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Boundedness Criteria for Solutions of Some Nonlinear Differential Equations of Second Order

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A B S T R A C T

Mathematical modelling phenomena of most applied sciences is associated with second order nonlinear differential equations, which are not easily solvable. Therefore, the study of behavior of the solutions has attracted the attention of many mathematicians worldwide. In the present work, we discuss some clear assumptions for the boundedness of all solutions of some non-linear differential equations of second order. The main tools in the proofs of our results are Gronwall's inequality and Bonnet's Theorem. The results obtained here extend and/or improve some of well-known results in the literature. Further, some illustrative examples are provided to show the applicability of the new results.

1 Introduction

In the recent years, there has been an increasing interest in studying the qualitative theory of solutions of nonlinear differential equations of second order. This is due to the fact that the second order nonlinear differential equations play an important role in many areas such as mechanics, engineering, economy, control theory, physics, chemistry, biology, medicine, atomic energy and information theory (see Ademola & Arawomo (2011), Ahmed and Ali (2019), Amhalhil (2021), Elabbasy & Elzeiny (2011), Saad et. al. (2013), Salhin (2019), Wong and Burton (1965) and the references cited therein). Boundedness theory as a part of the qualitative theory of non-linear differential equations has been extensively discussed by this time. An excellent summary of the results related to the problem of boundedness of solutions can be found in Athanassov (1987), Bihari (1957), Saker (2006) and Tunc (2010). One can also see the papers of Chang (1970), Graef and Spikes (1975), Hartman (1982) and Kroopnick (1995).

Consider the second order nonlinear differential equation of the form:

$$\left(r(t)\dot{x}(t)\right)^{\bullet} + q(t)g(x(t)) = p(t) \qquad (1.1)$$

Where r, q and p are real valued continuous functions on the half interval $[t_0,\infty)$, $t_0 \ge 0$, r is a positive function, g is a continuous function on the

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real line $\Re$ and satisfies the condition that

$xg(x) > 0 \quad for all \ x \neq 0.$

We recall that the solution $x(t)$ of Eq. (1.1) is called bounded if there exists a positive constant M_0 such that $|x(t)| \leq M_0$ for all $t \geq T \geq t_0$, This M_0 may be determined for each solution.

Wong (1966, 1967, 1968), Wong and Burton (1970), Waltman (1963) and Lalli (1969) discussed Eq. (1.1) in the case when $r(t) \equiv 1$ and derived many boundedness criteria. A primary purpose of the present paper is to contribute further in the direction of establishing sufficient conditions for all solutions of Eq. (1.1) to be bounded. As a consequence, we are able to extend and/or improve a number of well-known results in the literature. Besides, our new results will be illustrated by some examples.

Define

$$G(x) = \int_0^x g(v)\, dv \qquad \text{and} \qquad R(t) = \int_{t_0}^t \frac{|p(s)|}{r(s)} ds$$

It will be convenient to write Eq. (1.1) as the equivalent differential system

$$\begin{cases} \dot{x}(t) = y \\ \dot{y}(t) = \dfrac{p(t) - \dot{r}((t)y(t) - q(t)\, g(x(t))}{r(t)} \end{cases} \tag{1.2}$$

Before introducing our main results, we remind some basic results which are quite useful elements and in fact those results are interesting in their own rights.

2 Auxiliary Results

The next fundamental lemma, which is also known as Gronwall's inequality, will be needed. (see Bellman (1953), P. 35).

Lemma 2.1 If u and v are nonnegative real valued functions, c is a positive constant and if

$$u(t) \leq c + \int_{t_0}^t u(s)v(s)\, ds,$$

then

$$u(t) \leq c \exp\left(\int_{t_0}^t v(s)\, ds\right).$$

The following result is very useful to simplify the proofs of the obtained results here. (Also known as The Bonnet's Theorem, see Bartle (1976)).

Theorem 2.1 Let Q and R be continuous functions on $[a,b]$ with $Q \geq 0$. Then for some $c \in [a,b]$,

i. If Q is increasing, then

$$\int_a^b Q(x)R(x)\,dx = Q(b)\int_c^b R(x)\,dx,$$

ii. If Q is decreasing, then

$$\int_a^b Q(x)R(x)\,dx = Q(a)\int_a^c R(x)\,dx,$$

3 Main Results

Theorem 3.1. Suppose that

(1) $G(x)$ is bounded from below and $G(x) \to \infty$ as $|x| \to \infty$,

(2) $r(x)$ is bounded from above and non-decreasing on $[t_0, \infty)$ as $|x| \to \infty$

(3) $q(t)$ is positive and non-decreasing function on $[t_0, \infty)$,

(4) $\lim_{t \to \infty} R(t) < \infty$.

Then, every solution of Eq. (1.1) is bounded.

Proof. From the condition (1), there exists a constant $k_1 > 0$ such that $G(x) \geq -k_1$ for all $x \in \Re$, thus

$$G(x) + k_1 \geq 0 \text{ for all } x \in \Re$$

Now, define a function V as follows:

$$V(t)=\frac{G(x)+k_1}{r(t)}+\frac{y^2}{2q(t)} \quad ,t\in[t_0,\infty)$$

Then $$\dot{V}(t)=\frac{yg(x(t))}{r(t)}-\frac{(G(x)+k_1)\dot{r}(t)}{r^2(t)}+\frac{y\dot{y}}{q(t)}-\frac{y^2\dot{q}(t)}{2q(t)}$$

From which it follows by (1.2) that

$$\dot{V}(t)=\frac{yg(x(t))}{r(t)}-\frac{(G(x)+k_1)\dot{r}(t)}{r^2(t)}+\frac{yp(t)}{r(t)q(t)}-\frac{yg(x(t))}{r(t)}-\frac{\dot{r}(t)y^2}{r(t)q(t)}-\frac{y^2\dot{q}(t)}{2q(t)}$$

By the assumptions (1), (2) and (3) it can be shown easily that

$$\dot{V}(t)\le\frac{yp(t)}{r(t)q(t)} \quad ,t\ge t_0$$

Integrating the last inequality from t_0 to some $t\ge t_0$ we have

$$V(t)\le V(t_0)+\int_{t_0}^{t}\frac{y(s)p(s)}{r(s)q(s)}ds \quad ,t\ge t_0 \qquad (3.1)$$

which leads to

$$\frac{y^2+1}{2q(t)}\le\frac{1}{2q(t)}+V(t_0)+\int_{t_0}^{t}\frac{y(s)p(s)}{r(s)q(s)}ds \quad ,t\ge t_0$$

But we have $y\le\frac{1}{2}(y^2+1)$ for all $y\in\Re$, then

$$\frac{y}{q(t)}\le\frac{1}{2q(t)}+V(t_0)+\int_{t_0}^{t}\frac{y(s)p(s)}{r(s)q(s)}ds \quad ,t\ge t_0$$

Since $q(t)$ is a non-decreasing function, then by Theorem 2.1, we conclude

$$\frac{y(t)}{q(t)}\le\frac{1}{2q(t_0)}+V(t_0)+\int_{t_0}^{t}\frac{y(s)p(s)}{r(s)q(s)}ds \quad ,t\ge t_0$$

That is

$$\left|\frac{y}{q(t)}\right|\le\frac{y^2+1}{2q(t)}\le k_2+\int_{t_0}^{t}\frac{|p(s)|}{r(s)}\left|\frac{y(s)}{q(s)}\right|ds \quad ,t\ge t_0$$

where $k_2=\frac{1}{2q(t_0)}+V(t_0)$ is a positive constant, and as an application of Lemma 2.1, we get

$$\left|\frac{y}{q(t)}\right|\le k_2\exp\left(\int_{t_0}^{t}\frac{|p(s)|}{r(s)}ds\right)\le B<\infty \qquad (3.2)$$

By using (3.2) in (3.1), we obtain

$$V(t)\le V(t_0)+B\int_{t_0}^{t}\frac{|p(s)|}{r(s)}ds\le B_1<\infty.$$

It is now notable that $V(t)$ is bounded. But

$$V(t)\ge\frac{G(x)+k_1}{r(t)}$$

Since $r(t)$ is bounded and then $G(x)$ is bounded from which it follows that $x(t)$ is bounded too. The proof is complete.

Example 3.1: Consider the following differential equation:

$$\left[\left(\frac{t}{t+1}\right)\dot{x}(t)\right]^{\bullet}+(t^2+3t)x^3(t)=\left(\frac{t}{t+1}\right)e^{-5t} \text{ , } t\ge t_0>0 \qquad (3.3)$$

We note that

(i) $xg(x)=x^4>0 \ \ \forall x\ne 0$,

$G(x)=\int_0^x g(u)du=\frac{1}{4}x^4\ge 0>-k_1$, $k_1>0$ $G(x)\to\infty$ as $|x|\to\infty$

(ii) $r(t)=\frac{1}{t+1}>0, \dot{r}(t)=\frac{1}{(t+1)^2}>0$ *and*

$r(t)\le 1$ *for all* $t\ge t_0>0$

(iii) $q(t)=t^2+3t>0, \quad \dot{q}(t)=2t+3>0$

(iv) $R(t)=\int_{t_0}^{t}\frac{|p(s)|}{r(s)}ds=\frac{1}{5}\left(e^{-5t_0}-e^{-5t}\right), \quad \lim_{t\to\infty}R(t)<\infty$

Hence by Theorem 3.1, all solutions of Eq. (3.3) are bounded.

Theorem 3.2: Assume that conditions (1), (2) and (4) hold and assume in addition that

(5) $\gamma(t)=\dfrac{q(t)-1}{r(t)}$ is a positive and non-increasing function on $[t_0,\infty)$.

Then every solution of Eq. (1.1) is bounded.

Proof. From the condition (1), there exist $k_1>0$ such that $G(x)\ge -k_1$ for all $x\in R$, thus

$$G(x)+k_1\ge 0 \text{ for all } x\in R$$

Define the function V as follows:

$$V(t)=\frac{G(x)+k_1}{r(t)}+\frac{y^2}{2},\quad t\ge t_0$$

we obtain

$$\dot{V}(t)=\frac{yg(x(t))}{r(t)}-\frac{(G(x)+k_1)\dot{r}(t)}{r^2(t)}+y\dot{y}\qquad ,t\ge t_0$$

From which it follows with (1.2) that

$$\begin{aligned}\dot{V}(t)&=\frac{yg(x(t))}{r(t)}-\frac{(G(x)+k_1)\dot{r}(t)}{r^2(t)}+\frac{yp(t)}{r(t)}-\frac{yq(t)g(x(t))}{r(t)}-\frac{\dot{r}(t)y^2}{r(t)}\quad ,t\ge t_0\\&\le\frac{yg(x(t))}{r(t)}+\frac{yp(t)}{r(t)}-\frac{yq(t)g(x(t))}{r(t)}\\&\le\frac{yp(t)}{r(t)}-\frac{(q(t)-1)}{r(t)}(yg(x(t)))\end{aligned}$$

Integrating the last inequality from t_0 to some $t\ge t_0$ we have

$$\begin{aligned}V(t)&\le V(t_0)+\int_{t_0}^{t}\frac{y(s)p(s)}{r(s)}ds-\int_{t_0}^{t}\frac{(q(s)-1)}{r(s)}y(s)g(x(s))ds\\&\le V(t_0)+\int_{t_0}^{t}\frac{y(s)p(s)}{r(s)}ds-\int_{t_0}^{t}\gamma(s)y(s)g(x(s))ds\end{aligned}$$

since $\gamma(t)$ is a positive and non-increasing function on $[t_0,\infty)$, then by Theorem 2.1, for all $t\ge t_0$ there exists $a_t\in[t_0,t]$ such that

$$\begin{aligned}V(t)&\le V(t_0)+\int_{t_0}^{t}\frac{y(s)p(s)}{r(s)}ds-\gamma(t_0)\int_{t_0}^{a_t}g(x(s))\dot{x}(s)ds\\&\le V(t_0)+\int_{t_0}^{t}\frac{y(s)p(s)}{r(s)}ds-\gamma(t_0)\int_{x(t_0)}^{x(a_t)}g(u)du\\&\le V(t_0)+\int_{t_0}^{t}\frac{y(s)p(s)}{r(s)}ds-\gamma(t_0)\left[\int_{x(t_0)}^{0}g(u)du+\int_{0}^{x(a_t)}g(u)du\right]\\&\le V(t_0)+\int_{t_0}^{t}\frac{y(s)p(s)}{r(s)}ds+\gamma(t_0)G(x(t_0))-\gamma(t_0)G(x(a_t))\end{aligned}$$

Then we have

$$V(t)\le V(t_0)+\gamma(t_0)G(x(t_0))+k_1\gamma(t_0)+\int_{t_0}^{t}\frac{y(s)p(s)}{r(s)}ds \qquad (3.4)$$

which yields that

$$\frac{y^2+1}{2}\le\frac{1}{2}+V(t_0)+\gamma(t_0)G(x(t_0))+k_1\gamma(t_0)+\int_{t_0}^{t}\frac{y(s)p(s)}{r(s)}ds$$

But we have $y\le\frac{1}{2}(y^2+1)$, thus

$$y(t)\le\frac{1}{2}+V(t_0)+\gamma(t_0)G(x(t_0))+k_1\gamma(t_0)+\int_{t_0}^{t}\frac{y(s)p(s)}{r(s)}ds$$

Then

$$|y(t)|\le\frac{y^2+1}{2}\le k_3+\int_{t_0}^{t}\frac{|y(s)p(s)|}{r(s)}ds$$

where

$$k_3=\frac{1}{2}+V(t_0)+\gamma(t_0)G(x(t_0))+k_1\gamma(t_0)$$

is a positive constant, and as an application of Lemma 2.1, we get

$$|y(t)|\le k_3\exp\int_{t_0}^{t}\frac{|p(s)|}{r(s)}ds\le B_2<\infty \qquad (3.5)$$

By using (3.5) in (3.4), we obtain

$$V(t)\le V(t_0)+\gamma(t_0)G(x(t_0))+k_1\gamma(t_0)+B_2\int_{t_0}^{t}\frac{p(s)}{r(s)}ds$$

Therefore the last inequality above becomes

$$V(t)\le V(t_0)+B\int_{t_0}^{t}\frac{|p(s)|}{r(s)}ds\le B_1<\infty$$

which yields that

$$V(t) \le B_3 < \infty$$

Hence $V(t)$ is bounded. On the other hand we know that

$$V(t) \ge \frac{G(x)+k_1}{r(t)}$$

Since $r(t)$ is bounded, then $x(t)$ is bounded too. The proof is complete.

Example 3.2: Consider the following differential equation:

$$\left[\frac{t^2}{t^2+1}\dot{x}(t)\right]^{\cdot} + \left(\frac{t^3e^{-t}}{1+t^2}+1\right)\left(x^9(t)+\frac{6x^5(t)}{1+x^6(t)}\right) = \left(\frac{1}{1+t^2}\right)\frac{sin4t}{(\cos^2(2t)+1)}, \; t \ge t_0 \ge 1 \qquad (3.6)$$

We note that

(i) $$xg(x) = x\left(x^9(t)+\frac{6x^5(t)}{1+x^6(t)}\right) = x^{10}(t)+\frac{6x^6(t)}{1+x^6(t)} > 0 \; \forall x \ne 0$$

and $G(x) \to \infty$ as $|x| \to \infty$

(ii) $r(t) = \frac{t^2}{t^2+1} > 0$, $\dot{r}(t) = \frac{2t}{(t^2+1)^2} > 0$, $r(t) \le 1 \;\; \forall \; t \ge t_0 > 1$

(iii) $$R(t) = \int_{t_0}^{t}\frac{|p(s)|}{r(s)}ds = \int_{t_0}^{t}\frac{|sin4s|}{(\cos^2(2s)+1)s^2}ds \le \int_{t_0}^{t}\frac{ds}{s^2}ds = -\frac{1}{t}+\frac{1}{t_0},$$

$$\lim_{t\to\infty} R(t) < \infty$$

(iv) $\gamma(t) = \frac{q(t)-1}{r(t)} = te^{-t} > 0$ and $\dot{\gamma}(t) = e^{-t}(1-t) \le 0$

Hence by Theorem 3.2, all solutions of Eq. (3.6) are bounded.

Remark 3.1: Theorems 3.1 and 3.2 extend and improve some of the related results of Burton and Townsend (1968), Olehnik (1972) & (1973), Greaf and Spikes (1975), Waltman (1963) and Wong (1967).

4 Conclusion

Throughout this paper, we concerned with the boundedness characteristic of a class of nonlinear differential equations. In this direction, we determined some new sufficient conditions for all solutions of equation (1.1) to be bounded. Further we introduced some illustrative examples. A remark was also included to show the evidence of our main results.

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