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Oscillatory Solutions of Nonlinear Differential Equations of Second- Order with Variable Coefficients

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The oscillation characteristic of various components has attracted considerable interest, leading to extensive research on oscillatory models in various types of differential equations. In this research, oscillation property of nonlinear second-order differential equations are analyzed. Some analytical techniques are employed to establish new oscillation conditions. In addition, some illustrative examples are provided to support the new results. Our results improve and extend many previously established results in the field.

1 Introduction

Second-order nonlinear differential equations are crucial in various mathematical fields and have extensive applications across multiple disciplines. They are essential for modeling dynamic systems and offer numerous intriguing applications, particularly in engineering.

Recently, researchers have demonstrated increasing interest in determining appropriate conditions for oscillation in this class of equations. Traditional approaches typically address constant coefficients, but the introduction of variable coefficients brings both challenges and opportunities for new theoretical insights [12, 14].

Interest in this area has significantly increased, with researchers keen to explore the theoretical aspects alongside numerical methods. In many cases, exact solutions may be unattainable, which emphasizes the importance of studying qualitative properties within suitable functional spaces. This often involves simplifying the original equation into more familiar forms or developing numerical algorithms. Such investigations are essential for addressing these equations and contribute to our understanding of their oscillatory properties, which are key elements of the qualitative theory (see [1-19]). This article aims to examine the oscillations of all solutions of second-order nonlinear differential equations of a specified form:

$$[r(t)f(\dot{x}(t))]^{\cdot} + q(t)g\big(x(t)\big) = 0; \qquad (1.1)$$

$r, q \in C([t_0, \infty), \mathbb{R}), t_0 \geq 0$, $r > 0$, f is a continuous function on $\mathbb{R}$ with $yf(y) > 0$ for all $y \neq 0$ and g is continuously differentiable function on $\mathbb{R}$ except $x = 0$ with $xg(x) > 0$ and $g'(x) \geq k > 0$ for all $x \neq 0$. Also $f(u)$ is called superlinear if it satisfies:

$$\int_{\mathcal{E}}^{\infty} \frac{du}{f(u)} < \infty \quad and \quad \int_{-\mathcal{E}}^{-\infty} \frac{du}{f(u)} < \infty \qquad (1.2)$$

In this study, we focus specifically on the solutions of Eq. (1.1) that belonging to certain interval, which may depend on the particular solution. The solution $x(t)$ of the differential Eq. (1.1) is classified as an oscillator if it is neither eventually positive nor eventually negative. Equation (1.1) is considered an oscillator if all its solutions are oscillators. It is noteworthy that integral media techniques and Riccati substitution, are utilized to analyze the oscillatory characteristics of Eq. (1.1). For further information, refer to the works of Ahmed and Ali [3], Saad et al. [16], Onose [15] and Philos [17]. The particular case of equation (1. 1), that is to say, when $g\big(x(t)\big) = |x(t)|^{\gamma} signx(t)$, $\gamma > 0$ is particularly interesting. In fact, differential equations of the form

$$\big(r(t)\dot{x}(t)\big)^{\bullet} + q(t)|x(t)|^{\gamma} signx(t) = o, \quad t \geq t_0 \qquad (1.3)$$

and

$$\Big(r(t)f\big(\dot{x}(t)\big)\Big)^{\bullet} + q(t)|x(t)|^{\gamma} signx(t) = o, t \geq t_0 \qquad (1.4)$$

are serve as prototypes of equation (1.1). However, Ahmed et al. [2] and Ahmed and Ali [3] investigated equation (1.3), and deriving several oscillation criteria for different ranges of γ, i.e. ($\gamma > 1$, $\gamma > 0$) respectively. Their results were subsequently extended by Ahmed [1] for the equation (1.4) under some conditions where $\gamma > 0$.

In [16], Saad et al. studied the sub-linear form of equation (1.1) and introduced new sufficient conditions for oscillation. For more details, readers are encouraged to see the work of Atkinson [5], Philos [17] and wong [18] which investigated various types of equations (1.1). For additional details, the studies by Elabbasy and Elzeiny [8], Kim [13], and Al-Jaser et al. [4]. offer valuable insight. Previous studies have established various oscillation criteria for some specific forms of variable coefficient equations. However, it would be interesting to enhance and expand the finding results for the more general sup-linear form of equation (1.1), and this what our current research article concentrates on.

2 RESULTS

In the sequel, we need the following lemma, which includes Lemmas of Erbe [9], Greaf and Spikes [11] and Wong [18].

Lemma 2.1: Suppose that

(1) $r(t) \le k_1$ on $[t_0, \infty)$,

(2) $yf(y) \ge k_2(f(y))^2$ for all $y \in \mathbb{R}$, $k_2 > 0$.

(3) $\lim_{t\to\infty} \inf \int_T^t q(s)ds \ge 0 \quad for\ all\ large\ T$.

Every non-oscillatory solution to equation (1.1) which cannot be absolutely constant must satisfies $x(t)\dot{x}(t) > 0$ for all large t .

Proof: Assume that $x(t) > 0$ for $t \ge T_1 \ge t_0$. If the lemma is false, then either $\dot{x}(t) < 0$ for all large t or $\dot{x}(t)$ oscillates for all large t .

Case 1: we let $\dot{x}(t) < 0$ for all large t and may consider that T_1 is sufficiently large such that

$$\int_{T_1}^t q(s)ds \ge 0 \quad and \quad \dot{x}(t) < 0\ for\ all\ t \ge T_1.$$

Now, integrating the equation (1.1), we obtain

$$r(t)f(\dot{x}(t)) - r(T_1)f(\dot{x}(T_1)) + g(x(t))\int_{T_1}^t q(s)ds - \int_{T_1}^t \left[\dot{x}(s)\dot{g}(x(s))\int_{T_1}^s q(u)du\right]ds = 0$$

But $\quad g(x(t))\int_{T_1}^t q(s)ds \ge 0 \quad and$

$$-\int_{T_1}^t \left[\dot{x}(s)\dot{g}(x(s))\int_{T_1}^s q(u)du\right]ds \ge 0$$

Thus, for every $t \ge T_1$, we have

$$r(t)f(\dot{x}(t)) \le r(T_1)f(\dot{x}(T_1))$$

Using (1), we get it

$$f(\dot{x}(t)) \le \frac{r(T_1)f(\dot{x}(T_1))}{k_1} = -B \quad \text{where } B > 0.$$

Then

$$f(\dot{x}(t)) \le -B\ \ for\ t \ge T_1$$

$$\dot{x}(t) \le -A\ \ for\ t \ge T_1\ \ and\ \ A > 0.$$

$$x(t) \le x(T_1) - A(t - T_1)$$

From last inequality, we have $x(t) \longrightarrow -\infty$ as $t \longrightarrow \infty$, which is contrary to $x(t) > 0$ for $t \ge T_1$.

Case 2: If $\dot{x}(t)$ oscillates, then there exists $\{\tau_n\} \to \infty$ where $\dot{x}(\tau_n) = 0$, n=1,2,3,… for all $t \ge T_1$. Define

$$\omega(t) = \frac{r(t)f(\dot{x}(t))}{g(x(t))},\ t \ge T.$$

This and by the equation (1.1), we get

$$\dot{\omega}(t) = -q(t) - \frac{r(t)f(\dot{x}(t))\acute{g}(x(t))\dot{x}(t)}{g^2(x(t))},\ t \ge T.$$

From the conditions (1) and (2), we have that

$$\dot{\omega}(t) \le -q(t) - A_1\omega^2(t), \quad t \ge T.$$

Where $A_1 = kk_2/k_1$. Then,

$$q(t) \le -\dot{\omega}(t) - A_1\omega^2(t), \ \ t \ge T,$$

which leads to

$$\dot{\omega}(t) \le -q(t) \qquad \text{for all } t \ge T_1 \qquad (2.1)$$

Thus, for every $\tau_{n+1} \ge \tau_n$, we get

$$\int_{\tau_n}^{\tau_{n+1}} q(s)ds \le -\int_{\tau_n}^{\tau_{n+1}} \dot{\omega}(s)ds = -\omega(\tau_{n+1}) + \omega(\tau_n) = 0,$$

Since it contradicts with condition (3), then the proof of the lemma is complete.

Theorem 2.1: Assume that the conditions (1) - (3) hold,

(4) $\frac{f(y)}{y} \le k_3$ for all $y \ne 0$.

Assume that ρ be a positive continuously differentiable function on the interval $[t_0, \infty)$ such that $\dot{\rho} \ge 0$ and $(\dot{\rho}(t)r(t))^{\cdot} \le 0$ on $[t_0, \infty)$, and

(5) $\lim_{t\to\infty} sup \frac{1}{t^\beta} \int_{t_0}^{t} (t-s)^\beta \rho(s)\, q(s)ds = \infty, for\ some\ \beta \ge 0.$

Then, all solutions of the super linear Eq. (1.1) are oscillatory.

Proof: Without loss of generality, suppose that there exists a solution $x(t)$ of (1.1) satisfies $x(t) > 0$ on $[T_1, \infty)$ $for\ all\ T_1 \ge t_0 \ge 0$. From Lemma2.1, we obtain $\dot{x}(t) > 0$ on $[T_2, \infty)$ $for\ all\ T_2 \ge T_1$. Define

$$\omega(t) = \frac{\rho(t)\, r(t) f(\dot{x}(t))}{g(x(t))}, t \ge T_2.$$

$$\dot{\omega}(t) = \frac{\dot{\rho}(t)\, r(t) f(\dot{x}(t))}{g(x(t))} + \frac{\rho(t)[r(t) f(\dot{x}(t))]^{\cdot}}{g(x(t))} - \frac{\rho(t) r(t) f(\dot{x}(t)) \acute{g}(x(t))\dot{x}(t)}{g^2(x(t))}, t \ge T_2.$$

This and by the equation (1.1), we obtain

$$\dot{\omega}(t) \le \frac{\dot{\rho}(t)\, r(t) f(\dot{x}(t))}{g(x(t))} - \rho(t) q(t) \quad , t \ge T_2.$$

From the conditions (4), we have that

$$\dot{\omega}(t) \le k_3\, \dot{\rho}(t) \frac{r(t)\dot{x}(t)}{g(x(t))} - \rho(t) q(t) \quad , t \ge T_2.$$

Then

$$\rho(t) q(t) \le -\dot{\omega}(t) + k_3\, \dot{\rho}(t) r(t) \frac{\dot{x}(t)}{g(x(t))} \quad , t \ge T_2$$

Integrate the last inequality multiplied by $(t-s)^\beta$ $o$from T_2 to t, we have

$$\int_{T_2}^{t} (t-s)^\beta \rho(s)\, q(s)ds \le -\int_{T_2}^{t} (t-s)^\beta\, \dot{\omega}(s)ds + k_3 \int_{T_2}^{t} (t-s)^\beta\, \dot{\rho}(s) r(s) \frac{\dot{x}(s)}{g(x(s))}\, ds \qquad (2.2)$$

By the Bonnet's theorem, for each $t \ge T_2$, there exists $a_t \in [T_2, t]$ such that

$$-\int_{T_2}^{t} (t-s)^\beta\, \dot{\omega}(s)ds = -(t-T_2)^\beta \int_{T_2}^{a_t} \dot{\omega}(s)ds = -(t-T_2)^\beta [\omega(a_t) - \omega(T_2)]$$

$$\le (t-T_2)^\beta \omega(T_2) \qquad (2.3)$$

But $(t-s)^\beta(\dot{\rho}(t) r(t))$ is a decreasing function, so by applying the Bonnet's theorem again for each $t \ge T_2$, there exist $b_t \in [T_2, t]$ such that

$$\int_{T_2}^{t} (t-s)^\beta\, \dot{\rho}(t) r(s) \frac{\dot{x}(s)}{g(x(s))}\, ds = (t - T_2)^\beta \dot{\rho}(T_2) r(T_2) \int_{T_2}^{b_t} \frac{\dot{x}(s)}{g(x(s))}\, ds$$

$$= (t-T_2)^{\beta}\dot{\rho}(T_2)r(T_2)\int_{x(T_2)}^{x(b_t)}\frac{du}{g(u)} \quad (2.4)$$

Hence, from (2.3) and (2.4) in (2.2), we have

$$\int_{T_2}^{t}(t-s)^{\beta}\rho(s)\,q(s)ds \le (t-T_2)^{\beta}\omega(T_2) + k_3\,(t-T_2)^{\beta}\dot{\rho}(T_2)r(T_2)\int_{x(T_2)}^{x(b_t)}\frac{du}{g(u)},$$

Now, dividing by t^{β} and taking the upper limit as $t\to\infty$, we get

$$\lim_{t\to\infty} sup\frac{1}{t^{\beta}}\int_{T_2}^{t}(t-s)^{\beta}\rho(s)\,q(s)ds < \infty,$$

which leads to a contradiction with condition (5). Hence, the proof is complete.

Example 2.1: Consider the following differential equation

$$\left[\left(\frac{2t^2}{t^8+16}\right)\left(\dot{x}(t)+\frac{\dot{x}^7(t)}{\dot{x}^6(t)+1}\right)\right]^{\cdot} + \frac{1}{t^6}x^{11}(t) = 0,\ t \ge t_0 \ge 8\sqrt{48}.$$

We have that

(1) $0 < r(t) < 2\ \ for\ all\ t \ge t_0$.

(2) $0 < \dfrac{f(y)}{y} = 1 + \dfrac{y^6}{y^6+1} < 2 \quad for\ \ all\ y.$

(3) $g(x) = x^{11}\ \ and\ \int_{\varepsilon}^{\infty}\dfrac{dx}{g(x)} < \infty$

$and \qquad \int_{-\varepsilon}^{-\infty}\frac{dx}{g(x)} < \infty \ \ for\ all\ \varepsilon.$

(4) $\lim_{t\to\infty}\inf\int_{T}^{t}q(s)ds = \lim_{t\to\infty}\inf\int_{T}^{t}\dfrac{ds}{s^6} > 0.$

By taking $\rho(t)=t^5$ such that $\dot{\rho}(t)=5t^4$, $(r(t)\dot{\rho}(t))^{\cdot} \le 0\ for\ all\ t \ge t_0 \ge 8\sqrt{48}$ and

(5) $\lim_{t\to\infty} sup\dfrac{1}{t^{\beta}}\int_{T_2}^{t}(t-s)^{\beta}\rho(s)\,q(s)ds$

$$= \lim_{t\to\infty} sup\frac{1}{t}\int_{T_2}^{t}\frac{(t-s)}{s}ds = \infty \quad where\ \beta = 1.$$

By theorem 2.1, the solutions of equation are oscillatory.

Remark 2.1: Theorem 2.1 extends results of Wong and Yeh [19].

Theorem 2.2: If the conditions (1)- (3) are fulfilled and

(6) $0 < k_4 < \frac{f(y)}{y}$ for all $y \ne 0$.

(7) $\lim_{t\to\infty} sup\frac{1}{t}\int_{t_0}^{t}\frac{1}{r(s)}\int_{t_0}^{s}q(u)\,duds = \infty.$

Then, Eq. (1.1) is oscillatory.

Proof: Let $x(t)$ be a non-oscillatory solution of the eq. (1.1) and assume that $x(t) > 0$ for $t \ge T_1 \ge t_0$. Lemma 2.1, implies that $\dot{x}(t) > 0\ \ on\ [T_2,\infty)$ for all $T_2 \ge T_1$. From the inequality (2.1), we obtain

$$\int_{T_2}^{t}\dot{\omega}(s)ds \le -\int_{T_2}^{t}q(s)ds \qquad \text{for all } t \ge T_2.$$

$$\omega(t) \le \omega(T_2) - \int_{T_2}^{t}q(s)ds \qquad \text{for all } t \ge T_2.$$

So, we can get the following

$$\frac{f(\dot{x}(t))}{g(x(t))} \le \frac{\omega(T_2)}{r(t)} - \frac{1}{r(t)}\int_{T_2}^{t}q(s)ds \qquad (2.5)$$

Since $r(t) \geq k_5 > 0 \ \ then$

$$\frac{\mathbf{1}}{r(t)} \leq \frac{1}{k_5} > 0 \quad \text{for all} \quad t \geq T_2 \qquad \textbf{(}2.6)$$

Hence, by the condition (6) and from (2.6) in (2.5), we have

$$\frac{k_4 \dot{x}(t)}{g(x(t))} \leq \frac{\omega(T_2)}{k_5} - \frac{1}{r(t)} \int_{T_2}^{t} q(s) ds$$

Thus

$$\frac{k_4 \dot{x}(t)}{g(x(t))} \leq \frac{\omega(T_2)}{k_5} - \frac{1}{r(t)} \int_{T_2}^{t} q(s) ds$$

By integration, division by t and taking the limit superior as $t \to \infty$, we obtain

$$\lim_{t\to\infty} sup \frac{1}{t} \int_{x(T_2)}^{x(t)} \frac{du}{g(u)} \leq \lim_{t\to\infty} sup \ \frac{1}{t} \frac{\omega(T_2)}{k_4 k_5} (t - T_2) - \frac{1}{k_4} \lim_{t\to\infty} sup \frac{1}{t} \int_{T_2}^{t} \frac{1}{r(s)} \int_{T_2}^{s} q(u) duds \leq -\infty,$$

This is a contradiction. Then, the proof is completed.

Example 2.2: Applying the conditions of Theorem 2.2 to this equation as follows:

$$\left[\left(\frac{8t^2}{t^2+1}\right)\left(4\dot{x}(t) + \frac{\dot{x}^{11}(t)}{\dot{x}^4(t)+1}\right)\right]^{\cdot} + t^4 x^{13}(t) = 0, \quad t \geq t_0 > 0$$

$r(t) = \frac{8t^2}{t^2+1}$, $f(y) = 4y + \frac{y^{11}}{y^4+1}$, $q(t) = \ t^4$, $g(x) = x^{13}$ and

(1) $\lim_{t\to\infty} inf \int_{T}^{t} q(s) ds = \lim_{t\to\infty} inf \int_{T}^{t} s^4 ds = \infty > 0.$

(2) $$\lim_{t\to\infty} sup \frac{1}{t} \int_{T_2}^{t} \frac{1}{r(s)} \int_{T_2}^{s} q(u) duds = \lim_{t\to\infty} sup \frac{1}{t} \int_{T_2}^{t} \frac{s^2+1}{8s^2} \int_{T_2}^{s} u^4 duds = \infty.$$

Then, every solution of the equation is oscillatory.

Remark 2.2: Theorem 2.2 extends the results of Philos [17].

Theorem 2.3: In addition to conditions (1), (2) and (3), suppose that

(8) $0 < k_4 < \frac{f(y)}{y} \leq k_6$ for all $y \neq 0.$

Assume that ρ be a positive continuously differentiable function on the interval $[t_0, \infty)$ such that $\dot{\rho} \geq 0 \ and \ (\dot{\rho}(t) r(t))^{\cdot} \leq 0 \ on \ [t_0, \infty)$, and

(9) $\lim_{t\to\infty} \int_{t_0}^{t} \rho(s) q(s) \, ds = \infty.$

(10) $\lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{\rho(s) r(s)} \int_{t_0}^{s} \rho(u) q(u) \, duds = \infty$

Then, every solution of the superlinear equation (1.1) is oscillatory.

Proof: Let $x(t)$ be a non-oscillatory solution eq. (1.1) and assume that $x(t) > 0$ for $t \geq T_1 \geq t_0$. From Lemma 2.1, we know that $\dot{x}(t) > 0 \ on \ [T_2, \infty)$ for all $T_2 \geq T_1$.

Multiplying the equation (1.1) by $\frac{\rho(t)}{g(x(t))}$, we have

$$\frac{\rho(t)(r(t) f(\dot{x}(t)))^{\cdot}}{g(x(t))} + \rho(t) q(t) {=} 0$$

Integrating the last equation and by the condition (8), we have

$$k_4 \frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))} \le \frac{\rho(T_2)r(T_2)f(\dot{x}(T_2))}{g(x(T_2))} + k_6 \int_{T_2}^{t} \frac{\dot{\rho}(s)r(s)\dot{x}(s)}{g(x(s))} ds - \int_{T_2}^{t} \rho(s)q(s)ds \qquad (2.7)$$

Since $(\dot{\rho}(t)r(t))$ in the inequality (2.7) is a decreasing function, then by the Bonnet's theorem, for each $t \ge T_2$, there exists $\beta_t \in [T_2, t]$ such that

$$k_6 \int_{T_2}^{t} \frac{\dot{\rho}(t)r(s)\dot{x}(s)}{g(x(s))} ds = k_6\, \dot{\rho}(T_2)r(T_2) \int_{T_2}^{\beta_t} \frac{\dot{x}(s)}{g(x(s))} ds$$

$$= k_6\, \dot{\rho}(T_2)r(T_2) \int_{x(T_2)}^{x(\beta_t)} \frac{du}{g(u)} = k_6\, N_1 < \infty.$$

Let

$$C_1 = \frac{\rho(T_2)r(T_2)f(\dot{x}(T_2))}{g(x(T_2))} + N_1,$$

Thus, the inequality (2.7) becomes

$$k_4 \frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))} \le C_1 - \int_{T_2}^{t} \rho(s)q(s)ds \qquad (2.8)$$

But $\lim_{t\to\infty} \int_{T_2}^{t} \rho(s)q(s)\, ds = \infty$, then, there exists $t \ge T_3 \ge T_2$ achieves

$$\int_{T_3}^{t} \rho(s)q(s)\, ds \ge 2C_1$$

That's

$$C_1 \le \frac{1}{2} \int_{T_3}^{t} \rho(s)q(s)\, ds.$$

Thus, the inequality (2.8) becomes

$$\frac{\dot{x}(t)}{g(x(t))} \le -\frac{1}{2k_4} \frac{1}{\rho(t)r(t)} \int_{T_3}^{t} \rho(s)q(s)ds$$

Integrating this inequality, we obtain

$$\int_{x(T_3)}^{x(t)} \frac{du}{g(u)} \le -\frac{1}{2k_4} \int_{T_3}^{t} \frac{1}{\rho(s)r(s)} \int_{T_2}^{s} \rho(u)q(u)duds \longrightarrow -\infty \quad as \ t \longrightarrow \infty.$$

This is a contradiction, which completes the proof.

Example 2.3: Suppose we have the following equation:

$$\left[\left(\frac{1}{t^4}\right)\left(\frac{\dot{x}(t)}{\dot{x}^2(t)+1}\right)\right]^{\cdot} + (1+2cost)x^5(t) = 0, \quad t \ge t_0 > 1.$$

We have $r(t) = \frac{1}{t^4}$, $f(y) = \frac{y}{y^2+1}$, $q(t) = 1 + 2cost$, $g(x) = x^5$,

(1) $\lim_{t\to\infty} inf \int_T^t q(s)ds = \lim_{t\to\infty} inf \int_T^t (1 + 2\cos(s))ds = \infty > 0$.

Let $\rho(t) = t$ such that $\dot{\rho}(t) = 1$, $(r(t)\dot{\rho}(t))^{\cdot} \le 0 \; for \; all \; t \ge t_0 \ge 1$,

(2) $\lim_{t\to\infty} \frac{1}{t} \int_{T_2}^{t} \rho(s)q(s)ds = \lim_{t\to\infty} \frac{1}{t} \int_{T_2}^{t} s(1 + 2\cos(s))\, ds = \infty.$

(3) $\lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{\rho(s)r(s)} \int_{t_0}^{s} \rho(u)q(u)\, duds = \lim_{t\to\infty} \int_{t_0}^{t} s^3 \int_{t_0}^{s} u(1 + 2\cos(u))\, duds = \infty.$

So, Theorem 2.3, confirms the oscillation of the equation.

Remark 2.3: Theorem 2.3 extends the results of Grace and Lalli [10], Ahmed [1], Ahmed et al. [2] and Ahmed and Ali [3].

Remark 2.3: Theorem 2.3 extends the results of Grace and Lalli [10], Ahmed [1], Ahmed et al. [2] and Ahmed and Ali [3].

3 Discussion

A set of new oscillation conditions are stated and proved. Some of illustrative examples are provided to show the applications of the oscillation criteria and the comparisons between our results and studied previous results.

4 Conclusions

In this research, we have established and demonstrated a set of oscillation conditions that improve and extend the existing oscillation criteria and treating cases have not been discussed by known results. In addition, we have provided illustrative examples to support our work. Some notes are also included to highlight the importance of our main findings.

Acknowledgements

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