AL-TAHDI UNIVERSITY

FACULTY OF SCIENCE

DEPARTMET OF MATHEMATICS

(A study of Difference Equation on the form)

$$y_{n+1} = \frac{\alpha + \beta y_n^p + \gamma y_{n-1}^p + \delta y_{n-2}^p}{a + b y_n^q + c y_{n-1}^q + d y_{n-2}^q}, n = 0,1,.......$$

A DISSERTATION SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS.

BY

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Title of Thesis

A Study of Difference Equation on the Form

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بسم الله الرحمن الرحيم

((دعواهم فيها سبحانك اللهم وتحيتهم فيها سلام وأخر دعواهم

أن الحمد لله رب العالمين))(10)

صدق الله العظيم

الايه (10)من سورة يونس

شكر وتقدير

الحمد لله رب العالمين،اللهم لك الحمد كما ينبغي لجلال وجهك ولعظيم سلطانك ،والصلاة والسلام علي سيد الأولين والآخرين ،سيدنا محمد وعلي اله وصحبه أجمعين،

وبعد :

فأني أتقدم بخالص الشكر والتقدير إلى :الدكتور /نبيل زكي فريد علي تفضله بالإشراف على هذه الرسالة ،وعلى ما انفق من وقت ،وبذل من جهد لإتمام هذه الرسالة فله من الله سبحانه وتعالي خير الجزاء وأحسنه.

كما لا يفوتني أن أتوجه بالشكر والتقدير إلى/ أخي فرج و كل من ساهم في أعداد هذا البحث وقدم لي النصيحة والإرشاد والدعم المتواصل.

فلا املك ما أقوله لكل هؤلاء إلا الدعاء لهم بالخير ،فاللهم وفقهم جميعا وسدد خطاهم وضاعف لهم الأجر والثواب ،وثبتنا وإياهم بالقول الثابت في الحياة الدنيا وفي الآخرة ،انك سميع الدعاء.

"Contents"

Chapter 3: On some properties of solutions of third order recursive rational difference equations

Summary

The project of this thesis concerns some properties of solutions of differential equations (difference equations).

We believe that the results about third order rational difference equation are of paramount importance in their own right and furthermore we believe that these results offer prototypes towards the development of the basic theory of the global behavior of solutions of non-linear difference equation of third order.

Chapter(1)

Chapter 1

Preliminary Results and Definitions

1.1 Introduction

The primary purpose of this thesis is to study the behavior of some difference equations, where difference equations have received much attention of many scientists from various disciplines .Perhaps this is largely due to the advent of computers where differential equations are solved by using their approximate difference equation formulations . With the use of a computer one can easily experiment with difference equations and one can easily discover that such equations posses fascinating properties with a great deal of structure and regularity . Of course all computer observations and predictions must also be proven analytically . Therefore this is a fertile area of research , still in its infancy , with deep and important results.

Difference equations appear as natural descriptions of observed evolution phenomena because most measurements of time ivolving variables are discrete and as such these equations are in their own right important mathematical models. More importantly, difference equations also appear in the study of discretization methods for differential equations. Several results in the theory of difference equations have been obtained as more or less natural discrete analogues of corresponding results of differential equations. This is especially true in the case of lyapunov theory of stability. Nonetheless, the theory of difference equations is a lot richer

Than the corresponding theory of differential equations. For example, a simple difference equation resulting from a first order differential equation may have a phenomena often called appearance of "ghost" solutions or existence of chaotic orbits that can only happen for higher order differential equations and the theory of difference equations is interesting in itself.

The applications of the theory of difference equations is rapidly increasing to various fields such as numerical analysis, control theory, finite mathematics and computer science.

Thus, there is many reason for studying the theory of difference equations as a well deserved discipline.

Examples of discrete phenomenon in nature abound and yet somehow its continuous version has commandeered all our attention perhaps due to that special mechanism in human nature which permits us to notice what we have been conditioned . Although difference equations manifest themselves as mathematical models describing real life situations in probability theory, queuing problems, statistical problems, stochastic time series ,combinatorial analysis ,number theory , geometry , electrical networks , quanta in radiation , genetics in biology , economics , psy - Chology, sociology, etc...

The study of dynamics is the study of how things change over time Discrete dynamics is the study of quantities that change at discrete points in time, such as the size of a population from one year to the next, or the change in the genetic make-up of a population from one generation to the

Next. In general, we concurrently develop a model of some situation and to develop our mathematical theory, we will be able to add more components to our model.

The mean for studying change is to find a relationship between, what is happening now and what will be happened in the 'near' future; that is, cause and effect, By analyzing this relationship, we can often predict what will be happened in the distant future. The distant future is sometimes a given point in time, but more often is a limit as time goes to infinity. In doing our analysis, we will use many algebraic and calculus topics such as, factoring, exponentials and logarithms, solving systems of equations, and derivatives. We should also be able to apply discrete dynamics to any field in which things change, which is the most fields. The goal, then, is to not only learn mathematics, but to get develop a differently way of thinking about the world.

The mathematical modeling of several real-world phenomenon leads to differential or difference equations of various types depending on the nature of the phenomenon under consideration. The problem of obtaining solutions of such equations in terms of the elementary functions of analysis is not solvable for most equations because of the nonintegerability of the equation. Therefore, the fundamental problem in the theory of differential or difference equations is to deduce the qualitative properties of the solutions of a given equation from the analytic form of the equation. The numerical methods are other ways to know some qualitative properties about the solutions.

The oscillation and global asymptotic behavior of solutions are two such qualitative properties which are very important for applications in many areas such as control theory, mathematical biology, neural networks, etc.. It is impossible to use computer based (numerical) techniques to study the oscillation or the asymptotic behavior of all solutions of a given equation due to the global nature of these properties. Therefore, these properties have received the attention of several mathematicians, engineers and other scientists around the world. Since the numerical integration of differential equations, gives rise to difference equations.

Study of the analogous and nonanalogous Properties of both equations is of special importance for both theory and applications . Also, many notions from continuous case are used for its discrete version without any modifications . For example, the difference equation

$$\Delta y_{n-1} + p_n y_{n-k} = 0, k \in N. \tag{1.1}$$

Where Δ is the forward difference operator defined by

$$\Delta y_n = y_{n+1} - y_n$$

May be considered as a discrete version of the first order delay differentia Equation

$$\dot{y}(t) + p(t)y(t-\tau) = 0$$

So equation (1.1) is said to be a first order delay difference equation. But without mentioning the term delay, Equation (1.1) is a k^{th} order difference equation.

1.2 The basic definitions and elementary results

In the following, we present some basic definition and known results which will be useful in our study.

Now let I be an interval of real numbers and let

$$F: I^{k+1} \to I.$$

Where F is a continuously differentiable function.

Consider the difference equation

$$y_{n+1} = F(y_n, y_{n-1}, \ldots, y_{n-k}), n = 0,1,2,\ldots \qquad (1.2)$$

With the initial conditions $y_{-k}, y_{-k+1}, \ldots, y_0 \in I$.

The following definitions are given in [1-2],[11]and[26].

Definition 1.1

An ordinary difference equation is a relation of the form given by (1.2).

Definition 1.2

The order of a difference equation is the difference between the highest and lowest indices that appear in the difference equation.

The expression given by (1-2) is a k^{th} order difference equation if and only if the term y_n appears in the function F on the right- hand side.

Definition 1.3

A difference equation is linear if it can be put in the form:

$$y_{n+k} + a_1(n)y_{n+k-1} + a_2(n)y_{n+k-2} + \dots\dots + a_k(n)y_n = R_n$$

where $a_i(n), i = 1,\dots\dots,k$ and R_n are given functions of n.

Definition 1.4

A difference equation is non-linear if it is not linear.

Definition 1.5

A solution of equation (1.2) is a function $\Phi(n)$ that reduces the equation to an identity.

Definition 1.6

We say that $\bar{y}$ is an equilibrium point of Equation (1.2)if

$$F(\bar{y},\bar{y},\dots,\bar{y}) = \bar{y}$$

That is, the constant sequence $\{y_n\}_{n=-k}^{\infty}$ With

$$y_n = \bar{y} \quad \text{For all} \quad n \geq -k$$

Is a solution of Equation(1.2).

Definition (Stability)1.7:

(i)The equilibrium point $\bar{y}$ of Equation (1.2) is locally stable if for every $\varepsilon \rangle 0$ there exists $\delta \rangle 0$ such that for all $y_{-k}, y_{-k+1}, \dots, y_{-1}, y_0 \in I$ with

$$|y_{-k} - \bar{y}| + |y_{-k+1} - \bar{y}| + \dots\dots + |y_0 - \bar{y}| < \delta,$$

We have

$$|y_n - \bar{y}| < \varepsilon. \quad \text{For all } n \geq -k$$

(ii) The equilibrium point $\bar{y}$ of Equation (1.2) is locally asymptotically stable if $\bar{y}$ is locally stable solution of Equation (1.2) and there exists $\gamma > 0$ such that for all $y_{-k}, y_{-k+1}, \dots, y_{-1}, y_0 \in I$ With

$$\left|y_{-k}-\bar{y}\right|+\left|y_{-k+1}-\bar{y}\right|+\dots\dots+\left|y_0-\bar{y}\right|<\gamma,$$

We have

$$\lim_{n\to\infty} y_n = \bar{y}$$

(iii) The equilibrium point $\bar{y}$ of Equation (1.2) is global attractor if for all

$$y_{-k}, y_{-k+1}, \dots, y_{-1}, y_0 \in I$$

We have

$$\lim_{n\to\infty} y_n = \bar{y}$$

(iv) The equilibrium point $\bar{y}$ of Equation (1.2) is globally asymptotically stable if $\bar{y}$ is locally stable, and $\bar{y}$ is also a global attractor of Equation (1.2).

(v) The equilibrium point $\bar{y}$ of Equation (1.2) is unstable if $\bar{y}$ is not locally stable.

Definition (periodicity)1.8:

Let b be a point in the domain of F ,Then

(i) b is called a periodic point of F or Equation (1.2) if for some positive integer p, $F^p(b)=b$ Hence a point is p periodic if it is a fixed point of $F^p(b)=b$, that is, if it is an equilibrium point of the difference equation

$$y_{n+1} = H(y_n),$$

Where $H = F^p$

The periodic orbit of b, $O^+(b) = \{b, F(b), F^2(b), \ldots, F^{p-1}(b)\}$ is often called a p cycle.

(ii) b is called eventually p periodic if for some positive integer m, $F^m(b)$ is a periodic point .

In other words, b is eventually p periodic if

$$F^{m+p}(b) = F^m(b)$$

(iii)A Sequence $\{y_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $y_{n+p} = y_n$ for all $n \geq -k$ .A sequence $\{y_n\}_{n=-k}^{\infty}$ is said to be periodic with prime period p if p is the smallest positive integer having this property.

The linearized equation of Equation (1.2) about the equilibrium is the linear difference equation

$$z_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\bar{y}, \bar{y}, \ldots, \bar{y})}{\partial y_{n-i}} z_{n-i} \tag{1.3}$$

The characteristic equation associated with Equation (1.3)is

$$f(\lambda) = a_0\lambda^k + a_1\lambda^k + \ldots + a_{k-1}\lambda^k + a_k = 0 \tag{1.4}$$

Where

$$a_i = \frac{\partial F(\bar{y}, \bar{y}, \ldots, \bar{y})}{\partial y_{n-i}} \quad .$$

Theorem 1.1 (Linearized Stability Theorem[11])

(i) If all roots of (1.4) have modulus less than one, then equilibrium of (1.3) is locally asymptotically stable .

(ii) If at least one of the roots of (1.4) has modulus greater than one , then the equilibrium of (1.3) is unstable .

The equilibrium of (1.3) is called saddle point equilibrium if (1.4) has roots both inside and outside the unit disk.

Theorem 1.2 [11]

Assume that $p, q \in R$ and $K \in \{0,1,2,......\}$.

Then

$$|p|+|q|<1$$

Is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+1} + py_n + qy_{n-k} = 0 \, , n = 0,1,...$$

Remark 1.1 Theorem 1.2 [11] can be easily extended to a general linear equations of the form

$$y_{n+k} + p_1 y_{n+k-1} + ... + p_k y_n = 0 \, , n = 0,1,... \tag{1.5}$$

Where $p_1, p_2, ..., p_k$ and $k \in \{1,2,..\}$. Then Equation (1.5) is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1$$

Definition (Permanence) 1.9

The difference equation (1.2)

Is said to be permanent if there exist numbers m and M for any initial conditions $y_{-k}, y_{-k+1}, \ldots, y_{-1}, y_0 \in (0, \infty)$ there exists a positive integer N which depends on the initial conditions such that

$$m \le y_n \le M \text{ for all } n \ge N$$

Definition (1.10):

A nontrivial solution $y(n)$ of (1.2) is said to be oscillatory (a round zero).If for every positive integer N, there exists $n \ge N$ such that

$$y(n)y(n+1) \le 0.$$

Otherwise, the solution is said to be nonoscillatory .

The solution $y(n)$ is said to be oscillatory around an equilibrium point $\bar{y}$ if $(y(n) - \bar{y})$ is oscillatory around zero.

The following theorem was given in [20] and it is an important theorem which will be useful in our study of the following chapter.

Theorem 1.3[20]:

Let $F \in [I^{k+1}, I]$ for some interval I of real numbers and for some nonnegative integer K, and consider the difference equation (1.2).

Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Equation(1.2), and suppose that there exist constants $A \in I$ and $B \in I$ such that

$$A \le y_n \le B \text{ for all } n \ge -k$$

Let l_0 be a limit point of the sequence $\{y_n\}_{n=-k}^{\infty}$ .Then the following statements are true.

(i) There exists a solution $\{L_n\}_{n=-\infty}^{\infty}$ of equation (1.2),called a full limiting sequence of $\{y_n\}_{n=-k}^{\infty}$ such that $L_0 = l_0$,and such that for every $N \in \{\dots,-1,0,1,\dots\}$ L_N is a limit point of $\{y_n\}_{n=-k}^{\infty}$.

(ii) For every $i_0 \le -k$,there exists a subsequence $\{y_{r_i}\}_{i=0}^{\infty}$ of $\{y_n\}_{n=-k}^{\infty}$ such that

$$L_N = \lim_{i \to \infty} y_{r_i+N} \text{ for every } N \ge i_0$$

The theory of the Full Limiting sequence was indicated in [3],[4] and[20].

1.3 On the rational recursive sequence

$$y_{n+1} = \frac{\alpha + \beta y_n^p + \gamma y_{n-1}^p + \delta y_{n-2}^p}{a + b y_n^q + c y_{n-1}^q + d y_{n-2}^q},$$

The aim of this theses is to study the global behaviour and the periodic Solutions of particular cases of the following difference equation

$$y_{n+1} = \frac{\alpha + \beta y_n^p + \gamma y_{n-1}^p + \delta y_{n-2}^p}{a + b y_n^q + c y_{n-1}^q + d y_{n-2}^q}, n = 0,1,\dots\dots \qquad (E)$$

Where $\alpha, \beta, \gamma, \delta, A, B, C, D, p$ and q are nonnegative real numbers and the initial conditions y_{-2}, y_{-1} and y_0 are real numbers.

Particular cases of the equation (E) has been considered by many authors. These particular cases can be classified as follows:

$$y_{n+1} = \frac{a + by_{n-1}}{A + By_{n-2}}, n = 0,1,.... \qquad (E_1)$$

$$y_{n+1} = \frac{\alpha + \gamma y_n + \delta y_{n-2}}{a + by_n^q + cy_{n-1}^q + dy_{n-2}^q}, n = 0,1,.... \qquad (E_2)$$

$$y_{n+1} = \frac{p + y_{n-k}}{qy_n + y_{n-k}}, n = 0,1,.... \qquad (E_3)$$

$$y_{n+1} = \frac{py_{n-1} + y_{n-2}}{q + y_{n-2}}, n = 0,1,.... \qquad (E_4)$$

Amieh ,A.M., Ladas ,G.and KriK , V.,[5], studied the recuarsive sequence (E_1).

Camouzis ,E.,Ladas ,.G., and Vouliv ,H.,D.,[8], studied the recursive sequence (E_2).

The global behaviour of the equation (E_3) and particular cases of it were studied by [10].

The global attractor of the equation (E_4), studied by Grove ,E.,A ,. Ladas ,G., Perdescu,M., and Radin ,M.,[4].

In chapter 2 ,we discuses the boundedness ,periodicity ,and the global stability of the equilibrium points of the solutions of

$$y_{n+1} = \frac{\alpha y_{n-l} + \beta y_{n-2}}{Ay_{n-l} + By_{n-2}}, n = 0,1,....., l \in \{0,1\} \qquad (E_5)$$

Where α, β, A and $B \in (0,\infty)$ and where the initial values y_{-2}, y_{-1} and y_0 are real numbers.

In chapter 3,we discuse the global behaviour and the periodic solutions of the following equations

$$y_{n+1} = \frac{\alpha y_{n-1}}{\beta + \gamma y_{n-2}^{p}}, n = 0,1,.... \qquad (E_6)$$

Where α, β, γ and p are nonnegative real numbers and the initial conditions y_{-2}, y_{-1} and y_0 are nonnegative real numbers such that

$$\beta + \gamma y_{n-2}^{p})0, \forall n \geq 0.$$

And

$$y_{n+1} = \frac{b y_{n-1}^{2}}{A + B y_{n-2}} \qquad (E_7)$$

Where b, A and B are nonnegative real numbers and the initial conditions y_{-2}, y_{-1} and y_0 are nonnegative real numbers such that

$$A + B y_{n-2})0, \forall n \geq 0.$$

Chapter(2)

Chapter (2)

On the rational recursive sequence $y_{n+1} = \dfrac{\alpha y_{n-l} + \beta y_{n-2}}{Ay_{n-l} + By_{n-2}}, l \in \{0,1\}$

2.1 Introduction

Qualitive analysis difference equations are not only interesting in its own right. but it can provide insights into their continuous counterparts, namely, differential equations.

There is a class of nonlinear difference equations, known as the rational difference equations, each of which consists of the ratio of two polynomials in the sequence terms in the same form. There has been a lot of work concerning the global asymptotic behavior of solutions of rational difference equations [6-7,9-1o,12,13,15-19,21,23 ,24].

Related nonlinear, rational difference equations were investigated in [5],[8],[14],[22]and[25].

The study of theses equations is quite challenging is in rapid development.

This chapter the boundedness , periodicitly ,and the global stability of the equilibrium Points of the solutions of the following rational difference equations

$$y_{n+1} = \frac{\alpha y_{n-l} + \beta y_{n-2}}{Ay_{n-l} + By_{n-2}}, l \in \{0,1\} \tag{2.1}$$

Where α, β, A and $B \in (0, \infty)$ and where the initial values y_{-2}, y_{-1} and y_0 are real numbers.

2.2 Local stability of the equilibrium point of equation (2.1)

In this section we study the local stability character of the solutions of equation (2.1).Equation (2.1) has a unique positive equilibrium point and is given by

$$\bar{y} = \frac{\alpha + \beta}{A + B}$$

Let f:(0, ∞) × (0, ∞) ⟶ (0, ∞) be the continuous function defined by

$$f(u, v) = \frac{\alpha u + \beta v}{Au + Bv}$$

Therefore it follows that

$$\frac{\partial f(u, v)}{\partial u} = \frac{(\alpha B - \beta A)v}{(Au + Bv)^2}$$

$$\frac{\partial f(u, v)}{\partial v} = \frac{-(\alpha B - \beta A)u}{(Au + Bv)^2}$$

Then, we see that

$$\frac{\partial f(\bar{y}, \bar{y})}{\partial u} = \frac{(\alpha B - \beta A)}{(A + B)(\alpha + \beta)} = -p$$

$$\frac{\partial f(\bar{y}, \bar{y})}{\partial v} = -\frac{(\alpha B - \beta A)}{(A + B)(\alpha + \beta)} = -q$$

Then the Linearized equation of equation (2.1) about $\bar{y}$ is

$$z_{n+1} + pz_{n-l} + qz_{n-2} = 0, l \in \{0,1\} \qquad (2.2)$$

Whose characteristic equation is

$$\lambda^3 + p\lambda^{2-l} + q = 0 \qquad (2.3)$$

Theorem 2.2.1 Assume that

$$(\alpha + \beta)(A + B) > 2|\beta A - \alpha B|$$

Then the positive equilibrium point of equation (2.1) is locally asymptotically stable.

Proof: it is follows by Theorem 1.1 that, equation (2.2) is asymptotically stable if all roots if equation (2.3) lie in the open disc $|\lambda| < 1$ that is if

$$|p_1| + |q_1| < 1$$

Then,

$$\left|\frac{\alpha B - \beta A}{(A+B)(\alpha+\beta)}\right| + \left|\frac{\beta A - \alpha B}{(A+B)(\alpha+\beta)}\right| < 1$$

Thus,

$$|\alpha B - \beta A| + |\beta A - \alpha B| < (A+B)(\alpha+\beta)$$

Or

$$2|\alpha B - \beta A| < (A+B)(\alpha+\beta)$$

Then, the proof is complete.

2.3 Boundedness of solutions of equation (2.1)

Here, we study the permanence of equation (2.1).

Theorem 2.3.2 Every solutions of equation (2.1) is bounded and persists.

Proof: let $\{y_n\}_{n=-2}^{\infty}$ be solution of equation (2.1) It follows from equation (2.1) that

$$y_{n+1} = \frac{\alpha y_{n-l}}{Ay_{n-l} + By_{n-2}} + \frac{\beta y_{n-2}}{Ay_{n-l} + By_{n-2}}.$$

Then,

$$y_{n+1} \le \frac{\alpha}{A} + \frac{\beta}{B} = M \quad \text{for all } n \ge 0$$

Thus, $y_n \le M \quad \text{for all } n \ge 1$ (2.4)

Now, we wish to show that there exists $m > 0$ such that

$$y_n \ge m \text{ for all } n \ge 1$$

Suppose that

$$y_n = \frac{1}{x_n},$$

Will reduce equation (2.1) to the equivalent form

$$\frac{1}{x_{n+1}} = \frac{\frac{\alpha}{x_{n-1}} + \frac{\beta}{x_{n-2}}}{\frac{A}{x_{n-1}} + \frac{B}{x_{n-2}}},$$

$$= \frac{\alpha x_{n-2} + \beta x_{n-1}}{A x_{n-2} + B x_{n-1}},$$

Then,

$$x_{n+1} = \frac{A x_{n-2} + B x_{n-1}}{\alpha x_{n-2} + \beta x_{n-1}},$$

$$= \frac{A x_{n-2}}{\alpha x_{n-2} + \beta x_{n-1}} + \frac{B x_{n-1}}{\alpha x_{n-2} + \beta x_{n-1}}$$

$$\le \frac{A}{\alpha} + \frac{B}{\beta} = \frac{1}{m} \quad \text{for all } n \ge 0$$

Then, we obtain

$$y_{n+1} \ge m \qquad \text{for all } \quad n \ge 0$$

Thus

$$y_n \geq m \quad \text{for all} \quad n \geq 1 \tag{2.5}$$

From (2.4) and (2.5), we get

$$m \leq y_n \leq M \text{ for all } n \geq 1$$

Therefore, every solution of equation (2.1) is bounded and persists.

2.4 Periodicity of solution of equation (2.1)

In this section, we study the existence of prime period-two solution of equation (2.1).

Theorem 2.4.3 Equation (2.1) has positive prime period two solutions if and only if

$$4A\beta < (\alpha - \beta)(B - A) \text{ and } l = 1 \tag{a}$$

Proof: First suppose that there exists a prime period two solution

Of equation (2.1).

$$\ldots, p, q, p, q, \ldots$$

We see from equation (2.1) that

$$p = \frac{\alpha p + \beta q}{Ap + Bq}$$

and

$$q = \frac{\alpha q + \beta p}{Aq + Bp}$$

Then

$$Ap^2 + pqB = \alpha p + \beta q \tag{2.6}$$

and

$$Aq^2 + pqB = \alpha q + \beta p \tag{2.7}$$

Subtracting (2.6) from (2.7) gives

$$A(p^2 - q^2) = \alpha(p - q) + \beta(q - p)$$
$$= (p - q)(\alpha - \beta)$$

Since, $p \neq q$, it follows that

$$p + q = \frac{\alpha - \beta}{A} \tag{2.8}$$

Also, since p and q are positive, ($\alpha - \beta$) should be positive.

Again, adding (2.6) and (2.7) yields

$$2pqB + A(p^2 + q^2) = (\alpha + \beta)(p + q) \tag{2.9}$$

It follows by (2.8), (2.9) and the relation

$$p^2 + q^2 = (p + q)^2 - 2pq \quad \text{For all } p, q \in R$$

Thus

$$2pqB + A\left[\left(\frac{\alpha - \beta}{A}\right)^2 - 2pq\right] = (\alpha + \beta)\left(\frac{\alpha - \beta}{A}\right)$$

Then,

$$2pqB+\left[\frac{(\alpha-\beta)(\alpha-\beta)}{A}\right]-2pqA=\frac{(\alpha+\beta)(\alpha-\beta)}{A}$$

That

$$2pq(B-A)=\frac{(\alpha-\beta)}{A}2\beta$$

Again, since p and q are positive and $\alpha>\beta$, we see that $B>A$

Thus,

$$pq=\frac{\beta(\alpha-\beta)}{A(B-A)} \tag{2.10}$$

Now it is clear from equation (2.8) and equation (2.10) that p and q are the two positive distinct roots of the quadratic equation

$$t^2-\frac{\alpha-\beta}{A}t+\frac{\beta(\alpha-\beta)}{A(B-A)}=0. \tag{2.11}$$

and so

$$\left(\frac{\alpha-\beta}{A}\right)^2-\frac{4\beta(\alpha-\beta)}{A(B-A)}>0$$

Since $(\alpha-\beta)$ and $(B-A)$ are positive,

$$\frac{\alpha-\beta}{A}>\frac{4\beta}{B-A},$$

which is equivalent to

$$4A\beta<(\alpha-\beta)(B-A)$$

Therefore, inequality (a) holds.

Second, suppose that inequality (a) is true.

We will show that equation (2.1) has a prime periodic-two solutions

Assume that

$$p=\frac{\frac{\alpha-\beta}{A}-\sqrt{(\frac{\alpha-\beta}{A})^2-\frac{4\beta(\alpha-\beta)}{A(B-A)}}}{2}$$

and

$$q=\frac{\frac{\alpha-\beta}{A}+\sqrt{(\frac{\alpha-\beta}{A})^2-\frac{4\beta(\alpha-\beta)}{A(B-A)}}}{2}$$

We see from inequality (a) that

$$(\alpha-\beta)(B-A)>4\beta A$$

Then, $(\alpha-\beta)$ and $(B-A)$ have the same sign,

$$(\alpha-\beta)^2>\frac{4\beta A(\alpha-\beta)}{B-A}$$

which equivalents to

$$(\frac{\alpha-\beta}{A})^2>\frac{4\beta A(\alpha-\beta)}{A(B-A)}$$

Therefore, p and q are distinct positive real numbers.

Set

$$y_{-2}=p,\ y_{-1}=q \text{ and } y_0=p$$

We wish to show that

$$y_1 = y_{-1} = q \quad \text{and } y_2 = y_0 = p$$

It follows from equation (2.1) that

$$y_1 = \frac{\alpha y_{-1} + \beta y_{-2}}{Ay_{-1} + By_{-2}} = \frac{\alpha q + \beta p}{Aq + Bp}$$

$$y_1 = \frac{\alpha\left[\frac{\alpha-\beta}{A} + \sqrt{(\frac{\alpha-\beta}{A})^2 - \frac{4\beta(\alpha-\beta)}{A(B-A)}}\right] + \beta\left[\frac{\alpha-\beta}{A} - \sqrt{\left(\frac{\alpha-\beta}{A}\right)^2 - \frac{4\beta(\alpha-\beta)}{A(B-A)}}\right]}{A\left[\frac{\alpha-\beta}{A} + \sqrt{(\frac{\alpha-\beta}{A})^2 - \frac{4\beta(\alpha-\beta)}{A(B-A)}}\right] + B\left[\frac{\alpha-\beta}{A} - \sqrt{(\frac{\alpha-\beta}{A})^2 - \frac{4\beta(\alpha-\beta)}{A(B-A)}}\right]}$$

Dividing the denominator and numerator by $(\frac{\alpha-\beta}{A})$,

gives

$$y_1 = \frac{\alpha\left[1 + \sqrt{1 - \frac{4\beta A}{(\alpha-\beta)(B-A)}}\right] + \beta\left[1 - \sqrt{1 - \frac{4\beta A}{(\alpha-\beta)(B-A)}}\right]}{A\left[1 + \sqrt{1 - \frac{4\beta A}{(\alpha-\beta)(B-A)}}\right] + B\left[1 - \sqrt{1 - \frac{4\beta A}{(\alpha-\beta)(B-A)}}\right]}$$

$$= \frac{(\alpha+\beta) + (\alpha-\beta)\sqrt{1 - \frac{4\beta A}{(\alpha-\beta)(B-A)}}}{(A+B) + (A-B)\sqrt{1 - \frac{4\beta A}{(\alpha-\beta)(B-A)}}}$$

Multiplying the denominator and numerator by

$$(A+B) - (A-B)\sqrt{1 - \frac{4\beta A}{(\alpha-\beta)(B-A)}}$$

Gives

$$y_1 = \frac{(\alpha+\beta)(A+B)-(\alpha-\beta)(A-B)\left(1-\frac{4\beta A}{(\alpha-\beta)(B-A)}\right)}{(A+B)^2-(A-B)^2\left(1-\frac{4\beta A}{(\alpha-\beta)(B-A)}\right)}$$

,

$$+\frac{((\alpha-\beta)(A+B)-(\alpha+\beta)(A-B))\sqrt{1-\frac{4\beta A}{(\alpha-\beta)(B-A)}}}{(A+B)^2-(A-B)^2\left(1-\frac{4\beta A}{(\alpha-\beta)(B-A)}\right)}$$

$$y_1 = \frac{(\alpha+\beta)(A+B)-(\alpha-\beta)(A-B)+4\beta A}{(A+B)^2-(A-B)^2\left(1-\frac{4\beta A}{(\alpha-\beta)(B-A)}\right)}+$$

$$\frac{\left[\alpha(A+B-A+B)+\beta(-A-B-A+B)\sqrt{1-\frac{4\beta A}{(\alpha-\beta)(B-A)}}\right]}{(A+B)^2-(A-B)^2\left(1-\frac{4\beta A}{(\alpha-\beta)(B-A)}\right)}$$

$$y_1 = \frac{\alpha(A+B-A+B)+\beta(A+B+A-B-4A)}{A^2+2AB+B^2-A^2+2AB-B^2-\frac{4\beta A(A-B)}{\alpha-\beta}}+$$

$$\frac{2\alpha B-2\beta A\sqrt{1-\frac{4\beta A}{(\alpha-\beta)(B-A)}}}{A^2+2AB+B^2-A^2+2AB-B^2-\frac{4\beta A(A-B)}{\alpha-\beta}}$$

$$y_1 = \frac{2\alpha B-2\beta A+(2\alpha B-2\beta A)\sqrt{1-\frac{4\beta A}{(\alpha-\beta)(B-A)}}}{4AB-\frac{4\beta A(A-B)}{\alpha-\beta}}$$

$$= \frac{2(\alpha B - \beta A)\left[1 + \sqrt{1 - \frac{4\beta A}{(\alpha - \beta)(B - A)}}\right]}{\frac{4A}{\alpha - \beta}\left[B(\alpha - \beta) - \beta(A - B)\right]}$$

$$= \frac{2(\alpha B - \beta A)\left[1 + \sqrt{1 - \frac{4\beta A}{(\alpha - \beta)(B - A)}}\right]}{\frac{4A}{\alpha - \beta}(\alpha B - B\beta - \beta A + B\beta)}$$

$$= \frac{(\alpha B - \beta A)\left[1 + \sqrt{1 - \frac{4\beta A}{(\alpha - \beta)(B - A)}}\right]}{\frac{2A}{\alpha - \beta}(\alpha B - \beta A)}$$

$$= \frac{\left[1 + \sqrt{1 - \frac{4\beta A}{(\alpha - \beta)(B - A)}}\right]}{\frac{2A}{\alpha - \beta}}$$

$$= \frac{\frac{\alpha - \beta}{A} + \frac{\alpha - \beta}{A}\sqrt{1 - \frac{4\beta A}{(\alpha - \beta)(B - A)}}}{2}$$

$$= \frac{\frac{\alpha - \beta}{A} + \sqrt{\left(\frac{\alpha - \beta}{A}\right)^2 - \frac{4\beta(\alpha - \beta)}{A(B - A)}}}{2}$$

$$= q .$$

Similarly as before, one can easily show that

$$y_2 = p$$

Then it follows by induction that

$$y_{2n} = p \text{ and } y_{2n+1} = q \text{ for all } n \geq -1$$

Thus, equation (2.1) has the positive prime period-two solutions

$$\ldots, p, q, p, q \ldots$$

Where p and q are the distinct root of the quadratic equation (2.11) and the proof is complete.

Lemma 2.4.1 if l=0, then equation (2.1) has no periodic solution of prime period.

Proof: Assume for the sake of contradiction that there exist distinctive positive

real numbers p and q such that

$$\ldots p, q, p, q \ldots$$

Be two periodic solutions of equation (2.1).then, we see from equation (2.1)

that

$$p = \frac{\alpha q + \beta q}{Aq + Bq} = \frac{\alpha + \beta}{A + B}$$

and

$$q = \frac{\alpha p + \beta p}{Ap + Bp} = \frac{\alpha + \beta}{A + B}$$

Which implies

$$p = q$$

This is a contradiction.

2.5 Global stability of equation (2.1)

In this section, we investigate the global asymptotic stability of equation (2.1).

Theorem 2.5.4: The equilibrium point $\bar{y}$ is a global attractor of equation (2.1) if one of the following statements holds

$$\alpha B \geq \beta A \text{ and } B \geq \alpha \tag{2.12}$$

$$\alpha B \leq \beta A \text{ and } \alpha \geq \beta \tag{2.13}$$

Proof: let $\{y_n\}_{n=-m}^{\infty}$ be a solution of equation (2.1) and again let f be a function defined by

$$f(u,v) = \frac{\alpha u + \beta v}{Au + Bv}$$

We will prove (2.12) and (2.13) are similar and will be omitted.

Assume that (2.12) is true, then it is easy to see that the function f is non decreasing in u and non-increasing in v. thus from equation (2.1), we see that

$$y_{n+1} = \frac{\alpha y_{n-l} + \beta y_{n-k}}{Ay_{n-l} + By_{n-k}}$$

$$\leq \frac{\alpha y_{n-l} + \beta(0)}{Ay_{n-l} + B(0)} = \frac{\alpha}{A}.$$

Then

$$y_n \leq \frac{\alpha}{A} = M_1 \text{ for all } n \geq 1 \tag{2.14}$$

Also, we see from equation (2.1) that

$$y_{n+1} = \frac{\alpha y_{n-l} + \beta y_{n-2}}{A y_{n-l} + B y_{n-2}}$$

$$\geq \frac{\alpha(0) + \beta y_{n-2}}{A(0) + B y_{n-2}}$$

$$\geq \frac{\beta y_{n-2}}{B y_{n-2}} = \frac{\beta}{B}$$

Then

$$y_n \geq \frac{\beta}{B} = m \quad \text{for all} \quad n \geq 1 \qquad (2.15)$$

Then from (2.14) and (2.15), we see that

$$m_1 = \frac{\beta}{B} \leq y_n \leq \frac{\alpha}{A} = M_1 \text{ for all } \quad n \geq 1$$

It follows by the Method of Full Limiting sequence that there exist solutions $\{I_n\}_{n=-\infty}^{\infty}$ and $\{S_n\}_{n=-\infty}^{\infty}$ of equation (2.1) with

$$I = I_0 = \liminf_{n \to \infty} y_n \leq \limsup_{n \to \infty} y_n = S_0 = S,$$

Where

$$I_n, S_n \in [I, S], n = 0, -1, \ldots$$

It suffices to show that

$$I = S$$

Now, from equation (2.1) that

$$I = \frac{\alpha I_{-1-l} + \beta I_{-3}}{A I_{-1-l} + B I_{-3}} \geq \frac{\alpha I + \beta S}{A I + B S},$$

And so

$$\alpha I + \beta S - AI^2 \le BSI \tag{2.16}$$

Similarly, we see from equation (2.1) that

$$S = \frac{\alpha S_{-1-t} + \beta S_{-3}}{AS_{-1-t} + BS_{-3}} \le \frac{\alpha S + \beta I}{AS + BI}.$$

And so

$$\alpha S + \beta I - AS^2 \ge BSI \tag{2.17}$$

Therefore it follows from equations (2.16) and (2.17)

That

$$\alpha I + \beta S - AI^2 \le \alpha S + \beta I - AS^2$$

If and only if

$$\alpha(S - I) + \beta(I - S) + A(I + S)(I - S) \ge 0$$

Thus

$$(I - S)[A(I + S) + \beta - \alpha] \ge 0,$$

And so

$$I \ge S \text{ if } A(I + S) + \beta - \alpha \ge 0.$$

Now, we know by (2.12) that

$$\beta \ge \alpha,$$

And so it follows that

$$I \ge S.$$

Therefore

$$I = S.$$

This completes the proof.

Chapter(3)

Chapter 3

"On some properties of solution of third order difference equations"

3.1 Introduction

In this chapter, we will prove some results considering third order recursive rational difference equations, which particular cases of the following rational difference equation

$$y_{n+1} = \frac{\alpha + \beta y_n^p + \gamma y_{n-1}^p + \delta y_{n-2}^p}{a + b y_n^q + c y_{n-1}^q + d y_{n-2}^q}, n = 0,1,......$$

The results involve some elements of qualitative theory of difference equations, namely ,oscillation, periodicity, global attractively and boundedness nature.

We believe that the results of this chapter are of paramount importance in their own right and furthermore we believe that these results offer prototypes towards the development of basic theory of the global behavior of solution of non-linear difference equations of third order. The techniques and results of this chapter are also extremely useful in analyzing equations in the mathematical models of various biological systems and other applications.

3.2 On the dynamics of the recursive sequence: $y_{n+1} = \dfrac{\alpha y_{n-1}}{\beta + \gamma y_{n-2}^p}$

In this section, we investigate the global behavior and the periodic character of the solutions of the third order recursive difference equation

$$y_{n+1} = \frac{\alpha y_{n-1}}{\beta + \gamma y_{n-2}^{p}}, n = 0,1,... \qquad (3.1)$$

where the parameters α, β, γ and p are nonnegative real numbers and the initial conditions y_{-2}, y_{-1}, y_0 are nonnegative real number such that

$$\beta + \gamma y_{n-2}^{p} > 0, \forall n \geq 0.$$

By generalizing the results due to [5], the study of such equations are quite challenging and rewarding and is still in its infancy.

<u>The special cases</u> $\alpha\beta\gamma p = 0$

We examine the character of solution of (3.1) when one or more of the parameters of (3.1) are zero.

There are five such equations, namely,

$\alpha = 0$:

$$y_{n+1} = 0, n = 0,1,... \qquad (3.2)$$

$\beta = 0$:

$$y_{n+1} = \frac{\alpha y_{n-1}}{\gamma y_{n-2}^{p}}, n = 0,1,... \qquad (3.3)$$

$p = 0$:

$$y_{n+1} = \frac{\alpha}{\beta + \gamma} y_{n-1}, n = 0,1,... \qquad (3.4)$$

$\gamma = 0$:

$$y_{n+1} = \frac{\alpha}{\beta} y_{n-1}, n = 0.1... \quad (3.5)$$

$\beta = p = 0$

$$y_{n+1} = \frac{\alpha}{\gamma} y_{n-1}, n = 0.1.... \quad (3.6)$$

In each of the above five equations ,we assume that all parameters in the equations are positive. Equation (3.2) is trivial, equations (3.4), (3.5) and (3.6) are linear. Equation (3.3) can also be reduced to a linear difference equation by the charge of variables

$$y_n = e^{x_n}$$

<u>The dynamics of equation (3.1)</u>

We investigate the dynamics of (3.1) under the assumptions that all parameters in equation are positive and the initial conditions are nonnegative

The change of variables $y_n = \left(\frac{\beta}{\gamma}\right)^{\frac{1}{p}} x_n$ reduces (3.1) to the difference equation

$$x_{n+1} = \frac{r x_{n-1}}{1 + x_{n-2}^p}, n = 0,1,... \quad (3.7)$$

Where $$r = \frac{\alpha}{\beta} > 0 .$$

Note that $\overline{x}_1=0$ is always an equilibrium point of (3.7),when r>1,(3.7)also possesses the unique positive equilibrium $\overline{x}_2 = (r-1)^{\frac{1}{p}}$

<u>**Theorem 3.1**</u>

The following statements are true:

(1) The equilibrium point $\overline{x}_1=0$ of (3.7) is locally asymptotically stable if and only if r<1.

(2) The equilibrium point $\overline{x}_1=0$ of (3.7) is a saddle point if and only if r >1.

(3) When r >1, Then the positive equilibrium point $\overline{x}_2 = (r-1)^{\frac{1}{p}}$ of (3.7) is unstable.

<u>**Proof**</u>: The Linearized equation of (3.7) about the equilibrium point $\overline{x_1} = 0$ is

$$z_{n+1} = r z_{n-1}, n = 0,1,2,\ldots$$

So, the characteristic equation of (3.7) about the equilibrium point $\overline{x}_1=0$ is

$$\lambda^3 - r\lambda = 0 .$$

And hence, the proof of (1) and (2) follows from The Linearized Stability Theorem.

For (3), we assume that r >1, Then the linarized equation of (3. 7) about the equilibrium point $\bar{x}_2 = (r-1)^{\frac{1}{p}}$ has the form

$$z_{n+1} = z_{n-1} - \frac{p(r-1)}{r} z_{n-2}, n = 0,1,...$$

So, the characteristic equation of (3.7) about the equilibrium point

$\bar{x}_2 = (r-1)^{\frac{1}{p}}$ is

$$\lambda^3 - \lambda + \frac{p(r-1)}{r} = 0 \qquad (3.8)$$

It is clear that (3.8) has a root in the interval $(-\infty,-1)$ and so, $\bar{x}_2 = (r-1)^{\frac{1}{p}}$ is

unstable equilibrium point. This completes the proof.

Theorem 3.2.

Assume that r > 1, and let $\{x_n\}_{n=-2}^{\infty}$ be a solution of (3.7) such that

$$x_{-2}, x_0 \geq \bar{x}_2 = (r-1)^{\frac{1}{p}} \text{ and } x_{-1} < \bar{x}_2 = (r-1)^{\frac{1}{p}} \qquad (3.9)$$

or

$$x_{-2}, x_0 < \bar{x}_2 = (r-1)^{\frac{1}{p}} \text{ and } x_{-1} \geq \bar{x}_2 = (r-1)^{\frac{1}{p}} \qquad (3.10)$$

Then $\{x_n\}_{n=-2}^{\infty}$ oscillate about $\bar{x}_2 = (r-1)^{\frac{1}{p}}$.

with semi cycle of length one.

Proof: Assume that (3.9) holds [the case where (3.10) holds is similar and will be omitted].

Then

$$x_1 = \frac{rx_{-1}}{1+x_{-2}^p} < \overline{x_2} ,$$

and

$$x_2 = \frac{rx_0}{1+x_{-1}^p} \geq \overline{x_2} .$$

Then, the proof follows by induction.

Theorem 3. 3

Assume that r<1, Then the equilibrium point $\bar{x}_1$ of (3.7)is globally asymptotically stable.

Proof:

We know from Theorem (3.1) that the equilibrium point $\bar{x}_1$ =0 of (3.7) is locally asymptotically stable.

So let $\{x_n\}_{n=-2}^{\infty}$ be a solution of (3.7). It suffices to show that

$$\lim_{n\to\infty} x_n = 0$$

Since

$$0 < x_{n+1} = \frac{rx_{n-1}}{1+x_{n-2}^p} < rx_{n-1} < x_{n-1}$$

So

$$\lim_{n\to\infty} x_n = 0$$

This complete the proof.

Theorem 3. 4

Assume that r=1, Then (3.7) possesses the prime period-two solution

$$\dots, \varphi, 0, \varphi, 0, \dots \qquad (3.11)$$

with $\varphi > 0$. Furthermore every solution of (3.7) converges to a period two solution (3.11) with $\varphi \geq 0$.

Proof:

Let
$$\dots, \varphi, 0, \varphi, 0, \dots$$

be a period-two solution of (3.7). Then

$$\varphi = \frac{r\varphi}{1+\psi^{p}} \quad \text{and} \quad \psi = \frac{r\psi}{1+\varphi^{p}}$$

so

$$\psi\varphi = \frac{(\varphi-\psi)(r-1)}{\psi^{p-1}-\varphi^{p-1}} \geq 0$$

Which implies that

$$r-1 \leq 0.$$

If $r<1$. Then it implies that

$$\varphi < 0 \text{ or } \psi < 0,$$

which is impossible, So $r=1$.

To complete the proof, we assume that r=1 and let $\{x_n\}_{n=-2}^{\infty}$ be a solution of (3.7). Then,

$$x_{n+1} - x_{n-1} = \frac{-x_{n-1}x_{n-2}^{p}}{1+x_{n-2}^{p}} \leq 0$$

So the even terms of this solution decrease to a limit (say $\varphi \geq 0$), and the odd terms decrease to a limit (say $\psi \geq 0$). Thus

$$\varphi = \frac{\varphi}{1+\psi^{p}} \quad \text{and} \quad \psi = \frac{\psi}{1+\varphi^{p}},$$

which implies that

$$\varphi\psi^{p} = 0 \quad \text{and} \quad \psi\varphi^{p} = 0$$

This completes the proof.

Theorem 3. 5

Assume that r>1. Then (3.7) possesses unbounded solution.

Proof: From theorem 3.2, we can assume without loss of generality that the solution $\{x_n\}_{n=-2}^{\infty}$ of (3.7) is such that

$$x_{2n-1} < \bar{x} = (r-1)^{\frac{1}{p}} \text{ and } x_{2n} > \bar{x} = (r-1)^{\frac{1}{p}} \; \forall n \geq 0.$$

Then,

$$x_{2n+2} = \frac{rx_{2n}}{1+x_{2n-1}^{p}} > \frac{rx_{2n}}{1+(r-1)} = x_{2n},$$

and

$$x_{2n+3} = \frac{rx_{2n+1}}{1+x_{2n}^{p}} < \frac{rx_{2n+1}}{1+(r-1)} = x_{2n+1}.$$

from which it follows that,

$$\lim_{n\to\infty} x_{2n} = \infty \text{ and } \lim_{n\to\infty} x_{2n+1} = 0$$

Then, the proof is complete.

Remark.

If $p = 1$, the results of [6] follows directly.

3.3 On the dynamics of : $y_{n+1}=\dfrac{by_{n}^{2}}{A+By_{n-2}}$

The asymptotic stability, periodicity, and trichotomy character of

$$y_{n+1}=\frac{by_{n-1}}{A+By_{n-2}},n=0,1,... \tag{3.12}$$

Was investigated when A, B and b are non-negative parameters and with initial conditions concerning y_{-2}, y_{-1} and y_0 to be arbitrary non-negative real numbers, see[5].

Now, in this section we want to generalize the above results.
consider the third order non-linear rational difference equation

$$y_{n+1}=\frac{by_{n-1}^{2}}{A+By_{n-2}},n=0,1,.... \tag{3.13}$$

where the parameters A, B and b are non-negative real numbers and the initial conditions concerning y_{-2}, y_{-1} and y_0 to be arbitrary non-negative real numbers. We investigate the global stability and periodic nature of the solutions of (3.13).

The results which we develop for such equations are also useful in analyzing the equations in the mathematical models of various biological systems and other applications [8],[17],[19]and [27].

The special cases $bAB=0$

We examine the characters of solution (3.13) when one or more of the parameters are zero. There are three equations namely:

$$y_{n+1}=\frac{by_{n-1}^{2}}{\beta y_{n-2}},n=0,1,... \tag{3.14}$$

$$y_{n+1} = \frac{b}{A} y_{n-2}^2, n = 0,1,... \tag{3.15}$$

$$y_{n+1} = 0, n = 0,1,... \tag{3.16}$$

In each of the above cases we assume that all parameters in the equations are positive.

Equation (3. 16) are trivial and has a solution $y_{n+1} = 0$, for all n≥-2. The equations (3.14) and (3.15) are non-linear third and second order respectively, and the change of variables

$$y_n = e^{x_n}$$

reduce (3.14) and (3.15)to a third and second order linear difference equation respectively

Now ,We can investigate the dynamics of (3.13) under the assumptions that all parameters in equation (3.13)are positive and the initial conditions are non- negative.

The change of variables $y_n = \frac{A}{B} x_n$ reduces (3.13) to the difference equation

$$x_{n+1} = \frac{r x_{n-1}^2}{1 + x_{n-2}}, n = 0,1,... \tag{3.17}$$

Where

$$r = \frac{b}{B}$$

It is easily to seen that $\bar{x}_1 = 0$ is always an equilibrium point and when r>1,

we have also an equilibrium point

$$\bar{x}_2 = \frac{1}{r-1}.$$

An oscillation results:

Theorem 3.6:

Assume that r>I and let $\{x_n\}_{n=-2}^{\infty}$ be a solution of(3.17) such that either

$$x_{-2}, x_0 \geq \overline{x}_2 \text{ and } x_{-1} < \overline{x}_2 \quad (3.18)$$

Or

$$x_{-2}, x_0 < \overline{x}_2 \text{ and } x_{-1} \geq \overline{x}_2 \quad (3.19)$$

Then $\{x_n\}_{n=-2}^{\infty}$ oscillates about $\overline{x}_2$ with semi-cycle of length one.

Proof:

We will assume that (3.18) holds. The case where (3.19) holds is similar

and will be omitted.

Thus we obtain

$$x_1 < \overline{x}_2 \quad \text{and} \quad x_2 \geq \overline{x}_2$$

and this completes the proof.

Existence of prime period-two solutions

The following theorem shows that (3.17) has a prime-period two solutions.

Theorem 3.7:

Equation (3.17) has non-negative prime period-two solutions if and only if either

$$x_{-1} = 0 \text{ and } x_0 = \frac{1}{r} \quad (3.20)$$

or

$$x_0 = 0 \quad \text{and} \quad \frac{x_{-1}^2}{1+x_{-2}} = \frac{1}{r^2} \tag{3.21}$$

The period two solution must be in the form

$$\ldots, 0, \frac{1}{r}, 0, \frac{1}{r}, \ldots \tag{3.22}$$

<u>Proof</u>: Assume that

$$\ldots, \varphi, \psi, \varphi, \psi, \ldots$$

Is a non-negative period two solution of (3. 17).

Then

$$\varphi = \frac{r\varphi^2}{1+\psi} \quad \text{and} \quad \psi = \frac{r\psi^2}{1+\varphi} \tag{3. 23}$$

Hence,

$$\varphi - \psi = r(\varphi^2 - \psi^2)$$

From which, we can see that

$$\varphi + \psi = \frac{1}{r} \tag{3.24}$$

From (3.23) and (3.24), we get the period two solution in the form

$$\ldots, 0, \frac{1}{r}, 0, \frac{1}{r} \ldots$$

Now, let(3.17) has a period two solutions

$$\ldots, 0, \frac{1}{r}, 0, \frac{1}{r}, \ldots$$

Then ,we have either

$$x_1 = x_{-1} = 0 \quad \text{or} \quad x_1 = \frac{r x_{-1}^2}{1 + x_{-2}} = \frac{1}{r} \; .$$

If $x_{-1} = 0$,Then $x_2 = x_0 = \frac{1}{r}$,so(3. 20)holds.

If $\frac{r x_{-1}^2}{1 + x_{-2}} = \frac{1}{r}$,so $\frac{x_{-1}^2}{1 + x_{-2}} = \frac{1}{r^2}$

And so $x_2 = x_0 = 0$,and(3.21)holds.

This completes the proof.

Local and global stability

It is clearly that $\bar{x}_1 = 0$ is always an equilibrium solution of (3.17). Furthermore when r>1, (3.17) also possesses the positive equilibrium $\bar{x}_2 = \frac{1}{r-1}$.

Theorem 3.8:

For (3.17), we have the following results.

(1)The zero equilibrium point is locally asymptotic stable.

(2)Assume that r>1, then equilibrium point $\bar{x}_2 = \frac{1}{r-1}$ is unstable. In particular $\bar{x}_2$ is a saddle point.

Proof:

The linearized equation associated with (3.17) about $\bar{x}_i$.

has the form

$$z_{n+1} - \frac{2r\bar{x}_i}{1+\bar{x}_i} z_{n-1} + \frac{r\bar{x}_i^2}{(1+\bar{x}_i)^2} = 0, n = 0,1,\ldots$$

So the linearized equation of (3.6) about $\bar{x}_1=0$ is

$$z_{n+1} = 0, n = 0,1,....$$

and the characteristic equation about $\bar{x}_1=0$ is

$$\lambda^3 = 0.$$

So, The proof of (1)follows immediately form linearized stability theorem

The linearized equation of (3.6)is a bout $\bar{x}_2 = \frac{1}{r-1}$ is

$$z_{n+1} - 2z_{n-1} + \frac{1}{r} z_{n-2} = 0, n = 0,1,....$$

and The characteristic equation a bout $\bar{x}_2 = \frac{1}{r-1}$ is

$$\lambda^3 - 2\lambda + \frac{1}{r} = 0, \text{with } r>1,$$

Set,

$$f(\lambda) = \lambda^3 - 2\lambda + \frac{1}{r} \qquad (3.25).$$

Then $f(1) = -1 + \frac{1}{r} < 0$ and $\lim_{\lambda \to \infty} f(\lambda) = \infty$, so $f(\lambda)$ has at least a root in $(1,\infty)$

and the product of the roots of (3.25) is $\frac{1}{r} < 1$.

Hence, there exists a root in a unit disk.

This completes the proof.

Theorem 3.9:

Assume that $r \leq 1$, then the zero equilibrium of (3.17) is globally

asymptotically stable with basin

$$S = [0,\infty) \times [0,1]^2$$

where

$$(x_{-1}, x_0) \neq (0,1) \text{ and } (x_{-2}, x_{-1}, x_0) \neq (0,1,0). \qquad (3.26)$$

proof:

We know by Theorem 3.81 that $\bar{x}_1=0$ is locally asymptotically stable equilibrium point of (3.17), and so it suffices to show that $\bar{x}_1=0$ is a global attractor of (3.6) with basin $[0,\infty) \times [0,1]^2$ such that (3.26) hold.

So let $\{x_n\}_{n=-2}^{\infty}$ be a non-negative solution of (3.17). It is sufficient to show that

$$\lim_{n \to \infty} x_n = 0$$

Assume that $r \leq 1$ and (3.26) hold, we have

$$x_1 = \frac{rx_{-1}^2}{1+x_{-2}} \leq rx_{-1}^2 \leq x_{-1}^2 \leq 1$$

$$x_2 = \frac{rx_0^2}{1+x_{-1}} \leq rx_0^2 \leq x_0^2 \leq 1$$

$$x_3 = \frac{rx_1^2}{1+x_0} \leq rx_1^2 \leq x_1^2 \leq 1$$

So, by induction, we have

$$0 \leq x_n \leq 1, n = -1,0,1,...$$

and

$$x_{n+1} = \frac{rx_{n-1}^2}{1+x_{n-2}} \leq rx_{n-1}^2 \leq x_{n-1}^2 \leq 1$$

So , we have

$$\{x_{2n}\}_{n=-1}^{\infty} \quad \text{and} \quad \{x_{2n-1}\}_{n=0}^{\infty}$$

are non- increasing sequence and bounded.

Now, suppose that

$$\lim_{n\to\infty} x_{2n} = M \quad \text{and} \quad \lim_{n\to\infty} x_{2n-1} = L$$

So, we can write

$$M = \frac{rM^2}{1+L} \quad \text{and} \quad L = \frac{rL^2}{1+M}$$

Now, we want to prove that M=L=0 .

We must consider the following cases:

(1)If M= 0 and L≠ 0, Then L=1 /r ≥1, which is a contradiction to $\{x_{2n-1}\}_{n=0}^{\infty}$ is a non-increasing sequence.

(2) If M≠ 0 and L= 0, Then M $=\frac{1}{r}\geq 1$, which is a contradiction to $\{x_{2n}\}_{n=-1}^{\infty}$ is a non-increasing sequence.

(3)If M ≠0 and L≠ 0, Then we have

$$M=L=\frac{1}{r-1}<0,$$

Which is a contradiction. So, we must have L=M=0.

This completes the proof.

Theorem 3.10:

The zero equilibrium point of (3.17) is a globally asymptotically stable with the basin

$$S = [0,\infty) \times [0, \frac{1}{r}]^2$$

where

$$(x_{-2}, x_{-1}, x_0) \neq (0, \frac{1}{r}, 0) \text{ and } (x_{-1}, x_0) \neq (0, \frac{1}{r}) \tag{3.27}$$

Proof:

We know by Theorem 3.8 that $\bar{x}_1 = 0$ is locally asymptotically stable equilibrium point of (3.17), and so it suffices to show that $\bar{x}_1 = 0$ is a global attractor of (3.17) with basin $[0,\infty) \times [0, \frac{1}{r}]^2$, where (3.27) hold.

So, let $\{x_n\}_{n=-2}^{\infty}$ be a non-negative solution of (3.17).

It suffices to show that

$$\lim_{n\to\infty} x_n = 0.$$

It is easily to seen that

$$x_1 = \frac{rx_{-1}^2}{1 + x_{-1}} \le rx_{-1}^2 \le x_{-1}^2 \le \frac{1}{r}$$

$$x_2 = \frac{rx_0^2}{1 + x_{-1}} \le rx_0^2 \le x_0^2 \le \frac{1}{r}$$

$$x_3 = \frac{rx_1^2}{1 + x_0} \le rx_1^2 \le x_1^2 \le \frac{1}{r}$$

So, by induction, we have

$$0 \le x_n \le \frac{1}{r}, n = -1, 0, 1 \ldots$$

$$0 \le x_{n+1} = \frac{rx_{n-1}^2}{1 + x_{n-2}} \le rx_{n-1}^2 \le x_{n-1}^2 \le \frac{1}{r}, n = 0, 1, \ldots$$

Which means that $\{x_{2n}\}_{n=-1}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ are non-increasing and bounded.

Now, suppose that

$$M = \lim_{n\to\infty} x_{2n} \quad \text{and} \quad L = \lim_{n\to\infty} x_{2n+1}$$

So, we have

$$M = \frac{rM^2}{1+L} \quad \text{and} \quad L = \frac{rL^2}{1+M}$$

Now, we want to prove that M=L= 0. So, we must consider the following cases:

(1) If M= 0 and L≠ 0, Then L= $\frac{1}{r}$, which is a contradiction to $\{x_{2n-1}\}_{n=0}^{\infty}$ is a non- increasing sequence.

(2) If M≠ 0 and L= 0, Then M=$\frac{1}{r}$, which is a contradiction to $\{x_{2n}\}_{n=-1}^{\infty}$ is a non- increasing sequence.

(3)If M≠ 0 and L≠ 0, then. we have

$$L - M = \frac{1}{r-1}\begin{cases} > \frac{1}{r} & \text{if } r > 1 \\ < 0 & \text{if } r < 1 \end{cases}$$

which is a contradiction.

So, we must have

L=M=0.

This completes the proof.

Existence of unbounded solutions

In this part, we show that when r>1, (3.17) possesses unbounded solutions.

Theorem 3.11:

Assume that r>1 and $x_{-1}, x_0 \notin [0, \frac{1}{r}]$. Then (3.17) possessan unbounded solution.

In particular, every solution of (3.17) which oscillates about equilibrium $\bar{x}_2 = \frac{1}{r-1}$, with semicycle of length one is unbounded.

Proof:

We will prove that every solution $\{x_n\}_{n=-2}^{\infty}$ of (3.17) which oscillates with semi-cycles of length one is unbounded (see Theorem. 3.6) for the existence of solutions which oscillate with semi-cycles of length one).

Let r>0 and without loss of generality that

$\{x_n\}_{n=-2}^{\infty}$ such that

$$x_{2n-1} < \bar{x} \text{ and } x_{2n} > \bar{x} \quad \forall n \geq 0$$

Then

$$x_{2n+2} = \frac{r x_{2n}^2}{1 + x_{2n-1}} > \frac{r x_{2n}^2}{1 + \frac{1}{r-1}} = (r-1) x_{2n}^2 > x_{2n}$$

and

$$x_{2n+3} = \frac{r x_{2n+1}^2}{1 + x_{2n-2}} < \frac{r x_{2n+1}^2}{1 + \frac{1}{r-1}} = (r-1) x_{2n+1}^2 < x_{2n+1},$$

from which it follows that

$$\lim_{n \to \infty} x_{2n} = \infty \quad \text{and} \quad \lim_{n \to \infty} x_{2n+1} = 0$$

This completes the proof.

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(ملخص الرسالة)

يهتم موضوع هذه الرسالة بدراسة خواص حلول المعادلات التفاضلية (معادلات الفروق).

نحن نعتقد أن النتائج التي حصلنا عليها حول معادلات الفروق القياسية من الرتبة الثالثة لها أهمية بالغة في حد ذاتها علاوة علي ذلك أننا نعتقد هذه النتائج تقدم الحجر الأساس نحو تطور النظرية الأساسية للسلوك الشامل لحلول معادلات الفروق الغير خطية من الرتبة الثالثة.

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دراسة معادلات الفروق على الصورة

$$y_{n+1} = \frac{\alpha + \beta y_n^p + \gamma y_{n-1}^p + \delta y_{n-2}^p}{a + b y_n^q + c y_{n-1}^q + d y_{n-2}^q}, n = 0,1,......$$

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للطالبه

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د.نبيل زكي فريد

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جامعة التحدي

سرت

إن الدراسة ليست غاية في حد ذاتها
وإنما الغاية هي خلق الإنسان النموذجي الجديد

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كلية العلوم

قسم الرياضيات

عنوان البحث

دراسة معادلات الفروق على الصورة :

$$\left(x_{n+1}\right) = \frac{\alpha + Bx_{n}^{p} + \gamma x_{n-1}^{p} + \delta x_{n-2}^{p}}{a + bx_{n}^{q} + cx_{n-1}^{q} + dx_{n-2}^{q}}$$

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