$ we get

$$\left|\log\frac{f'(z)}{f'(0)}+\log\left(1-|z|^2\right)-2\log\left(1-|f(z)|^2\right)\right|\le 2\log\left(\frac{1+|z|}{1-|z|}\right)+\log\left(\frac{1-|f(z)|}{1+|f(z)|^3}\right),$$

and by exponentiation we finally obtain

$$\left|\frac{f'(z)}{f'(0)}\right|\left(\frac{1-|z|^2}{1-|f(z)|^2}\right)\le\left(\frac{1+|z|}{1-|z|}\right)^2\left(\frac{1-|f(z)|}{1+|f(z)|}\right)^2.$$

3. Conclusion

From this paper, we come up with the following conclusion. Two theorems concurring the distortion in conformal mapping in hyperbolic geometry have been proved. The publishers in the last years used different approaches in their proofs of these kind of theorems, like: ((1)) the differential geometric methods, ((2)) the general coefficient theorem, and ((3)) the control theory. In this paper, we prove those theorems by using new teenics not used before, that is, we used what so called Nehari-type inequalities. The results which us came up with ore those inequalities in our paper which are indicated by numbers (3) and (8).

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