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SOME PROPERTIES OF RELATIVELY COMPACT SPACE

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Abstract : We consider the property of relative compactness of subspaces of Hausdorff spaces We prove that , the property of being a relatively compact subspace of a Hausdorff spaces is strictly stronger than being a regular space and strictly weaker than being a Tychonoff space.

I - Introduction : All spaces under consideration are assumed to be Hausdorff topological spaces . Codensation is a one-to-one continuous map onto Cardinals are initial ordinals .

Symbols ω , Z , R , stand for the first infinite cardinal , the set of all integers and real line , respectively . A space , closed in every regular (Hausdorff) space containing it is called R - closed (H-closed) .

A subspace Y of space X is said to be relatively comact in X iff every open cover of X has a finite subcover of Y [see 6] .

A subspace Y of a space X is said to be relatively normal in X iff whenever F_1 and F_2 are closed subsets of Y an $CL_X F_1 \cap CL_X F_2 = \varnothing$, then there are disjoint open] subsets U_1 , and U_2 of X such that $F_1 \subset U_1$ and $F_2 \subset U_2$.

Every relatively compact subspace of a space X is relatively normal in X and every relatively normal subspace is a regular space [see 1]. On the other hand every subspace Y of a compact space K is relatively compact in K . Hence every Tychonoff space Y can be embedded into some space (e.g.$I^{w(Y)}$) as a relatively compact subspace , Therefor we could consider being a relatively compact subspace , as a separation property , between regularity and complete regularity. Below we shall show that our property is strictly stronger than regularity and strictly weaker than complete regularity. We also use the followig separation property.

Definition 1. 1 : A space X has the countable separation property iff whenever F is a closed subspace of X and $x \notin F$, there are open W_i, $i \in \omega$, such that for each $i \in \omega$, $x \notin W_i$, $F \subset W_i$ and $CL_X W_{i+1} \subset W_i$.

Clearly, every Tychonoff space has the countable separation proprty and each space with countable separation property is regular

Definition 1 . 2 : A space Y will be said to be **potentially compact**, if there is a space X such that Y is a subspace of X and Y isd relatively compact in X .

Thus we have,

Proposition 1 . 3 : Every potentially copact space is regular [see 1].

The folloing observation helps to identify several regular spaces wich are not relatively compact in any Hausdorff space.

Prpposition 1 . 4 : Let Y be an R-closed space wich is relatively compact in a space X , then Y is copact.

Proof : Choose arbitrary $x \in X \setminus Y$ **and let** $Y_1 = Y \cup \{x\}$. Clearly, Y is relatively compact in X . Hence Y is regular (see prop. 1.3). Then Y is closed in Y so $x \notin CL_X Y$. It follows that Y is closed in X . Thus Y is compact in itself, $i.e$ Y is compact.

So any regular R-cosed non-comact space is not relatively compact in any Hausdorff space containing it. One of the well-known examples with such properties is the Joens space over Y= $= (\omega_1 + 1) x (\omega_1 + 1) \setminus \{(\omega_1, \omega_1)\}$. [see 4]. [see also 5p.150-153].

Let $C = \omega_1 x \{0\}$, $D = \{0\} x \omega_1$, $\overline{Y}$ be the quotient space obtained from $Y x \omega$ by identifing C_{2n+1} with C_{2n+2} and D_{2n+2} with D_{2n+3} for each $n \in N$ and

$\omega : Y \times \omega \rightarrow \tilde{Y}$ be the natural quotient map. Let $\tilde{\tilde{Y}} = \tilde{Y} \cup \{z\}$ topologized as follows

$\tilde{Y}$ is an open subspace of $\tilde{\tilde{Y}}$ and $\{\{z\} \cup \bigcup_{n>k} Y_n ,\ k \in \omega\}$ is a base in z .

The resulting space is regular , not completly regular space [see 4 and 5] .

Proposition 1 . 5 : Let $\tilde{Y}$ be the Jones space over $(\omega_1 + 1) \times (\omega_1 + 1) \setminus \{(\omega_1, \omega_1)\}$, then $\tilde{Y}$ is not relatively compact in any Hausdorff space .

Proof : In view of proposition 1.4 , we need to prove only that $\tilde{Y}$ is R-closed . Assume the contrary . X is a regular space $\tilde{Y} \subset X$ and $x \in CL_X \tilde{Y} \setminus \tilde{Y}$. Clearly $x \in CL_X Y_n$ for some $n \in \omega$. now we need the folloing fact .

Claim 1.6 : Let X be a regulare space and $Y = (\omega_1 + 1) \times (\omega_1 + 1) \setminus \{(\omega_1, \omega_1)\} \subset X$. Then

$|X \setminus Y| \leq 1$. If , moreover $X \setminus Y \neq \varnothing$, then $CL_X\ Y = (\omega_1 + 1) \times (\omega_1 + 1) = \beta Y$

It follows that $x \in CL\ Y$ and by induction we have that $x \in CL\ Y$ for each $k \geq n$

so $x = z$, contradicting $x \notin \tilde{Y}$.

Rmark 1.7 : The above arguments also work to show that $\tilde{Y}$ is not relatively normal in any regular space .

To constract a non-Tychonoff space Y wich is relatively compact in some Hausdorff space X we need the following lemma .

Lemma 1.8 : There are a Hausdorff space X and a Tychonoff zero-demensional relatively compact subspace Y of X and two uncountable closed disjoint G_δ - sbsets F_1 and F_2 of a space Y such that $CL_X F_1 \cap CL_X F_2 = \varnothing$, F_1 and F_2 can be separated (in Y) by disjoint open sets , but whenever $f : Y \rightarrow R$ is a continuous function , then $|f^{-1}(0) \cap F| > \omega$ implies $|F_2 \setminus f^{-1}(0)| \leq \omega$ in particular F_1 , and F_2 cannot be seperated (inY) by a continuous real-valued function .

Proof : Let Y be be the set

$$[-1, 1] \times [0, 1] \setminus \{(-1, 0), (1, 0)\} .$$

Basic elements for topology of Y are either :

1). $\{x\}$ for $x \in (-1,1) \times (0,1]$,

2). $\{[-1,-1+\varepsilon) \times \{y\} ; 0<\varepsilon<1\}$ for $(x,y) \in \{-1\} \times (0,1]$; $\{(1-\varepsilon,1] \times \{y\};\ 0<\varepsilon<1$

for $(x,y) \in \{1\} \times (0,1]$

3). $\{\{(x+e(1-|x|)t,\ t)\ ;\ t \in [0,1] \setminus k,\ e \in \{-1/2,\ 0,\ 1/2\}\},\ k \in [(0,1]]^{<\omega}\}$ for

$(x,y) \in (-1,1) \times \{0\}$.

A typical neighborhood V_a of a point $(a,0)$ can be described in the following way .Tak the vertical line ℓ_0 ; $x=a$ through $(a,0)$ and the tow lines ℓ_+ and ℓ_- through $(a,0)$ symmetrical with respect to ℓ_0 having the slope $\pm 2/(1-|a|)$. Then V_a is the intersection of the union $\ell_0 \cup \ell_+ \cup \ell_-$ with the rectangle $[-1,1] \times [0,1]$ frome wich any finite set of points different from $(a,0)$ is removed .

Clearly Y is a Hausdorff zero-dimensional (hence Tychonoff) space . Let

$$F_1 = \{-1\} \times (0,1]$$

$$F_2 = \{1\} \times (0,1)$$

$$U_1 = [-1,-1+1/10) \times (0,1]$$

$$U_2 = (1-1/10,1] \times (0,1]$$

Then F_1 , F_2 are disjoint closed G_δ - subsets of Y , U_1 , U_2 are disjoint open neighborhood of F_1 and F_2 respectively . Moreover for

$$W_i = [-1,-i+1/2^{2i+1}) \times (0,1) : i \in \omega ,$$

we have

$$\bigcap_{i\in\omega} W = F_1 \text{ and } CL_X\, W_{i+1} \subset W_i ,$$

First we prove :

Claim 1. 9 : Let $f : Y \to R$ be a continuous function such that $|f^{-1}(0) \cap F_1| > \omega$,

then $|F_2 \setminus f^{-1}(0)| \leq \omega$

Proof : Assume the contrary $f : Y \to R$ is a continuous function such that

$|f^{-1}(0) \cap F_1| > \omega$ and $|F_2 \setminus f^{-1}(0)| > \omega$. Then there are $\varepsilon > 0$ and $\rho \in [F_2]^{<\omega}$, such that
$\forall \rho \in P$, $f(p) > 3\varepsilon$. Since f is continuous there are $\delta > 0$, $L \in [p]^{>\omega}$, $M \in [F_1]$, such that $f(x,y) > 2\varepsilon$ whenever $1-\delta < x < 1$, $y \in L$ and $f(x,y) < \varepsilon$ whenever $-1 < x < -1+\delta$, $y \in M$. By the definition of the base of Y and continuity of f we have $f(x,0) < \varepsilon$ for each $-1 < x < -1+\delta$ and $f(x,0) > 2\varepsilon$ for each $1-\delta < x < 1$.
Moreover there is a family $\{K_x ; x \in (-1,-1+\delta)\} \subset [(0,1)]^{<\omega}$ such that
$f(\cup\{x+e(1-|x|)t ; t \in [0,1] \setminus K_x, e \in \{-1/2,0,1/2\}\}, x \in (-1,-1+\delta)\}) \subset [0,\varepsilon)$

It follows that $f(x,0) < \varepsilon$ for each $-1 < x < -1 + 4/5\ \delta$ except for finitely many times.

Therefor $|\{x \in (-1, -1+ 6/5\ \delta) ; f(x,0) > \varepsilon\}| < \omega$. Applying the arument above finitely many times we obtainthat $|x \in (-1, 1/5\ \delta) ; f(x,0) > \varepsilon\}| < \omega$. Similarly starting from the right end of the segment $[-1,1]$, we can prove that

$|\{x \in (-1/5\ \delta, 1) ; f(x,0) < 2\varepsilon\}| < \omega$. This contradiction completes the proof of the claim.

Now we shall construct a space X Consider the Stone-Cech extension Y of the space Y.
Let

$$\tilde{G}_1 = CL_{\beta Y}(F_1) \setminus Y$$

$$\tilde{G}_2 = CL_{\beta Y}(F_2) \setminus Y$$

$$\tilde{G}_3 = \beta Y \setminus (\tilde{G}_1 \cup \tilde{G}_2 \cup Y)$$

Let $X = G_1 \cup G_2 \cup . G_3 \cup Y$ be the disjoint union of copies of sets $\tilde{G}_1, \tilde{G}_2, \tilde{G}_3, \tilde{Y}$.

Basic elements for topology on X are either ;

1). U for some open $U \subset Y$,

2). $\{g\} \cup (U \cap Y)$, for some $g \in G_3$ and some neighborhood U of g in βY

3). $\{g\} \cup (U \cap U_1)$, for some $g \in G_1$ and some neighborhood U of g in βY

4). $\{g\} \cup (U \cap U_2)$, for some $g \in G_2$ and some neighborhood U of g in βY

Now $U_1 \cap U_2 = \varnothing$ implies that X is a Hausdorff space. Clearly, every open cover γ of X induces an open γ' of βY members of which are union of at most two elements of γ . it follows that Y is relativelyy comact in X . Finally $CL_X\ F_1 = G_1$, $CL_X\ F_2 = G_2$ yields , $CL_X\ F_1 \cap CL_X\ F_2 = \varnothing$. Thus Y and X satisfy all the required conditions .

We now turn to the second example .

Example 1. 10 : There is a regular non-Tychonoff space Y with the countable separation property which is relatively compact in some Hausdorff space .

Proof : We use the notation of lemma 1.8 . Feed Y and X into the Jones Machine [see 1.5] Let $A = F_1$, $B = F_2$, $C = CL_X F_1$, $D = CL_X F_2$ and let $\bar{X}$ be the quotiant space obtained from $X \times \omega$ by identifying C_{2n+1} , with C_{2n+2} and D_{2n+2} with D_{2n+3} for each $n \in N$ an $q : X \times \omega \to \bar{X}$ be the natural quotient map . Let $\tilde{X} = \bar{X} \cup \{z\}$ topologized as follows . $\bar{X}$ is an open subspace of $\tilde{X}$, and $\{ \{z\} \cup \bigcup_{n>k} X_n ; k \in \omega \}$ is a base in z . Let $\tilde{Y} = q(Y \times \omega) \cup \{z\}$. Clearly $\tilde{X}$ is a Hausdorff space and $\tilde{Y}$ is a regular non-Tychonoff subspace of $\tilde{X}$ [see 4,5] .Since for each $n \in \omega$, $Y \times n$ is relatively compact in $X \times n$ and hence in $\tilde{X}$ and every neighborhood of z contains all except at most finitely many $Y \times n$, $\tilde{Y}$ is relatively compact in $\tilde{X}$. Finally , since $\tilde{Y} \setminus \{z\}$ is Tychonoff space and $F_1 = \bigcap_{i \in \omega} W_i$ where $CL_Y W_{i+1} \subset W_i$, it follows that Y has the countaible separation property .

Example 1.11 : There is aregular space Z wich is relatively compact in some Hausdorff space and has the countaable separation property , but wich is not functionally Hausdorff .

Example 1. 12 : There is a regular space Z wich is relatively compact in some Hausdorff space , such that all real -valued functions on Z are constant .(see 3)

II - Wat if X has some separation property stronger than Hausdorff ?

First , since every Hausdorff space can be embedded as a closed subspace into some semiregular space the followig assertion holds .

Proposition 2. 1 : A space Y can be embedded as a relatively compact subspace into a Hausdorff iand only if Y can be embedded as a relatively compact subspace into semiregular space .

On the other hand if Y is relatively compact in some Urysohn space ,then Y must be Tychonoff space . Indeed we have .

Proposition 2 .2 : Let Y be a dence relatively compact subspace of a space X , then X is H-closed .

Proof : Direct cheek .

Theorem 2.3 : Let Y be a dense relatively compact subspace of an Urysohn space X , then there is a compact space Z and condensation $f : X \to Z$, such that , for each $\bar{y} \in X$ the restiction $f\,|_{Y \cup \{y\}}$ of f to $Y \cup \{y\}$ is a homeomorphism of $Y \cup \{y\}$ onto the image .

Proof : Let Z be the semiregularization of a space Z , and let $f : X \to Z$ be the natural condensation . Then Z is a semiregular Urysohn space , and f (Y) is relatively compact in Z proposition 2.2 yields that Z is H-closed . So Z is semiregular Urysohn H-closed space . Hence Z is compact . Now take arbitrary $y \in X$, then $Y \cup \{y\}$ is relatively compact in X . Therefor the semiregularization of $Y \cup \{y\}$ is again $Y \cup \{y\}$.

It follows that $f\,|_{Y \cup \{y\}}$ is a homeomorphism .

Definition 2.4 : A subspace Y of a space X is said to be real-normal in X iff every two subspces of Y having disjoint closures in X can be separated in X by a continuous real-valued function .

Corollary 2.5 : Let Y be a dense relatively comact subspace of an Urysohn space X , then Y is Tychonoff , X is functionally Hausdorff and Y is real -normal in X .

Proof : We shall prove that Y is real-normal in X , other properties are obvious .

Let $F_1 , F_2 \subset Y$, $CL_X\ F_1 \cap CL_X\ F_2 = \varnothing$. Use notation of 2.3 , since for each $y \in X$,$f\,|_{Y \cup \{y\}}$ is a homeomorphism $CL_Z\ f(F_1) \cap CL_Z\ f(F_2) = \varnothing$

Hence $CL_Z\ f\ (F_1)$ and $CL_Z\ f(F_2)$ can be separation in compact space Z by a continuous real-valued function . Therefor , the same is true for F_1 and F_2 in X

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