Some Properties Preserved by Cleavability

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Abstract

In 1985 Arhangl' Skii introduced different types of cleavability as following :

A topological space X is said to be cleavable over a class of spaces $\mathcal{P}$ if for $A \subset X$ there exists a continuous mapping $f: X \to Y \in \mathcal{P}$ such that $f^{-1}f(A) = A$, $f(X)=Y$.

We study the case :

If $\mathcal{P}$ is a class of topological spaces with certain properties and if X is cleavable over $\mathcal{P}$ then $X \in \mathcal{P}$

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1. Preliminaries:

Definition(1) [1]A topological space X is said to be pointwise cleavable over a class of spaces $\mathcal{P}$ if for any $x \in X$ there exists a continuous mapping $f: X \to Y$ such that $f^{-1}f(x)=\{x\}$.

Definition(2) [1] A topological space X is said to be absolutely cleavable over a class of spaces $\mathcal{P}$ if $A \subset X$ and there exists an injective continuous mapping $f: X \to Y \in \mathcal{P}$, such that $f^{-1}f(A)= A$.

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Definition(3) [2] A topological space X is said to be double cleavable over a class of spaces $\mathcal{P}$ if for any $A \subset X$ and $B \subset X$ there exists a continuous mapping $f: X \to Y \in \mathcal{P}$, such that $f^{-1}f(A)=A$ and $f^{-1}f(B)=B$.

Definition(4) [3] A topological space X is said to be completely Hausdorff space if for every two distinct points x and y there are two disjoint open sets U and V such that $x \in U$, $y \in V$ and $cl(U) \cap cl(V) = \emptyset$.

Definition(5) [3] A topological space X is said to be perfectly normal if and only if it is normal and each closed set in it is $G_\delta - set$.

Remark (1) By an open [closed , perfect ,,….] point wise cleavable we mean that the continuous function $f: X \longrightarrow Y$ is an injective open [closed perfect , ….] respectively .

Proposition (1) Let X be a pointwise cleavable space over a class of (T_0,T_1,T_2-spaces) $\mathcal{P}$, then X is (T_0 ,T_1 , T_2-spaces) respectively.

We consider only the case of T_0-space , if X is a point wise cleavable space over a class of T_0-spaces ,then $X \in \mathcal{P}$,

Proof: Let $x \in X$, then there exist a T_0-space $Y \in \mathcal{P}$, and a continuous mapping $f: X \to Y \in \mathcal{P}$,such that $f^{-1}f(x) = \{x\}$. .This implies that for every y $\in Y$ with $x \neq y$,we have $f(x) \neq f(y)$, since Y is T_0-space, so there exists an open set G in Y contains one of the two points but not the other, let $f(x_1) \in G$, $f(y) \notin G$,then $f^{-1}f(x) \in f^{-1}(G)$, $f^{-1}f(y) \notin f^{-1}(G)$, since f is continuous ,then $f^{-1}(G)$ is an open set in X . Hence X is T_0-space

Proposition (2) Let X be a closed pointwise cleavable space over a class of completely Hausdorff spaces $\mathcal{P}$, then $X \in \mathcal{P}$.

Proof: Let $x \in X$, then there exist a completely Hausdorff space $Y \in \mathcal{P}$ and a closed continuous mapping $f: X \to Y \in \mathcal{P}$, such that $f^{-1}f(x) = \{x\}$. This implies that for every y $\in X$ with $x \neq y$,we have $f(x) \neq f(y)$. Since Y is completely Hausdorff , so there exist open sets G , H such that $f(x) \in G$, $f(y) \in H$ and $cl\,(G) \cap cl\,(H) = \emptyset$,then $f^{-1}f(x) \in f^{-1}(G)$ and $f^{-1}f(y) \in f^{-1}(H)$, this implies that $x \in f^{-1}(G)$, y $\in f^{-1}(H)$, since f is continuous , then $f^{-1}(G)$, $f^{-1}(H)$ are open sets of X and $cl(f^{-1}(G)) \cap cl(f^{-1}(H)) \subseteq f^{-1}(cl(G)) \cap f^{-1}(cl(H)) = f^{-1}(cl(G) \cap cl(H)) = f^{-1}(\emptyset)=\emptyset$. Hence X is completely Hausdorff .

Proposition (3) Let X be a closed absolutely cleavable space over a class of regular spaces $\mathcal{P}$, then $X \in \mathcal{P}$.

Proof: Let x be any point in X , and a closed subset F of X ,with $x \notin F$,since X is absolutely cleavable , there exists an injective continuous mapping $f:X \longrightarrow Y \in \mathcal{P}$, such that $f^{-1}f(F)=F$, and for every y $\in Y$ there exists $x \in X$ such that y$=f(x) \Longrightarrow f^{-1}(y)= x$, then $f(F)$ is closed subset of Y and $f(x) \notin f(F)$,since Y is regular, so there exist two open sets G and H of Y with $f(x) \in G$, $f(F) \subset H$, $G \cap H = \emptyset$, then $x \in f^{-1}(G)$, $f^{-1}f(F) \subset f^{-1}(H)$,this implies that $x \in f^{-1}(G)$, $F \subset f^{-1}(H)$,since f is continuous , then $f^{-1}(G)$, $f^{-1}(H)$ are open sets of X , and $f^{-1}(G) \cap f^{-1}(H)= f^{-1}(G \cap H)=f^{-1}(\emptyset) =\emptyset$ .Hence X is regular.

Proposition(4) Let X be a closed absolutely double cleavable space over a class of normal spaces $\mathcal{P}$, then $X \in \mathcal{P}$.

Proof: Suppose F_1, F_2 are two disjoint closed sets of X ,then there exists an injective closed continuous mapping $f:X \longrightarrow Y \in \mathcal{P}$, such that $f^{-1}f(F_1)=F_1$, $f^{-1}f(F_2)=F_2$, since f is closed then $f(F_1)$, $f(F_2)$ are two disjoint closed sets of Y,since Y is normal space ,so there exist two open sets U ,V such that $f(F_1) \subset U$, $f(F_2) \subset V$ and $U \cap V = \emptyset$, $f^{-1}f(F_1) \subset f^{-1}(U)$, $f^{-1}f(F_2) \subset f^{-1}(V)$,this implies that $F_1 \subset f^{-1}(U)$, $F_2 \subset f^{-1}(V)$, since f is continuous ,then $f^{-1}(U)$, $f^{-1}(V)$ are open sets of X and

$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset$. Hence X is normal space.

Proposition (5) Let X be a closed absolutely cleavable space over a class of perfectly normal spaces $\mathcal{P}$, then $X \in \mathcal{P}$

Proof: Suppose F be a closed subset of X, then there exists an injective closed continuous mapping $f:X \longrightarrow Y \in \mathcal{P}$,such that $f^{-1}f(F) = F$,then $f(F)$ is closed and it is G_δ-set in Y, it means that $f(F) = \bigcap_{i=1}^{\infty} G_i$,where G_i is open in Y , for each $i \in$ I , so $F = f^{-1} f(F) = f^{-1}(\bigcap_{i=1}^{\infty} G_i) = \bigcap_{i=1}^{\infty} f^{-1}(G_i)$. This implies that $F = \bigcap_{i=1}^{\infty} f^{-1}(G_i)$, where $f^{-1}(G_i)$ is open ,for each $i \in$ I , i-e F is a G_δ-set in X .Therefore X is perfectly normal space.

Proposition (6) Let X be an open absolutely double cleavable space over a class of connected spaces $\mathcal{P}$, then $X \in \mathcal{P}$.

Proof: Let X be not connected ,then $X = u_1 \cup u_2$ where u_1 and u_2 are disjoint non empty open sets in X, and there exists an injective open continuous mapping $f: X \longrightarrow Y \in \mathcal{P}$, such that $f^{-1}f(u_1) = u_1$, $f^{-1}f(u_2) = u_2$,then $f(X) = f(u_1 \cup u_2) = f(u_1) \cup f(u_2)$ .This implies that $Y = f(u_1) \cup f(u_2)$,then $f(u_1)$ and $f(u_2)$ are disjoint non empty open sets of Y , but Y is connected space which contradicts that Y is connected . Therefore X must be connected.

Proposition (7) Let X be an open cleavable space over a class of locally connected spaces $\mathcal{P}$, then $X \in \mathcal{P}$.

Proof: For $x \in X$, let U be a neighborhood of x ,since X is open cleavable, so there exists an open continuous mapping $f : X \longrightarrow Y \in \mathcal{P}$, such that $f^{-1}f(U)= U$, $f(x) \in Y$,with $f(x) = y$,since Y is Locally connected ,so there exists a connected neighborhood V of y in Y, since f is continuous , so $f^{-1}(V)$ is a connected neighborhood of x in X ,such that $x \in f^{-1}(V) \subset U$ .Hence X is locally connected.

Proposition (8) Let X be an open absolutely cleavable space over a class of compact spaces $\mathcal{P}$, then $X \in \mathcal{P}$.

Proof : Suppose $\{U_i\}_{i\in I}$ be an open cover of X ,since X is an open cleavable space **,so** there exists an injective open mapping , $f : X \longrightarrow Y \in \mathcal{P}$, such that $f^{1} f$ $\{U_i\}_{i\in I}$ =$\{U_i\}_{i\in I}$, then $\{f(U_i)\}_{i\in I}$ is an open cover of Y ,since Y is compact so there exists $I_0 \subset I$ such that $\{f(U_i)\}_{i\in I_0}$ is a finite sub cover of Y , then $\{f^{-1}f(U_i)\}$ is open for each $i \in I_0$, i.e. $\{f^{1}(f(U_i)\}_{i\in I_0}$ is a finite sub cover of X , but $f^{-1}f(U_i)=U_i$ for each $i \in I$, so $\{U_i\}_{i\in I_0}$ is a finite sub cover of X .Hence X is compact.

Proposition (9) Let $\boldsymbol{X}$ be a perfect cleavable space over a class of locally compact spaces $\mathcal{P}$, then $\boldsymbol{X} \in \mathcal{P}$.

Proof: Let $\boldsymbol{U}$ be a closed neighborhood of $\boldsymbol{x} \in \boldsymbol{X}$,since $\boldsymbol{X}$ is a perfect cleavable space ,so there exists a closed continuous mapping $\boldsymbol{f} : \boldsymbol{X} \longrightarrow \boldsymbol{Y} \in \mathcal{P}$, such that $\boldsymbol{f}^{-1}\boldsymbol{f}(\boldsymbol{U})=\boldsymbol{U}$,for any point y in $\boldsymbol{Y}$, $\boldsymbol{f}$(x)=y $\Leftrightarrow \boldsymbol{x}=\boldsymbol{f}^{-1}(\boldsymbol{y})$, since $\boldsymbol{f}$ is continuous and closed so $\boldsymbol{f}$ $(\boldsymbol{U})$ is a closed neighborhood of $\boldsymbol{y}$ in $\boldsymbol{Y}$, but $\boldsymbol{Y}$ is locally compact, so $\boldsymbol{f}$ $\boldsymbol{U}$ is compact neighborhood of $\boldsymbol{y}$, since $\boldsymbol{f}$ is perfect , so $\boldsymbol{f}^{-1}\boldsymbol{f}(\boldsymbol{U}) = \boldsymbol{U}$ is compact closed neighborhood of $\boldsymbol{x}$ in $\boldsymbol{X}$. Hence $\boldsymbol{X}$ is locally compact.

2. References

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