`GREAT SOCIALIST PEOPLE LIBYAN ARAB JAMAHIRIYA

AL-TAHADI UNIVERSITY
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DEPARTMENT OF MATHEMATICS

A STUDY OF THE BEHAVIOR OF THE SECOND ORDER AND FIRST-DEGREE ORDINARY DIFFERENTIAL EQUATION

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

﴿وَمَا تَشَاؤُونَ إِلَّا أَن يَشَاءَ اللَّهُ إِنَّ اللَّهَ كَانَ عَلِيماً حَكِيماً﴾

صدق الله العظيم

سورة الإنسان الآية 30

الإهداء

إلى من قال الله في حقهما

﴿وَاخْفِضْ لَهُمَا جَنَاحَ الذُّلِّ مِنَ الرَّحْمَةِ وَقُلْ رَبِّ ارْحَمْهُمَا كَمَا رَبَّيَانِي صَغِيرًا﴾

والديّ الكرام

الشكر والتقدير

الشكر أولاً و أخيراً لله عز و جل شكراً يليق بجلال وجهه و عظيم سلطانه على توفيقه لإتمام هذا الجهد المتواضع، والذي أسأله سبحانه و تعالى أن يكون عملاً صالحاً متقبلاً، أتقدم بالشكر لكل من ساعدني لإتمام هذا البحث, سواء بجهده أو مشورته أو تشجيعه، واخص بالشكر الجزيل و التقدير الوفير للدكتور المشرف:

نبيل زكي فريد على ماأبداهُ لي من توجيه و تشجيع لانجاز هذا العمل.

كما أتقدم بالشكر الجزيل للدكتور **سعد عبد العزيز مناع** الأمين المساعد للشؤون العلمية بالجامعة على ما أبداهُ لي من رحابة صدر واهتمام وجهوده للرقي بمستوى العملية التعليمية فله خالص الشكر والتقدير كما يسعدني أن أسجل شكري وامتناني لأعضاء هيئة التدريس قسم الرياضيات لجهودهم وتشجيعهم لي.

وأخر دعوانا أن الحمد لله رب العالمين والصلاة والسلام على اشرف الأنبياء والمرسلين سيدنا محمد وعلى أله وصحبه أجمعين.

Contents

SUMMARY

In this dissertation, The behavior of the solutions of the second order ordinary differential equations. is studied The problem of determining the oscillation and continuability criteria for second order nonlinear differential equation with variable coefficients is considered.

The results obtained in this dissertation is compared with the results that obtained before by some researchers, it is found that their results can be deduced as special cases from the equations which we studied in the present dissertation.

INTRODUCTIN

1.1 Introduction

Differential equations appear naturally in many fields of research specially in the natural sciences. The second – order ordinary differential equation are frequently used as mathematical models of most dynamic processes, in electro – mechanical systems.

Many phenomena in different branches of sciences are interpreted and solved by second order differential equations.

The study of the oscillation of second – order nonlinear ordinary differential equations with alternating coefficients is of special interest because many physical systems are modeled by second – order nonlinear ordinary differential equations.

For example the so – called Emden -Fowler equation arises in the study of gas dynamics and fluid mechanics. This equation appears also in the study of relativistic mechanics and in the study of certain chemical reactions.

The problem of determining oscillation criteria for second -order nonlinear differential equations has received a great deal of attention in the twenty years after the publication of the new classic paper by Atkinson [1].

Many authors use some different techniques in studying the oscillatory behavior of the second order linear differential equations, especially, what so-called averaging techniques that dates back to works of Wintner [37] and its generalization by Hartman [16].

In this dissertation we consider the problem of determining oscillation and continuability criteria for the second – order nonlinear perturbed differential equations of the form:

$$\left(r(t)\psi(x(t))\dot{x}(t)\right)^{\bullet} + g_1(t,x(t)) = H(t,\dot{x}(t),x(t)). \qquad \text{(E)}$$

The dissertation consists of many illustrative examples as an application of the obtained results and remarks. The dissertation comprises three chapters.

1.2 Objectives and organization of the dissertation

We remarked in the previous discussion that ordinary differential equations have important applications in many scientific domains. Specially the second – order differential equations are of great interest, the main aims of the research are built on:

1. Study and review of previous papers.
2. Study of the differential equation and theorems.

The dissertation is divided as follows: in chapter 1 a simple review for previous studies and the new researches, also it includes some basic definitions, elementary results that will be used in the second and third chapter. and the major results of the oscillation for the second – order ordinary differential equations that can be found in published studies. In chapter 2 the study of the oscillation equation (E) with $H(t,\dot{x}(t),x(t)) = 0$, the oscillation criteria that obtained in this chapter includes some of the earlier results which can be found in the published studies for equation (E) with $H(t,\dot{x}(t),x(t)) = 0$, that is, the results obtained before is a special case of my study and it will be illustrated by some examples. In chapter 3 we study the continuability and oscillation of the second – order ordinary differential equation. Oscillation criteria for solutions of equation (E) with alternating coefficients. These results extend and improve some oscillation results which were obtained before and these results will be illustrated by some examples, and finally we give summary and conclusion of dissertation.

CHAPTER I

PRELIMINARY RESULTS AND DEFINTIONS

1.1 Introduction

The goal of this dissertation is the study of the sufficient conditions for the oscillation of the solutions of the second – order nonlinear ordinary differential equation of the form:

$$\left(r(t)\psi(x(t))\dot{x}(t)\right)^{\cdot} + g_1(t, x(t)) = H(t, \dot{x}(t), x(t)), \tag{E}$$

where r is a positive continuous function on the interval $[t_0, \infty), t_0 \geq 0$, ψ is a positive continuous function on the real line R , g_1 is a continuous function on $R \times R$ with $\frac{g_1(t, x(t))}{g(x(t))} \geq q(t)$, *for all* $x \neq 0$ and $t \in [t_0, \infty)$ where g is continuously differentiable function on the real line R except possible at 0 with $xg(x) > 0$ and $g'(x) \geq l > 0$ *for all* $x \neq 0$ and q is a continuous function on the interval $[t_0, \infty)$, $t_0 \geq 0$, and H is a continuous function on $[t_0, \infty) \times R \times R$ with $\frac{H(t, \dot{x}(t), x(t))}{g(x(t))} \leq m(t)$ *for all* $x \neq 0$ and $t \geq t_0$.

On this study we restrict our attention only to the solutions of the differential equation (E) that exists on some interval $[t_x, \infty)$, where t_x may depend on the particular solution.

In this chapter we list some basic definitions, elementary results that will be used throughout the next chapters and the oscillation criteria that obtained by a great number of authors for the equation (E) and / or special cases of this equation.

2.2 Basic definitions and elementary results

Definition 1.2.1

A point $t = \tau \geq 0$ is called a zero of the solution $x(t)$ if $x(\tau) = 0$.

Definition 1.2.2

A solution $x(t)$ of the differential equation (E) is said to be oscillatory if it has arbitrary large zeros, otherwise it is said to be non – oscillatory.

Definition 1.2.3

Equation (E) is called oscillatory if all its solutions are oscillatory; otherwise it is called non – oscillatory.

The next theorem plays a great importance in the theory of oscillation of solutions of linear differential equations:

The Sturm separation theorem [33]

If $x_1(t)$ and $x_2(t)$ are linearly independent solutions of the equation

$$\left(r(t)\dot{x}(t)\right)^{\bullet} + q(t)x(t) = 0,$$

then between any two consecutive zeros of $x_1(t)$ there is precisely one zero of $x_2(t)$.

Therefore the solutions of the second – order linear differential equations are all oscillatory or all non – oscillatory. The story of nonlinear equations is not the same.

The nonlinear differential equations may have both oscillatory criteria.

The importance of the classification of the second – order differential equations into oscillatory categories is due to the following well – known fact:

A non – trivial solution of the second – order ordinary differential equation must change its sign whenever it vanishes, since $x(t)$ and $\dot{x}(t)$ can not vanish simultaneously (in this case the zeros of $x(t)$ is said to be isolated).

Definition 1.2.4

The differential equation (E) is called

(1) Super – linear if the function g satisfies that

$$0<\int_{\varepsilon}^{\infty}\frac{du}{g(u)}<\infty \quad \text{and} \quad 0<\int_{-\varepsilon}^{-\infty}\frac{du}{g(u)}<\infty, \qquad \textit{for all } \varepsilon>0 .$$

(2) Sub – linear if the function g satisfies that

$$0<\int_{0}^{\varepsilon}\frac{du}{g(u)}<\infty \quad \text{and} \quad 0<\int_{0}^{-\varepsilon}\frac{du}{g(u)}<\infty, \qquad \textit{for all } \varepsilon>0 .$$

(3) Mixed type if the function g satisfies that

$$0<\int_{0}^{\infty}\frac{du}{g(u)}<\infty \quad \text{and} \quad 0<\int_{0}^{-\infty}\frac{du}{g(u)}<\infty . \qquad (*)$$

If $g(u)=g_1(u)+g_2(u)$ or $g(u)=e^{|u|}\operatorname{sgn} u$,

where g_1 is super – linear and g_2 is sub – linear, then we see easily that condition (*) holds.

The following theorem is quite useful elemental in our study in the next chapters:

The Bonnet's theorem [2]

Suppose that h is a continuous function on $[a,b]$, ρ is a non negative function on the interval $[a,b]$, and ρ is an increasing function on $[a,b]$, then there exists a point $c \in [a,b]$ such that

$$\int_a^b \rho(s)h(s)ds = \rho(b)\int_c^b h(s)ds .$$

If ρ is a decreasing function on $[a,b]$, then there exists a point $c \in [a,b]$ such that

$$\int_a^b \rho(s)h(s)ds = \rho(a)\int_a^c h(s)ds .$$

This theorem is a part of the second mean value theorem of integrals [2].

1.3 The oscillation of the equation (E)

A special cases of the equation (E) has been discussed by many authors these special cases are categorized as follows

$$\ddot{x}(t) + q(t)x(t) = 0, \qquad (1)$$

$$\left(r(t)\dot{x}(t)\right)^{\bullet} + q(t)x(t) = 0, \qquad (2)$$

$$\ddot{x}(t) + q(t)g(x(t)) = 0, \qquad (3)$$

$$\left(r(t)\dot{x}(t)\right)^{\bullet} + q(t)g(x(t)) = 0, \qquad (4)$$

$$\left(r(t)\psi(x(t))\dot{x}(t)\right)^{\bullet} + q(t)g(x(t)) = 0, \qquad (5)$$

$$\left(r(t)\dot{x}(t)\right)^{\bullet} + q(t)g(x(t)) = H(t,x(t)), \qquad (6)$$

where r, q are continuous functions on $[t_0, \infty)$, $t_0 \geq 0$, and r is a positive continuous function on the interval $[t_0, \infty)$, $t_0 \geq 0$, ψ is a continuous function on R and g is continuous function on the real line R except possible at 0 with $xg(x) > 0$ and $g'(x) > 0$ *for all* $x \neq 0$ and H is a continuous function on $[t_0, \infty) \times R$.

The oscillation property of the second – order differential equation has been the subject of interest of many authors since the first paper by Fite [10].

The investigation of the oscillation of (E) may be done by many directions, among these, an often considered way is to determine integral tests involving function q in order to obtain oscillation criteria.

whenever, $\int^{\infty}$ is written, it is to be assumed that $\int^{\infty} = \lim_{t \to \infty} \int^{t}$, and that this limit exists in the extended real numbers.

<u>The oscillation of equation (1)</u>

This section is confined to the oscillation criteria for the second – order linear differential equation of the form (1). The oscillation of equation (1) has brought attention of many authors since the first paper by Fite [10].

Among the numerous papers dealing with this topic we refer in particular to Fite[10] where he introduced the following:

<u>Theorem 1.3.1 Fite [10]</u>

(1) Let $q(t) > 0$ *for all* $t \geq t_0$ and $\int_{t_0}^{\infty} q(s)ds = \infty$,

then every solution of the equation (1) is oscillatory.

Proof

Let $x(t)$ be a non oscillatory solution of equation (1) and assume that $x(t) > 0$ *for all* $t \geq t_0$.

Let $\dot{x}(t) < 0$ *for* $t \geq T$,

Now, integrating the equation (1), we have

$$\dot{x}(t) - \dot{x}(T) + \int_T^t q(s)x(s)ds = 0,$$

then, *for* $t \geq T$, we get

$$\dot{x}(t) + x(t)\int_T^t q(s)ds - \int_T^t \left[\dot{x}(s)\int_T^s q(u)du\right]ds = \dot{x}(T)$$

$$\therefore \ \dot{x}(t) \leq \dot{x}(T).$$

Integrating *for* $t \geq T$, we get

$$x(t) \leq x(T) + \dot{x}(T)(t - T),$$

$$\therefore x(t) \to -\infty \quad \text{as } t \to \infty.$$

which contradicts the assumption that $x(t) > 0$ *for* $t \geq T$.

If $\dot{x}(t)$ oscillates, then there exists sequence $\{t_n\} \to \infty$ such that $\dot{x}(t_n) = 0$ $(n = 1,2,3,\dots)$,

for all $t \geq T$,

define

$$\omega(t)=\frac{\dot{x}(t)}{x(t)}, \quad t\geq T \ .$$

for all $t \geq T$, we obtain

$$\dot{\omega}(t)=-q(t)-\omega^2(t), \qquad \text{where} \quad q(t)=\frac{\ddot{x}(t)}{x(t)},$$

then, we have

$$\dot{\omega}(t)\leq -q(t), \qquad \textit{for all } t\geq T \ ,$$

so, for every $\tau_{n+1}\geq \tau_n$, we get

$$\int_{\tau_n}^{\tau_{n+1}} q(t)dt \leq -\int_{\tau_n}^{\tau_{n+1}} \dot{\omega}(t)dt$$

$$=\omega(\tau_n)-\omega(\tau_{n+1})=0\,,$$

then, we have

$$\int_{\tau_n}^{\tau_{n+1}} q(t)dt\leq 0$$

This contradiction to the condition (1), then $\dot{x}(t) > 0$

Let $$\omega(t)=\frac{\dot{x}(t)}{x(t)}, \quad t\geq t_0$$

$$\therefore\ \dot{\omega}(t)\leq -q(t),$$

then, for all $t\geq t_0$, we obtain

$$q(t)\leq -\dot{\omega}(t).$$

Then. for $t \geq t_0$, we get

$$\int_{t_0}^{t} q(s)ds \leq -\int_{t_0}^{t} \dot{\omega}(s)ds,$$

then, we have

$$\int_{t_0}^{t} q(s)ds \leq -[\omega(t) - \omega(t_0)] \leq \omega(t_0).$$

$$\therefore \int_{t_0}^{t} q(s)ds < \infty \quad \text{as } t \to \infty.$$

This contradicts to the condition (1); hence, the proof is completed.

Theorem 1.3.2 Wintener [37]

Suppose that

(1) $\lim_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} (t-s)q(s)ds = \infty,$

then every bounded solution of equation (1) is oscillatory.

Proof

Without loss of generality, we may assume that there exists a solution of equation (1) such that $x(t) > 0$ on $[t_1, \infty)$ *for all* $t_1 \geq t_0 > 0$,

Define

$$\omega(t) = \frac{\dot{x}(t)}{x(t)}, \qquad t \geq t_1,$$

hence, *for* $t \geq t_1$, we have

$$\dot{\omega}(t) = -q(t) - \omega^2(t),$$

then, $for\ t \geq t_1$, we obtain

$$\int_{t_1}^{t}(t-s)q(s)ds \leq -\int_{t_1}^{t}(t-s)\dot{\omega}(s)ds\,.$$

then by integration by parts $for\ t \geq t_1$, we have

$$\int_{t_1}^{t}(t-s)q(s)ds \leq -\left\{(t-s)\omega(s)\Big|_{t_1}^{t} + \int_{t_1}^{t}\omega(s)ds\right\}$$

$$= (t-t_1)\omega(t_1) - \ln(x(t)) + \ln(x(t_1))\,.$$

$$\therefore\ \lim_{t\to\infty}\frac{1}{t}\int_{t_1}^{t}(t-s)q(s)ds \leq \lim_{t\to\infty}\left(1-\frac{t_1}{t}\right)\omega(t_1) < \infty,$$

this contradiction to the condition (1); hence, the proof is completed.

Remark 1.3.1

The theorem 1.3.2 extended the result of Fite [10] to an equation in which q is of arbitrary sign.

Example 1.3.1

Consider the differential equation

$$\ddot{x}(t) + (2 - 5\sin t)x(t) = 0, \qquad t > 0\,.$$

Theorem 1.3.2 ensures that the given equation is oscillatory. However, theorem 1.3.1 fails.

Theorem 1.3.3 Kamenev [17]

The equation (1) is oscillatory if

(1) $\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s)ds = \infty$, for some integer $n \geq 2$,

Remark 1.3.2

Kamenev [17] has proved a new integral criterion for the oscillation of the differential equation (1). based on the use of the n - th primitive of the coefficient $q(t)$, which has Wintner's result [37] as a particular case. .

Theorem 1.3.4 Philos [26]

Let ρ be a positive continuously differentiable function on the interval $[t_0, \infty)$ such that

(1) $\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} \frac{(t-s)^{n-3}}{\rho(s)} [(n-1)\rho(s) - (t-s)\dot{\rho}(s)]^2 ds < \infty$, for some integer $n \geq 3$,

(2) $\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} \rho(s) q(s) ds = \infty$,

then every solution of equation (1) is oscillatory.

Proof

Let $x(t)$ be a non oscillatory solution of equation (1) and say $x(t) \neq 0$ *for all* $t \geq T$.

Define $\omega(t) = \frac{\dot{x}(t)}{x(t)}$, $t \geq T$.

then, for $t \geq T$, we have

$$\dot{\omega}(t) = -q(t) - \omega^2(t), \quad \text{where } q(t) = \frac{\ddot{x}(t)}{x(t)}.$$

Then, we get

$$q(t) = -\dot{\omega}(t) - \omega^2(t), \quad \text{for every } t \geq T,$$

then, for $t \geq T$, we have

$$\int_T^t (t-s)^{n-1} \rho(s) q(s) ds = -\int_T^t (t-s)^{n-1} \rho(s) \dot{\omega}(s) ds - \int_T^t (t-s)^{n-1} \rho(s) \omega^2(s) ds$$

$$= (t-T)^{n-1} \rho(T) \omega(T) + \frac{1}{4} \int_T^t \frac{(t-s)^{n-3}}{\rho(s)} \left[(n-1)\rho(s) - (t-s)\dot{\rho}(s)\right]^2 ds$$

$$- \int_T^t \left[(t-s)^{\frac{n-1}{2}} \sqrt{\rho(s)}\, \omega(s) + \frac{(t-s)^{\frac{n-3}{2}} \left[(n-1)\rho(s) - (t-s)\dot{\rho}(s)\right]}{2\sqrt{\rho(s)}} \right]^2 ds$$

$$\int_T^t (t-s)^{n-1} \rho(s) q(s) ds \leq (t-T)^{n-1} \rho(T) \omega(T) + \frac{1}{4} \int_T^t \frac{(t-s)^{n-3}}{\rho(s)} \left[(n-1)\rho(s) - (t-s)\dot{\rho}(s)\right]^2 ds,$$

on the other hand, for every $t \geq T$, we get

$$\int_{t_0}^t (t-s)^{n-1} \rho(s) q(s) ds - \int_T^t (t-s)^{n-1} \rho(s) q(s) ds = \int_{t_0}^T (t-s)^{n-1} \rho(s) q(s) ds$$

$$\leq (t-t_0)^{n-1} \int_{t_0}^T \rho(s) |q(s)| ds.$$

Thus, we have

$$\frac{1}{t^{n-1}} \int_{t_0}^t (t-s)^{n-1} \rho(s) q(s) ds \leq \left(1 - \frac{T}{t}\right)^{n-1} \rho(T) \omega(T) + \left(1 - \frac{t_0}{t}\right)^{n-1} \int_{t_0}^T \rho(s) |q(s)| ds$$

$$+ \frac{1}{4t^{n-1}} \int_T^t \frac{(t-s)^{n-3}}{\rho(s)} \left[(n-1)\rho(s) - (t-s)\dot{\rho}(s)\right]^2 ds,$$

for all $t \geq T$, this gives

$$\limsup_{t\to\infty}\frac{1}{t^{n-1}}\int_{t_0}^{t}(t-s)^{n-1}\rho(s)q(s)ds \leq \rho(T)\omega(T)+\int_{t_0}^{T}\rho(s)|q(s)|ds$$

$$+\frac{1}{4}\limsup_{t\to\infty}\frac{1}{t^{n-1}}\int_{T}^{t}\frac{(t-s)^{n-3}}{\rho(s)}\left[(n-1)\rho(s)-(t-s)\dot{\rho}(s)\right]^2 ds$$

$$<\infty .$$

This contradicts to the condition (2); hence, the proof is complete.

Remark 1.3.3

Philos [26] improved Kamenev's result and by putting $\rho(t)=1$ in theorem 1.3.4 leads to Kamenev's result [17] (theorem 1.3.3).

Example 1.3.2

Consider the differential equation

$$\ddot{x}(t)+3t^3x(t)=0\,, \qquad t>0\,.$$

Theorem 1.3.4 ensures that the given equation is oscillatory where $\rho(t)=t^3$, n=3.

Theorem 1.3.5 Yan [41]

Suppose that there exists an integer $n\geq 3$ with

(1) $\limsup_{t\to\infty}\frac{1}{t^{n-1}}\int_{t_0}^{t}(t-s)^{n-1}q(s)ds<\infty$.

Let $\Omega(t)$ be a continuous function on $[t_0,\infty)$ with

(2) $\liminf_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s)ds \geq \Omega(T)$, for every $T \geq t_0$

then equation (1) is oscillatory if

(3) $\int_{t_0}^{\infty} \Omega_+(s)ds = \infty$, where $\Omega_+(t) = \max\{\Omega(t), 0\}$, $t \geq t_0$.

Remark 1.3.4

Yan [41] presented another new oscillation theorem for equation (1).

Theorem 1.3.6 Philos [30]

Let H and h be continuous functions

$$h, H : D = \{(t,s) : t \geq s \geq t_0\} \to R,$$

and H has a continuous and non positive partial derivative on D with respect to the second variable such that $H(t,t) = 0$, *for* $t \geq t_0$, $H(t,s) > 0$ *for* $t > s \geq t_0$

and $$\frac{-\partial H(t,s)}{\partial s} = h(t,s)\sqrt{H(t,s)} \quad \textit{for all } (t,s) \in D,$$

then equation (1) is oscillatory if

(1) $\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)q(s) - \frac{1}{4}h^2(t,s) \right] ds = \infty$.

Proof

Let $x(t)$ be a non oscillatory solution of equation (1) and assume that $x(t) > 0$

for $t \geq t_0$.

define

$$\omega(t) = \frac{\dot{x}(t)}{x(t)}, \qquad t \geq t_0 ,$$

thus, for every $t \geq t_0$, we obtain

$$\dot{\omega}(t) = -q(t) - \omega^2(t),$$

then, for every $t \geq T$, we have

$$\int_T^t H(t,s)q(s)ds = -\int_T^t H(t,s)\dot{\omega}(s)ds - \int_T^t H(t,s)\omega^2(s)ds,$$

hence, *for* $t \geq T \geq t_0$, we get

$$\int_T^t H(t,s)q(s)ds = H(t,T)\omega(T) - \int_T^t h(t,s)\sqrt{H(t,s)}\omega(s)ds - \int_T^t H(t,s)\omega^2(s)ds ,$$

and hence,

$$\int_T^t H(t,s)q(s)ds = H(t,T)\omega(T) + \int_T^t \frac{1}{4}h^2(t,s)ds - \int_T^t \left[\sqrt{H(t,s)}\omega(s) + \frac{1}{2}h(t,s)\right]^2 ds ,$$

$$\therefore \int_T^t \left[H(t,s)q(s) - \frac{1}{4}h^2(t,s)\right]ds \leq H(t,T)\omega(T). \tag{1-1}$$

Dividing inequality (1-1) by $H(t,T)$ and taking the upper limit as $t \to \infty$, we obtain

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)q(s) - \frac{1}{4}h^2(t,s)\right]ds < \infty ,$$

which contradicts to condition(1); hence, the proof is complete.

Remark 1.3.5

Philos [30] extended the Kamenev's result [17].

Theorem 1.3.7 Philos [28]

Let H and h as in theorem 1.3.6, moreover, suppose that

$$(1)\ 0 < \inf_{s \geq t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right] \leq \infty,$$

$$(2)\ \limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t} h^2(t,s)ds < \infty,$$

assume that $\Omega(t)$ be a continuous function on $[t_0,\infty)$ with

$$(3)\ \int_{t_0}^{\infty}\Omega_+^2(s)ds = \infty, \quad \text{Where } \Omega_+(t) = \max\{\Omega(t),0\},$$

then equation (1) is oscillatory if

$$(4)\ \limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left[H(t,s)q(s) - \frac{1}{4}h^2(t,s)\right]ds \geq \Omega(T), \quad \text{for every } T \geq t_0.$$

Proof

Let $x(t)$ be a non oscillatory solution of equation (1) and assume that $x(t) \neq 0$ *for* $t \geq T \geq t_0$.

define

$$\omega(t) = \frac{\dot{x}(t)}{x(t)}, \qquad t \geq T.$$

then for every $t \geq T$, we obtain

$$\dot{\omega}(t) = -q(t) - \omega^2(t),$$

then, for every $t \geq T$, we have

$$\int_T^t H(t,s)q(s)ds = -\int_T^t H(t,s)\dot{\omega}(s)ds - \int_T^t H(t,s)\omega^2(s)ds,$$

hence, *for* $t \geq T \geq t_0$, we get

$$\int_T^t H(t,s)q(s)ds = H(t,T)\omega(T) - \int_T^t h(t,s)\sqrt{H(t,s)}\omega(s)ds - \int_T^t H(t,s)\omega^2(s)ds,$$

and hence,

$$\int_T^t H(t,s)q(s)ds = H(t,T)\omega(T) + \int_T^t \frac{1}{4}h^2(t,s)ds - \int_T^t \left[\sqrt{H(t,s)}\omega(s) + \frac{1}{2}h(t,s)\right]^2 ds,$$

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)q(s) - \frac{1}{4}h^2(t,s)\right]ds \leq \omega(T)$$

$$-\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[\sqrt{H(t,s)}\omega(s) + \frac{1}{2}h(t,s)\right]^2 ds,$$

$$\omega(T) \geq \limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)q(s) - \frac{1}{4}h^2(t,s)\right]ds + \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[\sqrt{H(t,s)}\omega(s) + \frac{1}{2}h(t,s)\right]^2 ds,$$

then by condition (4) *for* $T \geq t_0$, we have

$$\omega(T) \geq \Omega(T) + \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[\sqrt{H(t,s)}\omega(s) + \frac{1}{2}h(t,s)\right]^2 ds.$$

this shows that

$$\omega(T) \geq \Omega(T) \quad , \quad for\ T \geq t_0 .$$

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[\sqrt{H(t,s)}\omega(s) + \frac{1}{2}h(t,s) \right]^2 ds < \infty ,$$

hence,

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,s)\omega^2(s)ds + \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t h(t,s)\sqrt{H(t,s)}\omega(s)ds < \infty ,$$

i.e., we have

$$\liminf_{t\to\infty} \left[U(t) + V(t) \right] < \infty , \tag{1-2}$$

where $U(t) = \frac{1}{H(t,T)} \int_T^t H(t,s)\omega^2(s)ds \quad , \quad t \geq T ,$

and $\quad V(t) = \frac{1}{H(t,T)} \int_T^t h(t,s)\sqrt{H(t,s)}\omega(s)ds .$

Now, suppose that

$$\int_T^\infty \omega^2(s)ds = \infty , \tag{1-3}$$

from condition (1), we have

$$\liminf_{t\to\infty} \int_T^t \frac{H(t,s)}{H(t,T)} \omega^2(s)ds = \infty ,$$

thus, $\quad \lim_{t\to\infty} U(t) = \infty .$ (1-4)

Now consider a sequence $\{T_n\}$; $n = 1,2,3,\ldots$ in $[t_0,\infty)$ with $\lim_{n\to\infty} T_n = \infty$ and such that

$$\lim_{n\to\infty}[U(T_n)+V(T_n)] = \liminf_{t\to\infty}[U(t)+V(t)].$$

By (1-2) there exists a constant A such that

$$U(T_n) + V(T_n) \le A \quad ; \qquad n = 1,2,3,\ldots . \tag{1-5}$$

Furthermore, (1-4) guarantees that

$$\lim_{n\to\infty} U(T_n) = \infty , \tag{1-6}$$

and hence (1-5), gives

$$\lim_{n\to\infty} V(T_n) = -\infty . \tag{1-7}$$

By taking into account (1-6) from (1-5), we derive

$$1+\frac{V(T_n)}{U(T_n)} < \frac{A}{U(T_n)} < \frac{1}{2} \qquad \text{for all n is sufficiently large.}$$

Thus, we get $$\frac{V(T_n)}{U(T_n)} < -\frac{1}{2} \qquad \text{for all large n.} \tag{1-8}$$

from (1-7) and (1-8), we have

$$\lim_{n\to\infty}\frac{V^2(T_n)}{U(T_n)} > \lim_{n\to\infty}\frac{1}{4}\frac{U^2(T_n)}{U(T_n)} = \frac{1}{4}\lim_{n\to\infty}U(T_n) = \infty , \tag{1-9}$$

on the other hand, by Schwarz inequality, we have for any positive integer n

$$V^2(T_n) = \frac{1}{H^2(T_n,T)}\left[\int_T^{T_n} h(T_n,s)\sqrt{H(T_n,s)}\,\omega(s)\,ds\right]^2$$

$$\leq \frac{1}{H(T_n,T)}\int_T^{T_n} h^2(T_n,s)ds + \frac{1}{H(T_n,T)}\int_T^{T_n} H(T_n,s)\omega^2(s)ds.$$

Then, we have

$$V^2(T_n) \leq \frac{1}{H(T_n,T)}\int_T^{T_n} h^2(T_n,s)ds U(T_n),$$

or $$\frac{V^2(T_n)}{U(T_n)} \leq \frac{1}{H(T_n,T)}\int_T^{T_n} h^2(T_n,s)ds,$$

so, (1-9) becomes,

$$\lim_{n\to\infty} \frac{1}{H(T_n,T)}\int_T^{T_n} h^2(T_n,s)ds = \infty.$$

Thus, *for all* $t \geq T$, we obtain

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^{t} h^2(t,s)ds = \infty,$$

this contradicts to condition (2).

Thus, (1-3) fails, and hence

$$\int_T^{\infty} \omega^2(s)ds < \infty \quad , \quad \textit{for all } T \geq t_0 .$$

since $\omega(T) \geq \Omega(T)$, we have

$$\int_{t_0}^{\infty} \Omega_+^2(s)ds \leq \int_{t_0}^{\infty} \omega^2(s)ds < \infty .$$

This contradicts to the condition (3); hence, the proof is completed.

Remark 1.3.6

also, Philos [30] extended and improved Yan's result [41].

Although there is extensive literature on the topic of oscillation criteria of equation (1) a complete satisfactory answer has not yet been obtained because, as far as we know, necessary and sufficient conditions ensuring the oscillatory nature of equation (1), in which only the function q is involved, did not appear in the published studies.

The oscillation of equation (2)

This section is confined to the study of the oscillation of the equation (2). It is interesting to discuss conditions on the alternating coefficient q(t) that are sufficient for all solutions of equation (2) to be oscillatory. An interesting case is that of finding oscillations criteria of equation (2) which involve the average behavior of the integral of q.The problem has received the attention of many authors in recent years.

Among numerous papers dealing with such averaging techniques of the oscillation of equation (2), we mention the following:

Theorem 1.3.8 Moore [22]

Suppose that the function ρ satisfies $\rho \in C^2([t_0,\infty))$, $\rho(t) > 0$

(1) $\int_{t_0}^{\infty} \frac{ds}{r(s)\rho^2(s)} = \infty$,

(2) $\int_{t_0}^{\infty} \rho(s)\left[(r(s)\dot{\rho}(s))^{\cdot} + \rho(s)q(s)\right]ds = \infty$.

then equation (2) is oscillatory.

Remark 1.3.7

Moore [22] gave the previous oscillation criteria for the equation (2).

Theorem 1.3.9 Popa [32]

If r(t) is bounded above and

(1) $\limsup_{t\to\infty}\frac{1}{t^{n-1}}\int_{t_0}^{t}(t-s)^{n-1}q(s)ds=\infty$, n is an integer $n>2$,

then the equation (2) is oscillatory.

Theorem 1.3.10 Popa [32]

(1) Let $\frac{\dot r(t)}{r(t)}$ is bounded and

(2) $\limsup_{t\to\infty}\frac{1}{t^{n-1}}\int_{t_0}^{t}(t-s)^{n-1}\frac{q(s)}{r(s)}ds=\infty$, n is an integer $n>2$,

then equation (2) is oscillatory.

Remark 1.3.8

Emil Popa [32] extended Kamenev's oscillation criterion to apply on equation of the form (2).

The oscillation of equation (3)

This section is confined to the oscillation criteria for the second order nonlinear differential equation of the form (3). The oscillation of equation (3) has brought the attention of many authors since the first paper by Atkinson [1].The prototype of equation (3) is so called Emden – Fowler equation

$$\ddot x(t)+q(t)|x(t)|^{\gamma}\operatorname{sgn}x(t)=0,\quad \gamma>0 \qquad (3\text{-a})$$

Clearly equation (3-a) is sub-linear if $\gamma<1$ and super-linear if $\gamma>1$.

The oscillation problem for second order nonlinear differential equation is of particular interest. Many physical systems are modeled by nonlinear ordinary differential equations. For example, equation (3-a) appears in the study of gas dynamics and fluid mechanics, nuclear physics and chemical reacting systems.

The study of Emden – Fowler equation originates from earlier theorems concerning gas dynamics in as trophies around the turn of the century. For more details for the equation we refer to the paper by Sevelo [34] for a detailed account of second order nonlinear oscillation and its physical motivation.

There has recently been an increase in studying the oscillation for equations (3) and (3-a). We list some of more important oscillation criteria as follows:

Theorem 1.3.11 Atkinson [1]

Suppose that

(1) $q(t) > 0$ on $[t_0, \infty)$ and $g(x) = x^{2n+1}, n = 1,2,3,\ldots$,

(2) $\int_{t_0}^{\infty} sq(s)ds = \infty$,

then equation (3) is oscillatory.

Remark 1.3.9

The previous theorem gives the necessary and sufficient conditions for oscillation of the equation (3) with $g(x) = x^{2n+1}, n = 1,2,3,\ldots$.

Theorem 1.3.12 Waltman [36]

Suppose that

(1) $g(x) = x^{2n+1}, n = 1,2,3,\ldots$

(2) $\int_{t_0}^{\infty} q(s)ds = \infty$,

then every solution of equation (3) is oscillatory.

Remark 1.3.10

Waltman [36] extended Wintner's result [37] (which presented to equation (3)) for the equation which considered by Atkinson [1] without any restriction on the sign of $q(t)$.

Theorem 1.3.13 Kiguradze [18]

The equation (3-a) is oscillatory for $\gamma > 1$ if

(1) $\int^{\infty} \rho(t)q(t)dt = \infty$,

for $\rho(t)$ is a positive continuous and concave function.

Remark 1.3.11

Kiguradze [18] established the previous theorem for the Emden–Fowler equation (3-a).

Theorem 1.3.14 Wong [38]

Let $\gamma > 1$ equation (3-a) is oscillatory if

(1) $\liminf_{t\to\infty} \int_{t_0}^{t} q(s)ds > -\infty$,

(2) $\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} (t-s)q(s)ds = \infty$.

Remark 1.3.12

Wong [38] extended Wintner's oscillation criteria [37] to apply on the equation of the form equation (3-a).

Theorem 1.3.15 Onose [25]

Suppose that

(1) $\limsup_{t\to\infty} \int_{t_0}^{t} q(s)ds = \infty$,

(2) $\limsup_{t\to\infty} \int_{t_0}^{t}\int_{t_0}^{s} q(u)duds = \infty$.

Then equation (3-a) is oscillatory for $0 < \gamma < 1$.

Proof

Assume the contrary, then there exists a solution $x(t)$ which may be assumed to be positive on $[T_1,\infty)$ for some $T_1 \geq t_0 > 0$, we distinguish three cases for the behavior of $\dot{x}(t)$.

Case 1

Suppose that $\dot{x}(t)$ is oscillatory on $[T_1,\infty)$ then there exists a sequence $\{t_n, n = 1,2,3,...\}$ such that $\dot{x}(t_n) = 0$ and $t_n \to \infty$ as $n \to \infty$, dividing equation (3-a) by $x^{\gamma}(t)$ and integrating from t_k *to* t where k is some integer, we obtain

$$\frac{\dot{x}(t)}{x^{\gamma}(t)} + \gamma\int_{t_k}^{t}\frac{\dot{x}^2(s)}{x^{\gamma+1}(s)}ds + \int_{t_k}^{t} q(s)ds = 0$$

$$\frac{\dot{x}(t)}{x^{\gamma}(t)} + \gamma\int_{t_k}^{t}\left(\frac{\dot{x}(s)}{x^{\beta}(s)}\right)^2 ds + \int_{t_k}^{t} q(s)ds = 0, \qquad \text{where } \beta = \frac{\gamma+1}{2}, \tag{1-10}$$

Integrating (1-10) once more from t_k to t as follows

$$\int_{t_k}^{t}\frac{\dot{x}(s)}{x^{\gamma}(s)}ds+\gamma\int_{t_k}^{t}\int_{t_k}^{s}\left(\frac{\dot{x}(u)}{x^{\beta}(u)}\right)^2 duds+\int_{t_k}^{t}\int_{t_k}^{s}q(u)duds=0,$$

$$\therefore\ \frac{x^{-\gamma+1}(t)}{1-\gamma}+\gamma\int_{t_k}^{t}\int_{t_k}^{s}\left(\frac{\dot{x}(u)}{x^{\beta}(u)}\right)^2 duds+\int_{t_k}^{t}\int_{t_k}^{s}q(u)duds=\frac{x^{-\gamma+1}(t_k)}{1-\gamma}$$

$$\therefore\ \frac{x^{-\gamma+1}(t)}{1-\gamma}+\gamma\int_{t_k}^{t}\int_{t_k}^{s}\left(\frac{\dot{x}(u)}{x^{\beta}(u)}\right)^2 duds+\int_{t_k}^{t}\int_{t_k}^{s}q(u)duds=c_1, \qquad (1\text{-}11)$$

where $c_1=\dfrac{x^{-\gamma+1}(t_k)}{1-\gamma}$.

Therefore (1-11) yields,

$$\int_{t_k}^{t}\int_{t_k}^{s}q(u)duds\le c_1.$$

Now, taking the upper limit as $t\to\infty$, we get

$$\limsup_{t\to\infty}\int_{t_k}^{t}\int_{t_k}^{s}q(u)duds<\infty.$$

This contradicts to the condition (2).

Case 2

Suppose that $\dot{x}(t)>0$ for $t\ge T_2\ge T_1$, dividing equation (3-a) by $x^{\gamma}(t)$ and integrating from $T_2 \ge T_1$ to t, we obtain

$$\frac{\dot{x}(t)}{x^{\gamma}(t)}+\gamma\int_{T_2}^{t}\frac{\dot{x}^2(s)}{x^{\gamma+1}(s)}ds+\int_{T_2}^{t}q(s)ds=c_2,$$

where $c_2 = \dfrac{\dot{x}(T_2)}{x^{\gamma}(T_2)}$.

$$\frac{\dot{x}(t)}{x^{\gamma}(t)} + \gamma \int_{T_2}^{t} \left(\frac{\dot{x}(s)}{x^{\beta}(s)} \right)^2 ds + \int_{T_2}^{t} q(s)ds = c_2 , \qquad (1\text{-}12)$$

then, *for all* $t \geq T_2$, we obtain

$$\int_{T_2}^{t} q(s)ds \leq c_2 ,$$

by taking the upper limit as $t \to \infty$, we get

$$\limsup_{t \to \infty} \int_{T_2}^{t} q(s)ds < \infty .$$

This contradicts to the condition (1).

Case 3

Suppose that $\dot{x}(t) < 0$ *for* $t \geq T_2 \geq T_1$ then from equation (1-12)

Since $\qquad \gamma \int_{T_2}^{t} \left(\frac{\dot{x}(s)}{x^{\beta}(s)} \right)^2 ds > 0$

Integrating equation (1-12) *for* $t \geq T_2$, we obtain,

$$\left. \frac{x^{1-\gamma}(s)}{1-\gamma} \right|_{T_2}^{t} + \int_{T_2}^{t} \int_{T_2}^{s} q(u)duds \leq c_2(t - T_2) ,$$

then, *for all* $t \geq T_2$, we have

$$\frac{x^{1-\gamma}(t)}{1-\gamma} - \frac{x^{1-\gamma}(T_2)}{1-\gamma} + \int_{T_2}^{t} \int_{T_2}^{s} q(u)duds \leq c_2(t - T_2) .$$

This implies that

$$\int_{T_2}^{t}\int_{T_2}^{s} q(u)duds \le c_2(t-T_2)+\frac{x^{1-\gamma}(T_2)}{1-\gamma}, \qquad c_2<0,$$

taking the upper limit as $t \to \infty$, we obtain

$$\limsup_{t\to\infty}\int_{T_2}^{t}\int_{T_2}^{s} q(u)duds = -\infty .$$

This contradicts to the condition (2); hence, the proof is completed.

Example 1.3.3

Consider the differential equation

$$\ddot{x}(t)+(1+2\sin t)|x(t)|^{\gamma}\, sign x(t)=0.$$

Theorem 1.3.15 ensures that the given equation is oscillatory.

Theorem 1.3.16 Onose [25]

Suppose that

(1) $\liminf_{t\to\infty}\int_{t_0}^{t} q(s)ds \ge 0$,

(2) $\limsup_{t\to\infty}\int_{t_0}^{t} q(s)ds = \infty$,

then equation (3) is oscillatory.

Theorem 1.3.17 Onose [25]

Suppose that

(1) $g'(x) > k > 0 \quad \textit{for all } x \ge 0$,

(2) $\limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(u)\,du\,ds = \infty$,

(3) $\liminf_{t\to\infty} \int_{t_0}^{t} q(s)\,ds > -\lambda > -\infty \quad \lambda > 0$.

Then the super linear differential equation (3) is oscillatory.

__Proof__

Suppose that a solution $x(t)$ of equation (3) is a non oscillatory may $x(t) > 0$ on $[t_0, \infty)$ for some $t_0 > 0$, dividing equation (3) by $g(x(t))$ and integrating we obtain

$$\frac{\dot{x}(t)}{g(x(t))} + \int_{t_0}^{t} g'(x(s))\left(\frac{\dot{x}(s)}{g(x(s))}\right)^2 ds + \int_{t_0}^{t} q(s)\,ds = c_3\,, \qquad (1\text{-}13)$$

where $c_3 = \dfrac{\dot{x}(t_0)}{g(x(t_0))}$.

Integrating (1-13) from t_0 *to* t, we have

$$\int_{t_0}^{t} \frac{\dot{x}(s)}{g(x(s))}\,ds + \int_{t_0}^{t}\int_{t_0}^{s} g'(x(s))\left(\frac{\dot{x}(u)}{g(x(u))}\right)^2 du\,ds + \int_{t_0}^{t}\int_{t_0}^{s} q(u)\,du\,ds = c_3 t + c_4\,, \qquad (1\text{-}14)$$

where c_3 *and* c_4 are constants.

Now consider three cases of the behavior of $\dot{x}(t)$.

Case 1

If $\dot{x}(t)$ is oscillatory on $[t_1, \infty)$ then there exists a sequence $\{t_n, n = 1,2,3,...\}$ such that $\dot{x}(t_n) = 0$ as $t_n \to \infty$ from this fact (1-13) and condition (3) we have

$$\int_{t_0}^{t} g'(x(s))\left(\frac{\dot{x}(s)}{g(x(s))}\right)^2 ds \quad \text{is finite,} \tag{1-15}$$

from (1-15) and condition (1) we see that

$$\int_{t_0}^{t}\left(\frac{\dot{x}(s)}{g(x(s))}\right)^2 ds \le N^2 ,$$

where N is a positive constant.

By Schwarz's inequality we have

$$\left|\int_{x(t_0)}^{x(t)} \frac{du}{g(u)}\right|^2 = \left|\int_{t_0}^{t} \frac{\dot{x}(s)}{g(x(s))} ds\right|^2 \le t^2 \int_{t_0}^{t}\left[\frac{\dot{x}(s)}{g(x(s))}\right]^2 ds \le t^2 N^2 \quad , t \ge t_0, \tag{1-16}$$

from (1-14) and (1-16) we have

$$\limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(u)duds < \infty .$$

This contradicts to the condition (2).

Case 2

Suppose that $\dot{x}(t) > 0$ then from (1-14) and condition (3), we have

$$\limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(u)duds < \infty .$$

This contradicts to the condition (2).

Case 3

Suppose that $\dot{x}(t) < 0$ then by (1-13) and condition (3), we have

$$\frac{-\dot{x}(t)}{g(x(t))} \geq -(c_3+\lambda)+\int_{t_0}^{t} g'(x(s))\left(\frac{\dot{x}(s)}{g(x(s))}\right)^2 ds. \tag{1-17}$$

If $\int_{t_0}^{t} g'(x(s))\left(\frac{\dot{x}(s)}{g(x(s))}\right)^2 ds$ is finite as $t \to \infty$

we can deduce a contradiction as a similar way in the case when $\dot{x}(t)$ oscillatory.

Otherwise by multiplying (1-17) by

$$\left(\frac{-g'(x(t))\dot{x}(t)}{g(x(t))}\right) \Big/ \left\{-(c_3+\lambda)+\int_{t_0}^{t} g'(x(s))\left(\frac{\dot{x}(s)}{g(x(s))}\right)^2 ds\right\},$$

and integrating from T to t, we obtain,

$$\ln\left[-(c_3+\lambda)+\int_{t_0}^{t} g'(x(s))\frac{\dot{x}^2(s)}{g^2(x(s))}ds\right] \geq \int_{T}^{t}\frac{(-\dot{x}(s))g'(x(s))}{g(x(s))}ds$$

$$\geq -\int_{T}^{t}\frac{g'(u)}{g(u)}du \geq \ln\frac{g(x(T))}{g(x(t))}$$

$$\left[-(c_3+\lambda)+\int_{t_0}^{t} g'(x(s))\frac{\dot{x}^2(s)}{g^2(x(s))}ds\right] \geq \frac{g(x(T))}{g(x(t))}.$$

This together with (1-17) yields

$$\frac{-\dot{x}(t)}{g(x(t))} \geq \frac{g(x(T))}{g(x(t))}. \tag{1-18}$$

It follows from (1-18) that

$$\dot{x}(t) \le -g(x(T)) = -A, \qquad A > 0,$$

Therefore, $$x(t) \le -A(t-T) + x(T)$$

$$x(t) \to -\infty \qquad as\ t \to \infty,$$

which contradicts the assumption that $x(t) > 0$.

Remark 1.3.13

Onose [25] proved the theorem of Wong (theorem 1.3.14) for the sub – linear Emden - Fowler differential equation and also studied the extension of Wong's result [38] to the more general super – linear differential equation of the form of equation (3) as in the previous three theorems.

Theorem 1.3.18 Yeh [42]

Suppose that

(1) $\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s) ds = \infty$ for some integer $n > 2$,

then equation (3) is oscillatory.

Remark 1.3.14

C.C.Yeh [42] established new integral criteria for the equation (3) which has Wintner's result [37] as a particular case.

Theorem 1.3.19 Philos [27]

let ρ be a positive continuous differentiable function on the interval $[t_0,\infty)$ such that

(1) $\gamma\rho(t)\ddot{\rho}(t) + (t-\gamma)^{+t}\rho(t) \le 0 \qquad for\ all\ t \ge t_0$.

Then equation (3-a) is oscillatory if

(2) $\limsup_{t\to\infty}\frac{1}{t^{n-1}}\int_{t_0}^{t}(t-s)^{n-1}\rho(s)q(s)ds=\infty$ for some integer $n\geq 2$ and $0<\gamma<1$.

Remark 1.3.15

Philos [27] gave new oscillatory criteria for the differential equation (3-a) with $0<\gamma<1$.

Theorem 1.3.20 Philos [28]

Suppose that ρ be a positive twice continuously differentiable function on $[t_0,\infty)$ such that

$$\dot\rho(t)\geq 0 \text{ and } \ddot\rho(t)\leq 0 \text{ on } [t_0,\infty)$$

(1) $\lim_{t\to\infty}\int_{t_0}^{t}\rho(s)q(s)ds>-\infty$

(2) $\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}(t-s)\rho(s)q(s)ds=\infty$.

Then equation (3) is oscillatory.

Remark 1.3.16

Philos [28] improved Onose's result [25] for equation (3).

Theorem 1.3.21 Wong and Yeh [39]

Suppose that

(1) $\liminf_{t\to\infty}\int_{t_0}^{t}q(s)ds\geq 0$ for large T.

and there exists a positive concave function ρ on $[t_0,\infty)$ such that

(2) $\limsup_{t\to\infty} \frac{1}{t^{\beta}} \int_{t_0}^{t} (t-s)^{\beta} \rho(s) q(s) ds = \infty$ for some $\beta \ge 0$,

then the super linear differential equation (3) is oscillatory.

Proof

Without loss of generality, we assume that there exists a solution $x(t)$ such that $x(t) > 0$ on $[T_0, \infty)$ *for* $T_0 \ge 0$,it follows from Wong's lemma $\dot{x}(t) > 0$ on $[T_1, \infty)$ *for* $T_1 \ge T_0$.

Define

$$\omega(t) = \frac{\rho(t)\dot{x}(t)}{g(x(t))} \quad , \textit{for all } t \ge T_1$$

Then, *for* $t \ge T_1$ we get

$$\dot{\omega}(t) = \frac{\dot{\rho}(t)\dot{x}(t)}{g(x(t))} + \frac{\rho(t)\ddot{x}(t)}{g(x(t))} - \frac{\rho(t)\dot{x}^2(t)g'(x(t))}{g^2(x(t))}$$

$$\dot{\omega}(t) = \frac{\dot{\rho}(t)\dot{x}(t)}{g(x(t))} - \rho(t)q(t) - \frac{\omega^2(t)}{\rho(t)} g'(x(t)). \quad \text{where } q(t) = \frac{\ddot{x}(t)}{g(x(t))}.$$

Hence, *for all* $t \ge T_1$, we have

$$\int_{T_1}^{t} (t-s)^{\beta} \rho(s) q(s) ds \le -\int_{T_1}^{t} (t-s)^{\beta} \dot{\omega}(s) ds + \int_{T_1}^{t} (t-s)^{\beta} \dot{\rho}(s) \frac{\dot{x}(s)}{g(x(s))} ds . \qquad (1\text{-}19)$$

By the Bonnet's theorem, we see for each $t \ge T_1$, there exists ε_1 *and* $\eta_1 \in [T_1, t]$ such that

$$-\int_{T_1}^{t}(t-s)^{\beta}\dot{\omega}(s)ds = -(t-T_1)^{\beta}\int_{T_1}^{\varepsilon_1}\dot{\omega}(s)ds$$

$$= -(t-T_1)^{\beta}\left[\omega(\varepsilon_1)-\omega(T_1)\right] \le (t-T_1)^{\beta}\omega(T_1), \tag{1-20}$$

and

$$\int_{T_1}^{t}(t-s)^{\beta}\dot{\rho}(s)\frac{\dot{x}(s)}{g(x(s))}ds = (t-T_1)^{\beta}\dot{\rho}(T_1)\int_{T_1}^{\eta_1}\frac{\dot{x}(s)}{g(x(s))}ds$$

$$= (t-T_1)^{\beta}\dot{\rho}(T_1)\int_{x(T_1)}^{x(\eta_1)}\frac{du}{g(u)} \le (t-T_1)^{\beta}\dot{\rho}(T_1)\int_{x(T_1)}^{\infty}\frac{du}{g(u)} \tag{1-21}$$

It follows from (1-19), (1-20) and (1-21) that

$$\int_{T_1}^{t}(t-s)^{\beta}\rho(s)q(s)ds \le (t-T_1)^{\beta}\omega(T_1)+(t-T_1)^{\beta}\dot{\rho}(T_1)\int_{x(T_1)}^{\infty}\frac{du}{g(u)}, \qquad \forall t \ge T_1.$$

Dividing this inequality by t^{β} and taking limit supremum on both sides, we obtain

$$\limsup_{t\to\infty}\frac{1}{t^{\beta}}\int_{t_0}^{t}(t-s)^{\beta}\rho(s)q(s)ds < \infty,$$

This contradicts to the condition (2); hence, the proof is completed.

Remark 1.3.17

F.H.Wong and C.C.Yeh [39] proved an analogous result of Wong's result for equation (3-a) to the more general equation (3).

Example 1.3.4

Consider the differential equation

$$\ddot{x}(t) + \frac{1}{t^2} x^3(t) = 0 \quad , t > 0$$

Theorem 1.3.21 ensures that the given equation is oscillatory where $\rho(t) = t$, $\beta = 1$.

Theorem 1.3.22 Philos and Purnaras [31]

Suppose that

(1) $\liminf_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s) ds > -\infty$ for some integer $n \geq 2$.

(2) $\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \left[\int_{t_0}^{s} q(u) du \right]^2 ds = \infty$,

then the sub – linear differential equation (3) is oscillatory.

The oscillation of equation (4)

This section is confined to the oscillation criteria for the second order nonlinear differential equation of the form (4).

Theorem 1.3.23 Bhatia [3]

Suppose that

(1) $\int_{t_0}^{\infty} \frac{ds}{r(s)} = \infty$,

(2) $\int_{t_0}^{\infty} q(s) ds = \infty$.

Then equation (4) is oscillatory.

Remark 1.3.18

Bhatia [3] presented the previous oscillation criteria for the general equation (4) which contains as a special case Waltman's result [36] for the nonlinear case.

Theorem 1.3.24 EL – Abbasy [7]

Suppose that

(1) $\liminf_{t\to\infty} \int_{t_0}^{t} \rho(s)q(s)ds > -\infty$,

(2) $\limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\left[\int_{t_0}^{s} \rho(u)q(u)du\right]^2 ds = \infty$,

where $\rho:[t_0,\infty)\to(0,\infty)$ is continuously differentiable function such that

$\dot{\rho}(t)\ge 0$, $(r(t)\rho(t))^{\cdot}\ge 0$, $(r(t)\rho(t))^{\cdot}\le 0$ and $(r(t)\dot{\rho}(t))^{\cdot}\le 0$,

then equation (4) is oscillatory.

Remark 1.3.19

EL – Abbasy [7] improved and extended the result of Philos [27] to equation (4).

The oscillation of equation (5)

This section is confined to the study of the oscillation of the equation (5).

Theorem 1.3.25 Grace [12]

Suppose that

(1) $\frac{g'(x)}{\psi(x)} \ge k > 0 \quad for\ x \ne 0$,

(2) $\int^{\infty}\frac{du}{g(u)} < \infty$ and $\int^{-\infty}\frac{du}{g(u)} < \infty$.

Moreover, assume that there exists a differentiable function

$$\rho:[t_0,\infty)\rightarrow(0,\infty),$$

and the continuous functions

$$h,H:D=\{(t,s):t\geq s\geq t_0\}\rightarrow R,$$

and H has a continuous and non positive partial derivative on D with respect to the second variable such that

$$H(t,t)=0 \quad for\ t\ \geq\ t_0\ , \quad H(t,s)>0 \quad for\ t\ >\ s\ \geq\ t_0\ ,$$

and

$$\frac{-\partial H(t,s)}{\partial s}=h(t,s)\sqrt{H(t,s)} \quad for\ all\ (t,s)\in\ D.$$

(3) $$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\rho(s)q(s)-\frac{r(s)\rho(s)}{4k}\left(h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right)^2\right]ds=\infty,$$

then equation (5) is oscillatory.

<u>**Theorem 1.3.26 Grace [12]**</u>

Let condition (1) from theorem 1.3.25 hold and let the functions H,h and ρ be define as in theorem 1.3.25 and moreover, suppose that

(4) $$0<\inf_{s\geq t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right]\leq\infty,$$

(5) $$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}r(s)\rho(s)\left[h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right]ds<\infty,$$

if there exists a continuous function Ω on $[t_0,\infty)$ such that

$$(6)\ \limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\rho(s)q(s)-\frac{r(s)\rho(s)}{4k}\left[h(t,s)-\frac{\dot\rho(s)}{\rho(s)}\sqrt{H(t,s)}\right]^2\right]ds\ge\Omega(T)$$

for every $T\ge t_0$,

$$(7)\ \lim_{t\to\infty}\int_{t_0}^t\frac{\Omega_+^2(s)}{r(s)\rho(s)}ds=\infty,\quad \text{where } \Omega_+(t)=\max\{\Omega(t),0\},$$

then every solution of equation (5) is oscillatory.

<u>**Theorem 1.3.27 Grace [12]**</u>

Suppose that the condition (1) from theorem 1.3.25 hold, and

$$(8)\ \int^{\infty}\frac{\psi(u)}{g(u)}du<\infty \quad\text{and}\quad \int^{-\infty}\frac{\psi(u)}{g(u)}du<\infty,$$

and the functions H, h and ρ are defined as in theorem 1.3.25 and

$$\dot\rho(t)\ge 0 \text{ and } (r(t)\dot\rho(t))^{\cdot}\le 0 \quad \text{for } t\ge t_0$$

moreover, suppose that

$$(9)\ \liminf_{t\to\infty}\int_{t_0}^t\rho(s)q(s)ds>-\infty,$$

$$(10)\ \int_{t_0}^{\infty}\frac{1}{r(s)\rho(s)}ds=\infty,$$

then equation (5) is oscillatory if there exists a continuous function Ω on $[t_0,\infty)$ such that condition (6) from theorem 1.3.26 hold, and

(11) $\int_{t_0}^{\infty} \frac{\Omega_+^2(s)}{r(s)\rho(s)} ds = \infty$, where $\Omega_+(t) = \max\{\Omega(t), 0\}$

Proof

Let $x(t)$ be a non oscillatory solution of equation (5), say $x(t) > 0$ *for* $t \geq t_0$.

Define

$$\omega(t) = \frac{\rho(t) r(t) \psi(x(t)) \dot{x}(t)}{g(x(t))}, \qquad t \geq t_0 .$$

thus, for every $t \geq t_0$, we obtain

$$\dot{\omega}(t) = -\rho(t) q(t) + \frac{\dot{\rho}(t) r(t) \psi(x(t)) \dot{x}(t)}{g(x(t))} - \frac{1}{\rho(t) r(t)} \omega^2(t) \frac{g'(x(t))}{\psi(x(t))}, \qquad \text{where } q(t) = \frac{[r(t)\psi(x(t))\dot{x}(t)]^{\bullet}}{g(x(t))},$$

$$\dot{\omega}(t) \leq -\rho(t) q(t) + \frac{\dot{\rho}(t)}{\rho(t)} \omega(t) - \frac{k}{\rho(t) r(t)} \omega^2(t), \qquad \text{where } k = \frac{g'(x)}{\psi(x)}. \tag{1-22}$$

Integrating (1-22) from t *to* t_0, we have

$$\omega(t) \leq \omega(t_0) - \int_{t_0}^{t} \rho(u) q(u) du + \int_{t_0}^{t} \frac{\dot{\rho}(u) r(u) \psi(x(u)) \dot{x}(u)}{g(x(u))} du - \int_{t_0}^{t} \frac{k}{\rho(u) r(u)} \omega^2(u) du. \tag{1-23}$$

Now, using the Bonnet theorem for a fixed $s \geq t_0$ and for some $\xi_1 \in [t_0, s]$ such that

$$\int_{t_0}^{t} \frac{\dot{\rho}(u) r(u) \psi(x(u)) \dot{x}(u)}{g(x(u))} du = \dot{\rho}(t_0) r(t_0) \int_{t_0}^{\xi_1} \frac{\psi(x(u)) \dot{x}(u)}{g(x(u))} du$$

$$= \dot{\rho}(t_0) r(t_0) \int_{x(t_0)}^{x(\xi_1)} \frac{\psi(y)}{g(y)} dy.$$

and, since $\dot{\rho}(t_0) \geq 0$, and

$$\int_{x(t_0)}^{x(\xi_1)} \frac{\psi(y)}{g(y)} dy < \left\{ \begin{array}{ll} 0 & if\, x(\xi_1) < x(t_0) \\ \int_{x(t_0)}^{\infty} \frac{\psi(y)}{g(y)} dy & if\, x(\xi_1) > x(t_0) \end{array} \right\},$$

then, we have

$$\int_{t_0}^{t} \frac{\dot{\rho}(u) r(u) \psi(x(u)) \dot{x}(u)}{g(x(u))} du \leq k_1 ,$$

with $k_1 = \dot{\rho}(t_0) r(t_0) \int_{x(t_0)}^{\infty} \frac{\psi(y)}{g(y)} dy$,

then, we have

$$-\omega(t) \geq -\omega(t_0) + k_1 + \int_{t_0}^{t} \rho(u) q(u) du + \int_{t_0}^{t} \frac{k}{\rho(u) r(u)} \omega^2(u) du .$$

Suppose that

$$\int_{t_0}^{\infty} \frac{k}{\rho(u) r(u)} \omega^2(u) du = \infty . \qquad (1\text{-}24)$$

By condition (1) we see that

$$\int_{t_0}^{\infty} \frac{1}{\rho(u) r(u)} \frac{g'(x(u)}{\psi(x(u))} \omega^2(u) du = \infty .$$

By condition (9), and (1-23), it follows that for some constant L

$$-\omega(t) \geq L + \int_{t_0}^{t} \frac{1}{\rho(u) r(u)} \frac{g'(x(u))}{\psi(x(u))} \omega^2(u) du , \qquad (1\text{-}25)$$

We choose $t_1 \geq t_0$, such that

$$A = L + \int_{t_0}^{t_1} \frac{1}{\rho(u)r(u)} \frac{g'(x(u))}{\psi(x(u))} \omega^2(u)du > 0,$$

then (1-25) ensures that the function ω is negative on $[t_1, \infty)$.

Now (1-25) gives

$$\frac{1}{\left[L + \int_{t_0}^{t} \frac{1}{\rho(u)r(u)} \frac{g'(x(u))}{\psi(x(u))} \omega^2(u)du\right]} \geq \frac{-1}{\omega(t)}$$

$$\therefore \left(\frac{1}{\rho(t)r(t)} \frac{g'(x(t))}{\psi(x(t))} \omega^2(t)\right)\left[L + \int_{t_0}^{t} \frac{1}{\rho(u)r(u)} \frac{g'(x(u))}{\psi(x(u))} \omega^2(u)du\right]^{-1} \geq \frac{-1}{\rho(t)r(t)} \frac{g'(x(t))}{\psi(x(t))} \omega(t)$$

$$\geq \frac{-g'(x(t))\dot{x}(t)}{g(x(t))} \quad , for\ t \geq t_1,$$

and consequently, *for all* $t \geq t_1$

$$\ln \frac{1}{A}\left[L + \int_{t_0}^{t} \frac{1}{\rho(u)r(u)} \frac{g'(x(u))}{\psi(x(u))} \omega^2(u)du\right] \geq \ln \frac{g(x(t_1))}{g(x(t))}.$$

Hence,

$$\left[L + \int_{t_0}^{t} \frac{1}{\rho(u)r(u)} \frac{g'(x(u))}{\psi(x(u))} \omega^2(u)du\right] \geq A \frac{g(x(t_1))}{g(x(t))}.$$

So, (1-25) yields

$$\omega(t) \leq -\left[L + \int_{t_0}^{t} \frac{1}{\rho(u)r(u)} \frac{g'(x(u))}{\psi(x(u))} \omega^2(u)du\right],$$

then, we get

$$\frac{\rho(t)r(t)\psi(x(t))\dot{x}(t)}{g(x(t)))} \leq -A\frac{g(x(t_1))}{g(x(t))}$$

and so,

$$\psi(x(t))\dot{x}(t) \leq \frac{-C}{\rho(t)r(t)} \quad , for\ t \geq t_1 ,$$

where $C = Ag(x(t_1))$.

Thus, we have

$$\int_{x(t_1)}^{x(t)} \psi(s)ds \leq -C\int_{t_1}^{t} \frac{du}{\rho(u)r(u)} \to -\infty \quad \text{as } t \to \infty .$$

This a contradiction to the fact that $x(t) > 0\ \ for\ t \geq t_0$, and hence (1-24) fails.

Now we suppose that

$$\int_{t_0}^{\infty} \frac{1}{\rho(u)r(u)}\omega^2(u)du < \infty .$$

By inequality (1-22), we have

$$\int_{t_0}^{t} H(t,u)\rho(u)q(u)du \leq H(t,t_0)\omega(t_0) - \int_{t_0}^{t} \frac{kH(t,u)}{\rho(u)r(u)}\omega^2(u)du$$

$$- \int_{t_0}^{t} \sqrt{H(t,u)}\left[h(t,u) - \frac{\dot{\rho}(u)}{\rho(u)}\sqrt{H(t,u)}\right]\omega(u)du$$

$$\leq H(t,t_0)\omega(t_0) + \int_{t_0}^{t} \frac{\rho(u)r(u)}{4k}\left[h(t,u) - \frac{\dot{\rho}(u)}{\rho(u)}\sqrt{H(t,u)}\right]^2 du$$

$$-\int_{t_0}^{t}\left\{\sqrt{\frac{kH(t,u)}{\rho(u)r(u)}}\omega(u)+\frac{1}{2}\sqrt{\frac{\rho(u)r(u)}{k}}\left(h(t,u)-\frac{\dot{\rho}(u)}{\rho(u)}\sqrt{H(t,u)}\right)\right\}^2 du.$$

then, we get

$$\int_{t_0}^{t}\left[H(t,u)\rho(u)q(u)-\frac{\rho(u)r(u)}{4k}\left(h(t,u)-\frac{\dot{\rho}(u)}{\rho(u)}\sqrt{H(t,u)}\right)^2\right]du \le H(t,t_0)\omega(t_0)$$

Dividing by $H(t,t_0)$ and taking the limit supremum, we obtain

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,u)\rho(u)q(u)-\frac{\rho(u)r(u)}{4k}\left(h(t,u)-\frac{\dot{\rho}(u)}{\rho(u)}\sqrt{H(t,u)}\right)^2\right]du \le \omega(t_0).$$

By condition (6), we have

$$\omega(t_0)\ge\Omega(t_0)+\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left\{\sqrt{\frac{kH(t,u)}{\rho(u)r(u)}}\omega(u)+\frac{1}{2}\sqrt{\frac{\rho(u)r(u)}{k}}\left(h(t,u)-\frac{\dot{\rho}(u)}{\rho(u)}\sqrt{H(t,u)}\right)\right\}^2 du.$$

This shows that

$$\omega(t_0)\ge\Omega(t_0),$$

then, *for* $t \ge t_0$, we have

$$\Omega^2(t_0)\le\omega^2(t_0).$$

then, we get

$$\int_{t_0}^{\infty}\frac{\Omega_+^2(s)}{\rho(s)r(s)}ds\le\int_{t_0}^{\infty}\frac{1}{\rho(s)r(s)}\omega^2(s)ds<\infty,$$

This is a contradiction; hence, completes the proof.

The oscillation of equation (6)

In this section we constrict our considerations to study the oscillation equation (6). Many authors are concerned with the oscillation criteria of solutions of the homogeneous second order nonlinear differential equations. However, the non homogeneous equations, receive little attention.

Theorem 1.3.28 Greaf, Rankin and Spikes [15]

Suppose that

(1) $\lim\limits_{t \to \infty} \int\limits_{t_0}^{t} \frac{ds}{r(s)} = \infty$,

(2) $\lim\limits_{t \to +\infty} \int\limits_{t_0}^{t} \left(q(s) - p(s)\right) ds = \infty$.

Then equation (6) is oscillatory.

Remark 1.3.20

Greaf, Rankin and Spikes [15] give the previous theorem for the non homogeneous equation (6).

Example 1.3.5

Consider the differential equation

$$\left(t\dot{x}(t)\right)^{\cdot} + tx^3(t) = \frac{x^3(t)}{t^2(1 + x^2(t))}, \qquad t > 0.$$

Theorem 1.3.28 ensures that the given equation is oscillatory.

Theorem 1.3.29 Greaf, Rankin and Spikes [15]

Suppose that

(1) $\int_{t_0}^{\infty} \frac{ds}{r(s)} = \infty$,

(2) $\lim_{t\to\infty} \int_{t_0}^{t} (q(s) - p(s))ds < \infty$,

(3) $\liminf_{t\to\infty} \int_{t_0}^{t} (q(s) - p(s))ds \geq 0$, for all large t

(4) $\lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{r(s)} \int_{t_0}^{s} (q(u) - p(u))duds = \infty$,

then the super linear differential equation (6) is oscillatory.

Theorem 1.3.30 Greaf, Rankin and Spikes [15]

Suppose that

(1) $\frac{H(t, x(t))}{g(x(t))} \leq p(t), \quad for\ x \neq 0$,

(2) $\int_{t_0}^{\infty} \frac{M}{r(s)} ds - \int_{t_0}^{\infty} \frac{1}{r(s)} \int_{t_0}^{s} (q(u) - p(u))duds = -\infty$ for every constant M,

then the sub linear differential equation (6) is oscillatory.

Proof

Suppose that $x(t)$ is a solution of equation (6) with $x(t) > 0$, for $t \geq T \geq t_0$, dividing equation (6) by $g(x(t))$ we obtain

$$\frac{[r(t)\dot{x}(t)]^{\cdot}}{g(x(t))} = -q(t) + \frac{H(t, x(t))}{g(x(t))}, \tag{1-26}$$

then, for $t \geq T$, we have

$$\left[\frac{r(t)\dot{x}(t)}{g(x(t))}\right]^{\cdot} = \frac{(r(t)\dot{x}(t))^{\cdot}}{g(x(t))} - \frac{r(t)\dot{x}^2(t)g'(x(t))}{g^2(x(t))}. \tag{1-27}$$

It follows from (1-26) and (1-27), we get

$$\left[\frac{r(t)\dot{x}(t)}{g(x(t))}\right]^{\cdot} = -q(t) + \frac{H(t,x(t))}{g(x(t))} - \frac{r(t)\dot{x}^2(t)g'(x(t))}{g^2(x(t))},$$

therefore, for $t \geq T$, we have

$$\left[\frac{r(t)\dot{x}(t)}{g(x(t))}\right]^{\cdot} \leq p(t) - q(t). \quad \text{where} \quad p(t) = \frac{H(t,x(t))}{g(x(t))}.$$

Then, for $t \geq T$, integrating the above inequality, we obtain

$$\frac{r(t)\dot{x}(t)}{g(x(t))} \leq \frac{r(T)\dot{x}(T)}{g(x(T))} - \int_T^t [q(s) - p(s)]ds.$$

Now, multiplying the last inequality by $\frac{1}{r(t)}$, we have

$$\frac{\dot{x}(t)}{g(x(t))} \leq \frac{M}{r(t)} - \frac{1}{r(t)}\int_T^t [q(s) - p(s)]ds,$$

where $M = \dfrac{r(T)\dot{x}(T)}{g(x(T))}$ is a constant.

and integrating again yields

$$\int_T^t \frac{\dot{x}(s)}{g(x(s))}ds \leq \int_T^t \frac{M}{r(s)}ds - \int_T^t \frac{1}{r(s)}\int_T^s [q(u) - p(u)]duds.$$

From condition (2), we have

$$I(t)=\int_T^t \frac{\dot{x}(s)}{g(x(s))}ds \to -\infty \qquad \text{as} \quad t\to\infty .$$

But $$I(t)=\int_{x(T)}^{x(t)} \frac{du}{g(u)},$$

and if $x(t)>x(T)$ for large t, then I(t) >0.

This is a contradiction.

Hence, for large t, $x(t)<x(T)$, so

$$I(t)=\int_{x(T)}^{x(t)} \frac{du}{g(u)}=-\int_{x(t)}^{x(T)} \frac{du}{g(u)}=\left[\int_0^{x(t)} \frac{du}{g(u)}-\int_0^{x(T)} \frac{du}{g(u)}\right]\geq -\int_0^{x(T)} \frac{du}{g(u)}>-\infty.$$

This is again a contradiction; hence, the proof is completed.

Example 1.3.6

Consider the differential equation

$$\left(t^2\dot{x}(t)\right)^{\cdot}+x^{\frac{1}{3}}\left(\frac{1}{2}+t^2\right)=\frac{x^{\frac{5}{3}}}{4\left[|x|+1\right]}$$

Theorem 1.3.30 ensures that the given equation is oscillatory.

Theorem 1.3.31 Greaf, Rankin and Spikes [15]

Suppose that

(1) $r(t)\leq r_1$,

(2) $\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}\left(q(u)-p(u)\right)du\,ds=\infty$,

then all solutions of equation (6) are oscillatory.

Theorem 1.3.32 Greaf, Rankin and Spikes [15]

Suppose that condition (1) from theorem 1.3.31 holds. and

(3) $\liminf_{t\to\infty} \int_T^t (q(u) - p(u))du > -\lambda$ for all large T.

(4) $\limsup_{t\to\infty} \frac{1}{t} \int_T^t \int_T^s [q(u) - p(u)] du ds = \infty$.

then equation (6) is oscillatory.

Theorem 1.3.33 Greaf, Rankin and Spikes [15]

Suppose that

(1) $r(t) = 1$,

(2) $\lim_{t\to\infty} \int_{t_0}^t s(q(s) - p(s))ds = \infty$,

then the super linear differential equation (6) is oscillatory.

Theorem 1.3.34 EL – Abbasy [6]

Suppose that

(1) $r(t) = 1$,

(2) $G(x) = \int_0^x g(u)du \to \infty$ as $|x| \to \infty$,

(3) $p(t)$ is continuous real-valued function in every finite interval,

(4) $q(t) > 0$ for $t \geq t_0 > 0$,

Let $\rho(t) > 0$ such that

(5) $\lim_{t\to\infty} \int_{t_0}^{t} (\rho(s)|p(s)| - \frac{\dot{q}(s)}{q(s)})\, ds < \infty$.

(6) $\lim_{t\to\infty} \int_{t_0}^{t} \frac{|p(s)|}{\rho(s)q(s)}\, ds < \infty$,

(7) $\lim_{t\to\infty} \int_{t_0}^{t} \frac{|p(s)|}{\rho(s)}\, ds < \infty$,

(8) $\left(\frac{1}{\rho(t)}\right)^{\cdot}$ is positive and decreasing for $t \geq t_0 > 0$,

(9) $\lim_{|x|\to\infty} \left|\frac{g(x)}{x}\right| = \infty$,

(10) $\liminf_{t\to\infty} \int_{t_0}^{t} \frac{q(s)}{\rho(s)}\, ds > -\infty$.

(11) $\limsup_{t\to\infty} \int_{t_0}^{t} (t-s)\frac{q(s)}{\rho(s)}\, ds = \infty$.

then all bounded solutions of equation (6) are oscillatory.

Remark 1.3.21

EL-Abbasy [6] gives the previous theorem for non homogeneous equation.

CHAPTER 2

THE OSCILLATION OF THE EQUATION

$$\left(r(t)\psi(x(t))\dot{x}(t)\right)^{\cdot} + g_1(t,x(t)) = 0\,.$$

2.1 Introduction

In this chapter we shall study the oscillatory behavior of the solution of the differential equation of the form

$$\left(r(t)\psi(x(t))\dot{x}(t)\right)^{\cdot} + g_1(t,x(t)) = 0\,. \tag{E}$$

where r is a positive continuous function on the interval $[t_0,\infty)$ $t_0 \geq 0$, ψ is a positive continuous function on the real line R and g_1 is a continuous function on $R \times R$,with $\dfrac{g_1(t,x(t))}{g(x(t))} \geq q(t)$, *for all* $x \neq 0$ and $t \in [t_0,\infty)$, where g is continuously differentiable function on the real line R except possible at 0 with $xg(x) > 0$ and $g'(x) \geq l > 0$ *for all* $x \neq 0$ and q is a continuous function on the interval $[t_0,\infty)$, $t_0 \geq 0$.

Throughout this study we restrict our attention only to the solution of the differential equation (E) which exists on some interval $[t_0,\infty)$ $t_0 \geq 0$ may depend on a particular solution.

2.2 Oscillatory of the solutions

In the present section we shall state and prove some sufficient oscillation criteria of the solutions of the equation (E).

Theorem 2.1

Suppose that

(1) $\frac{1}{\psi(x)} \geq l_1 > 0$, *for all* $x \in R$,

(2) $\lim_{t\to\infty} \frac{1}{R(t)} = k_1 \in [0,\infty)$, where $R(t) = \int_{t_0}^{t} \frac{ds}{r(s)}$ *for* $t > t_0$.

Furthermore, let for some integer $n \geq 3$

(3) $\limsup_{t\to\infty} \frac{1}{R^{n-1}(t)} \int_{t_0}^{t} [R(t) - R(s)]^{n-1} q(s) ds = \infty$,

Then equation (E) is oscillatory.

Proof

Let $x(t)$ be a non oscillatory solution of the differential equation (E) and that $x(t) \neq 0$, *for* $t \geq T_2 \geq t_0$,

define

$$\omega(t) = \frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))}, \qquad t \geq T_2,$$

then, for every $t \geq T_2$, we have

$$\dot{\omega}(t) = \frac{-g_1(t, x(t))}{g(x(t))} - \frac{\omega^2(t) g'(x(t))}{r(t)\psi(x(t))}$$

From the condition (1), for all $t \geq T_2$, we obtain

$$\dot{\omega}(t) \leq -q(t) - H_1 \frac{\omega^2(t)}{r(t)}, \qquad t \geq T_2$$

$$\dot{\omega}(t) \leq -q(t) - k\frac{\omega^2(t)}{r(t)}, \qquad t \geq T_2,$$

where $k = ll_1$ is a positive constant.

Hence, for every $t \geq T_2$, we have

$$\int_{T_2}^{t}[R(t)-R(s)]^{n-1}q(s)ds \leq -\int_{T_2}^{t}[R(t)-R(s)]^{n-1}\dot{\omega}(s)ds - k\int_{T_2}^{t}[R(t)-R(s)]^{n-1}\frac{\omega^2(s)}{r(s)}ds$$

$$\leq [R(t)-R(T_2)]^{n-1}\omega(T_2) - \int_{T}^{t}k[R(t)-R(s)]^{n-1}\frac{\omega^2(s)}{r(s)}ds$$

$$-\int_{T_2}^{t}(n-1)[R(t)-R(s)]^{n-2}\frac{\omega(s)}{r(s)}ds$$

$$\leq [R(t)-R(T_2)]^{n-1}\omega(T_2) + \frac{(n-1)^2}{4k}\int_{T_2}^{t}\frac{[R(t)-R(s)]^{n-3}}{r(s)}ds$$

$$-\int_{T_2}^{t}\left\{\sqrt{\frac{k[R(t)-R(s)]^{n-1}}{r(s)}}\omega(s) + \frac{(n-1)}{2\sqrt{k}}\sqrt{\frac{[R(t)-R(s)]^{n-3}}{r(s)}}\right\}^2 ds.$$

then, for all $t \geq T_2$, we get

$$\int_{T_2}^{t}[R(t)-R(s)]^{n-1}q(s)ds \leq [R(t)-R(T_2)]^{n-1}\omega(T_2) + \frac{(n-1)^2}{4k}\int_{T_2}^{t}\frac{[R(t)-R(s)]^{n-3}}{r(s)}ds,$$

then, for $t \geq T_2$, we have

$$\int_{T_2}^{t}[R(t)-R(s)]^{n-1}q(s)ds \le [R(t)-R(T_2)]^{n-1}\omega(T_2)$$

$$+\frac{(n-1)^2}{4k(n-2)}[R(t)-R(T_2)]^{n-2}. \qquad (2\text{-}1)$$

Now, we know that

$$\int_{t_0}^{t}[R(t)-R(s)]^{n-1}q(s)ds = \int_{t_0}^{T_2}[R(t)-R(s)]^{n-1}q(s)ds+\int_{T_2}^{t}[R(t)-R(s)]^{n-1}q(s)ds$$

Dividing this inequality by $\frac{1}{R^{n-1}(t)}$, we have

$$\frac{1}{R^{n-1}(t)}\int_{t_0}^{t}[R(t)-R(s)]^{n-1}q(s)ds = \frac{1}{R^{n-1}(t)}\int_{t_0}^{T_2}[R(t)-R(s)]^{n-1}q(s)ds$$

$$+\frac{1}{R^{n-1}(t)}\int_{T_2}^{t}[R(t)-R(s)]^{n-1}q(s)ds$$

$$\le \frac{1}{R^{n-1}(t)}\int_{t_0}^{T_2}R^{n-1}(t)q(s)ds+\left(1-\frac{R(T_2)}{R(t)}\right)^{n-1}\omega(T_2)$$

$$+\frac{(n-1)^2}{4k(n-2)R(t)}\left(1-\frac{R(T_2)}{R(t)}\right)^{n-2}. \qquad (2\text{-}2)$$

then,

$$\limsup_{t\to\infty}\frac{1}{R^{n-1}(t)}\int_{t_0}^{t}[R(t)-R(s)]^{n-1}q(s)ds \le \left(1-k_1R(T_2)\right)^{n-1}\omega(T_2)$$

$$+\frac{k_1(n-1)^2}{4k(n-2)}\left(1-k_1R(T_2)\right)^{n-2}+\int_{t_0}^{T_2}q(s)ds<\infty,$$

then, we get

$$\limsup_{t\to\infty}\frac{1}{R^{n-1}(t)}\int_{t_0}^{t}[R(t)-R(s)]^{n-1}q(s)ds<\infty .$$

This contradicts to the condition (3); hence, the proof is completed.

Remark 2.1

Theorem 2.1 extends the results of Obriska and A. Zulova [23].

Example 2.1

Consider the differential equation

$$\left[\left(\frac{1}{t}\right)\left(\frac{1+x^4(t)}{2+x^4(t)}\right)\dot{x}(t)\right]^{\cdot}+x^3(t)\left(1+x^2(t)\right)=0\ ,\ \mathrm{t}>0$$

We note that

(1) $r(t)=\frac{1}{t}>0 \quad \forall t\geq t_0>0$,

(2) $\psi(x)=\frac{1+x^4}{2+x^4}>0$ and $\frac{1}{\psi(x)}=\frac{2+x^4}{1+x^4}\geq 1 \quad \forall x\in R$,

(3) $\frac{g_1(t,x(t))}{g(x(t))}=\frac{x^3(t)\left(1+x^2(t)\right)}{x^3(t)}=1+x^2(t)\geq 1=q(t) \quad$ *for all* $x\neq 0$ and $t\in[t_0,\infty)$

and $xg(x)=x^4>0$ and $g'(x)=3x^2>0 \quad$ *for all* $x\neq 0$,

(4) $R(t)=\int_{t_0}^{t}\frac{ds}{r(s)}=\int_{t_0}^{t}sds=\left.\frac{s^2}{2}\right|_{t_0}^{t}=\frac{t^2}{2}-\frac{t_0^2}{2} \qquad$ *for* $t>0$ and $\lim_{t\to\infty}\frac{1}{R(t)}=0\in[0,\infty)$,

Let n=3, we have

$$(5)\ \limsup_{t\to\infty}\frac{1}{R^{n-1}(t)}\int_{t_0}^{t}[R(t)-R(s)]^{n-1}q(s)ds = \limsup_{t\to\infty}\frac{4}{(t^2-t_0^2)^2}\int_{t_0}^{t}\left[\frac{t^2}{2}-\frac{s^2}{2}\right]^2 ds$$

$$=\limsup_{t\to\infty}\frac{4}{(t^2-t_0^2)^2}\int_{t_0}^{t}\left[\frac{t^4}{4}-\frac{t^2s^2}{2}+\frac{s^4}{4}\right]ds$$

$$=\limsup_{t\to\infty}\frac{4}{(t^2-t_0^2)^2}\left[\frac{t^4s}{4}-\frac{t^2s^3}{6}+\frac{s^5}{20}\right]_{t_0}^{t}$$

$$=\limsup_{t\to\infty}\frac{4}{(t^2-t_0^2)^2}\left[\frac{t^5}{4}-\frac{t^5}{6}+\frac{t^5}{20}-\frac{t^4t_0}{4}+\frac{t^2t_0^2}{6}-\frac{t_0^5}{20}\right]=\infty,$$

it follows from Theorem 2.1 that the given equation is oscillatory.

Theorem 2.2

Suppose that (1) holds, and furthermore

let for some integer $n\geq 2$

$$(4)\ \limsup_{t\to\infty}\frac{1}{t^n}\int_{t_0}^{t}\left[(t-s)^n q(s)-\frac{n^2}{4k_1}(t-s)^{n-2}r(s)\right]ds=\infty.$$

then equation (E) is oscillatory.

Proof

Let $x(t)$ be a non oscillatory solution of equation (E) and that $x(t)\neq 0\ \ for\ t\geq T_1\geq t_0$.

Define

$$\omega(t)=\frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))}, \quad t\ge T_3 .$$

then, for every $t\ge T_3$, we obtain

$$\dot{\omega}(t)\le -q(t)-H_1\frac{\omega^2(t)}{r(t)}, \qquad t\ge T_3$$

$$\dot{\omega}(t)\le -q(t)-k_2\frac{\omega^2(t)}{r(t)} .$$

Hence, for every $t\ge T_3$, we obtain

$$\int_{T_3}^{t}(t-s)^n q(s)ds\le -\int_{T_3}^{t}(t-s)^n\dot{\omega}(s)ds-k_2\int_{T_3}^{t}(t-s)^n\frac{\omega^2(s)}{r(s)}ds$$

$$\le (t-T_3)^n\omega(T_3)-\int_{T_3}^{t}n(t-s)^{n-1}\omega(s)ds-\int_{T_3}^{t}k_2(t-s)^n\frac{\omega^2(s)}{r(s)}ds .$$

then, for $t\ge T_3$, we get

$$\int_{T_3}^{t}(t-s)^n q(s)ds\le (t-T_3)^n\omega(T_3)+\frac{n^2}{4k_2}\int_{T_3}^{t}(t-s)^{n-2}r(s)ds-\int_{T_3}^{t}\left\{\sqrt{\frac{k_2(t-s)^n}{r(s)}}\,\omega(s)+\frac{1}{2}\frac{n(t-s)^{n-1}}{\sqrt{\frac{k_2(t-s)^n}{r(s)}}}\right\}^2 ds$$

$$\le (t-T_3)^n\omega(T_3)+\frac{n^2}{4k_2}\int_{T_3}^{t}(t-s)^{n-2}r(s)ds , \qquad (2\text{-}3)$$

then, for all $t\ge T_3$, we have

$$\int_{T_3}^{t}\left[(t-s)^n q(s)-\frac{n^2}{4k_2}(t-s)^{n-2}r(s)\right]ds \le (t-T_3)^n\omega(T_3)$$

$$\le (t-t_0)^n\omega(T_3), \qquad t \ge T_3 .$$

Now, we know that

$$\int_{t_0}^{t}\left[(t-s)^n q(s)-\frac{n^2}{4k_2}(t-s)^{n-2}r(s)\right]ds = \int_{t_0}^{T_3}\left[(t-s)^n q(s)-\frac{n^2}{4k_2}(t-s)^{n-2}r(s)\right]ds$$

$$+\int_{T_3}^{t}\left[(t-s)^n q(s)-\frac{n^2}{4k_2}(t-s)^{n-2}r(s)\right]ds$$

$$\le \int_{t_0}^{T_3}(t-s)^n q(s)ds+(t-t_0)^n\omega(T_3).$$

By the Bonnet theorem for a fixed $c_t \in [t_0, T_3]$ such that

$$\int_{t_0}^{T_3}(t-s)^n q(s)ds=(t-t_0)^n\int_{t_0}^{c_t}q(s)ds .$$

Then, for $t \ge T_3$, we get

$$\int_{t_0}^{t}\left[(t-s)^n q(s)-\frac{n^2}{4k_2}(t-s)^{n-2}r(s)\right]ds \le (t-t_0)^n\int_{t_0}^{c_t}q(s)ds+(t-t_0)^n\omega(T_3). \tag{2-4}$$

Now if we divide (2-4) by t^n take the upper limit as $t\to\infty$, we get

$$\limsup_{t\to\infty}\frac{1}{t^n}\int_{t_0}^{t}\left[(t-s)^n q(s)-\frac{n^2}{4k_2}(t-s)^{n-2}r(s)\right]ds<\infty .$$

This contradicts to the condition (4); hence, the proof is completed.

Remark 2.2

Theorem 2.2 extends results of Ohriska and A.Zulova [23].

Example 2.2

Consider the differential equation

$$\left[\left(\frac{x^2(t)+2}{x^2(t)+4}\right)\dot{x}(t)\right]^{\cdot} + x(t)(1+x^4(t)) = 0 \quad , t>0$$

We note that

(1) $r(t)=1>0$, $t \ge t_0 > 0$,

(2) $\psi(x)=\dfrac{x^2+2}{x^2+4}>0$ and $\dfrac{1}{\psi(x)}=\dfrac{x^2+4}{x^2+2}\ge 1 \qquad \forall x \in R$.

(3) $\dfrac{g_1(t,x(t))}{g(x(t))}=\dfrac{x(t)(1+x^4(t))}{x(t)}=1+x^4(t)\ge 1=q(t) \qquad$ *for* $x \ne 0$ and $t \in [t_0,\infty)$,

and $xg(x)=x^2>0$ and $g'(x)=1>0$ *for all* $x \ne 0$.

Let $n \ge 2$, we get

(4) $$\limsup_{t\to\infty}\frac{1}{t^n}\int_{t_0}^{t}\left[(t-s)^n-\frac{n^2}{4k_2}(t-s)^{n-2}r(s)\right]ds=\limsup_{t\to\infty}\frac{1}{t^n}\int_{t_0}^{t}\left[(t-s)^n-\frac{n^2}{4}(t-s)^{n-2}\right]ds$$

$$=\limsup_{t\to\infty}\frac{1}{t^n}\left[-\frac{(t-s)^{n+1}}{(n+1)}-\frac{n^2(t-s)^{n-1}}{4(n-1)}\right]_{t_0}^{t}$$

$$=\limsup_{t\to\infty}\frac{1}{t^n}\left[\frac{(t-t_0)^{n+1}}{(n+1)}+\frac{n^2(t-t_0)^{n-1}}{(n-1)}\right]$$

$$= \limsup_{t\to\infty}\left[\frac{t}{n+1}\left(1-\frac{t_0}{t}\right)^{n+1}+\frac{n^2}{4t}\left(1-\frac{t_0}{t}\right)^{n-1}\right]=\infty,$$

it follows from Theorem 2.2 that the given equation is oscillatory.

<u>**Theorem 2.3**</u>

Suppose that

(5) $0 < l_2 \le \psi(x(t)) \le l_1 \qquad$ *for all* $x \in R$,

and moreover, assume that there exists a differentiable function

$$\rho:[t_0,\infty)\to(0,\infty),$$

and the continuous functions

$$h,H: D \equiv \{(t,s): t\ge s\ge t_0\}\to R,$$

where H has a continuous and non positive partial derivative on D with respect to the second variable such that

$$H(t,t)=0 \qquad \textit{for } t \ge t_0,\ \ H(t,s)>0 \qquad \textit{for } t>s\ge t_0,$$

and $\dfrac{-\partial H(t,s)}{\partial s}=h(t,s)\sqrt{H(t,s)} \qquad$ *for all* $(t,s)\in D$,

(6) $\limsup_{t\to\infty}\left[X(t,t_0)-\dfrac{1}{4c_1}Y(t,t_0)\right]=\infty,$

where $\quad X(t,t_0)=\dfrac{1}{H(t,t_0)}\displaystyle\int_{t_0}^{t}H(t,s)\rho(s)q(s)ds.$

and $\quad \Upsilon(t,t_0) = \frac{1}{H(t,t_0)} \int_{t_0}^{t} r(s)\rho(s)\left[\gamma(s)\sqrt{H(t,s)} + l_3 h(t,s)\right]^2 ds,$

then equation (E) is oscillatory.

Proof

Let $x(t)$ be a non oscillatory solution of equation (E) and assume that $x(t) > 0$

for all $t \geq T_1 \geq t_0$.

Define

$$\omega(t) = \rho(t)\frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))}, \qquad t \geq T_1.$$

then, for every $t \geq T_1$, we obtain

$$\dot{\omega}(t) = -\frac{\rho(t)g_1(t,x(t))}{g(x(t))} + \frac{\dot{\rho}(t)r(t)\psi(x(t))\dot{x}(t)}{g(x(t))} - \frac{\rho(t)r(t)\psi(x(t))\dot{x}^2(t)g'(x(t))}{g^2(x(t))}.$$

Therefore, for all $t \geq T_1$, we have

$$\dot{\omega}(t) \leq -\rho(t)q(t) + \frac{\dot{\rho}(t)}{\rho(t)}\omega(t) - \frac{1}{l_3}\frac{l}{r(t)\rho(t)}\omega^2(t). \qquad t \geq T_1$$

$$\dot{\omega}(t) \leq -\rho(t)q(t) - \frac{1}{l_3}\left[\frac{l}{r(t)\rho(t)}\omega^2(t) + \gamma(t)\omega(t)\right].$$

where $\gamma(t) = -\frac{l_3\dot{\rho}(t)}{\rho(t)}$.

Then, for all $t \geq T_1 \geq t_0$, we obtain

$$\int_{T_1}^{t} H(t,s)\rho(s)q(s)ds \leq -\int_{T_1}^{t} H(t,s)\dot{\omega}(s)ds - \frac{1}{l_3}\int_{T_1}^{t}\left[\frac{lH(t,s)}{r(s)\rho(s)}\omega^2(s) + \gamma(s)H(t,s)\omega(s)\right]ds.$$

$$\therefore \int_{T_1}^{t} H(t,s)\rho(s)q(s)ds \leq -\left[H(t,s)\omega(s)\Big|_{T_1}^{t} - \int_{T_1}^{t}\frac{\partial H(t,s)}{\partial s}\omega(s)ds\right]$$

$$-\frac{1}{l_3}\int_{T_1}^{t}\left[\frac{lH(t,s)}{r(s)\rho(s)}\omega^2(s) + \gamma(s)H(t,s)\omega(s)\right]ds.$$

Then, for $t \geq T_1$, we have

$$\int_{T_1}^{t} H(t,s)\rho(s)q(s)ds \leq H(t,T_1)\omega(T_1) - \int_{T_1}^{t} h(t,s)\sqrt{H(t,s)}\omega(s)ds - \frac{1}{l_3}\int_{T_1}^{t}\left[\frac{lH(t,s)}{r(s)\rho(s)}\omega^2(s) + \gamma(s)H(t,s)\omega(s)\right]ds$$

$$\leq H(t,T_1)\omega(T_1) - \frac{1}{l_3}\int_{T_1}^{t}\left\{\frac{lH(t,s)}{r(s)\rho(s)}\omega^2(s) + \left[l_3 h(t,s)\sqrt{H(t,s)} + \gamma(s)H(t,s)\right]\omega(s)\right\}ds$$

$$\leq H(t,T_1)\omega(T_1) + \int_{T_1}^{t}\frac{r(s)\rho(s)}{4ll_3}\left[\gamma(s)\sqrt{H(t,s)} + l_3 h(t,s)\right]^2 ds$$

$$-\frac{1}{l_3}\int_{T_1}^{t}\left[\sqrt{\frac{lH(t,s)}{r(s)\rho(s)}}\omega(s) + \sqrt{\frac{r(s)\rho(s)}{4l}}\left(l_3 h(t,s) + \gamma(s)\sqrt{H(t,s)}\right)\right]^2 ds$$

$$\leq H(t,T_1)\left[\omega(T_1) - J(t,T_1)\right] + \int_{T_1}^{t}\frac{r(s)\rho(s)}{4c_1}\left[\gamma(s)\sqrt{H(t,s)} + l_3 h(t,s)\right]^2 ds, \qquad (2\text{-}5)$$

where

$$J(t,T_1) = \frac{1}{l_3 H(t,T_1)}\int_{T_1}^{t}\left[\sqrt{\frac{lH(t,s)}{r(s)\rho(s)}}\omega(s) + \sqrt{\frac{r(s)\rho(s)}{4l}}\left(l_3 h(t,s) + \gamma(s)\sqrt{H(t,s)}\right)\right]^2 ds \text{ and}$$

$c_1 = H_3$ is a positive constant.

moreover (2-5) implies that for $t \ge T_1$ we have

$$\int_{T_1}^{t} H(t,s)\rho(s)q(s)ds - \frac{1}{4c_1}\int_{T_1}^{t} r(s)\rho(s)\left[\gamma(s)\sqrt{H(t,s)} + l_3 h(t,s)\right]^2 ds \le H(t,T_1)\omega(T_1),$$

then, for $t \ge T_1$, we get

$$H(t,T_1)\left[X(t,T_1) - \frac{1}{4c_1}Y(t,T_1)\right] \le H(t,T_1)\omega(T_1)$$

$$\le H(t,t_0)\left|\omega(T_1)\right|, \qquad for\, T_1 \ge t_0. \tag{2-6}$$

In view of (2-5) and (2-6) we can easily obtain that

$$H(t,t_0)\left[X(t,t_0) - \frac{1}{4c_1}Y(t,t_0)\right] = \int_{t_0}^{T_1}\left\{H(t,s)\rho(s)q(s) - \frac{r(s)\rho(s)}{4c_1}\left[\gamma(s)\sqrt{H(t,s)} + l_3 h(t,s)\right]^2\right\}ds$$

$$+ \int_{T_1}^{t}\left\{H(t,s)\rho(s)q(s) - \frac{r(s)\rho(s)}{4c_1}\left[\gamma(s)\sqrt{H(t,s)} + l_3 h(t,s)\right]^2\right\}ds$$

$$\le H(t,t_0)\int_{t_0}^{T_1}\left|\rho(s)q(s)\right|ds + H(t,t_0)\left|\omega(T_1)\right|,$$

then, for $t \ge T_1$, we have

$$\limsup_{t\to\infty}\left[X(t,t_0) - \frac{1}{4c_1}Y(t,t_0)\right] \le \int_{t_0}^{T_1}\left|\rho(s)q(s)\right|ds + \left|\omega(T_1)\right|.$$

This contradicts the condition (6); hence, the proof is completed.

Remark 2.3

Theorem 2.3 extends the results of A.Tiryaki and A.Zafar [35].

Example 2.3

Consider the differential equation

$$\left[t\left(6+\frac{x^2(t)}{1+x^2(t)}\right)\dot{x}(t)\right]' + x(t)\left(t^2+x^4(t)\right)=0, \quad t>0$$

We note that

(1) $r(t)=t>0, \qquad t \geq t_0 > 0.$

(2) $0<6\leq \psi(x(t))=6+\dfrac{x^2}{1+x^2}\leq 7 \qquad$ *for all* $x\in R$,

(4) $\dfrac{g_1(t,x(t))}{g(x(t))}=\dfrac{x(t)\left(t^2+x^4(t)\right)}{x(t)}=t^2+x^4(t)\geq t^2=q(t) \qquad$ *for* $x\neq 0$ and $t\in[t_0,\infty)$

and $xg(x)=x^2>0$ and $g'(x)=1>0 \qquad$ *for* $x\neq 0$.

Let $\rho(t)=\dfrac{1}{t}$ and $H(t,s)=(t-s)^2$

Then $\dfrac{\partial H(t,s)}{\partial s}=-2(t-s), \quad h(t,s)=2, \quad t>0.$

(5) $$X(t,t_0)=\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\rho(s)q(s)ds=\frac{1}{(t-t_0)^2}\int_{t_0}^{t}(t-s)^2 s\,ds$$

$$=\frac{1}{(t-t_0)^2}\int_{t_0}^{t}\left[t^2s-2ts^2+s^3\right]ds$$

$$= \frac{1}{(t-t_0)^2}\left[\frac{t^2s^2}{2} - \frac{2ts^3}{3} + \frac{s^4}{4}\right]_{t_0}^{t}$$

$$= \frac{1}{(t-t_0)^2}\left[\frac{t^4}{12} - \frac{t^2t_0^2}{2} + \frac{2tt_0^3}{3} - \frac{t_0^4}{4}\right].$$

(6) $\mathrm{Y}(t,t_0) = \frac{1}{H(t,t_0)}\int_{t_0}^{t} r(s)\rho(s)\left[\gamma(s)\sqrt{H(t,s)} + l_3 h(t,s)\right]^2 ds$

$$\gamma(s) = -\frac{l_3\dot{\rho}(s)}{\rho(s)} = -7\cdot\frac{(-1)}{s} = \frac{7}{s},$$

$$\mathrm{Y}(t,t_0) = \frac{1}{(t-t_0)^2}\int_{t_0}^{t}\left[\frac{7}{s}(t-s)+14\right]^2 ds = \frac{49}{(t-t_0)^2}\int_{t_0}^{t}\left[\frac{t}{s}+1\right]^2 ds$$

$$= \frac{49}{(t-t_0)^2}\int_{t_0}^{t}\left[\frac{t^2}{s^2} + \frac{2t}{s} + 1\right]ds$$

$$= \frac{49}{(t-t_0)^2}\left[\frac{-t^2}{s} + 2t\ln s + s\right]_{t_0}^{t}$$

$$= \frac{49}{(t-t_0)^2}\left[2t\ln t + \frac{t^2}{t_0} - 2t\ln t_0 - t_0\right].$$

(7) $\limsup_{t\to\infty}\left[\mathrm{X}(t,t_0) - \frac{1}{4c_1}\mathrm{Y}(t,t_0)\right]$

$$= \limsup_{t\to\infty}\left[\frac{1}{(t-t_0)^2}\left\langle\frac{t^4}{12} - \frac{t^2t_0^2}{2} + \frac{2tt_0^3}{3} - \frac{t_0^4}{4} - \frac{7}{2}t\ln t - \frac{7t^2}{4t_0} + \frac{7}{2}t\ln t_0 + \frac{7t_0}{4}\right\rangle\right] = \infty,$$

it follows from Theorem 2.3 that the given equation is oscillatory.

Theorem 2.4

Suppose that (I) holds, and moreover, assume that there exists a differentiable function

$$\rho : [t_0, \infty) \to (0, \infty),$$

and the continuous functions

$$h, H : D \equiv \{(t, s) : t \ge s \ge t_0\} \to R,$$

where H has a continuous and non positive partial derivative on D with respect to the second variable such that

$H(t,t) = 0$ *for* $t \ge t_0$, $H(t,s) > 0$ *for* $t > s \ge t_0$.

and $$\frac{-\partial H(t,s)}{\partial s} = h(t,s)\sqrt{H(t,s)} \qquad \text{for all } (t,s) \in D,$$

$$(7)\ \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} r(s)\rho(s)\left[h(t,s) - \frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right]^2 ds < \infty,$$

$$(8)\ \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)q(s)ds = \infty,$$

then equation (E) is oscillatory.

Proof

Let $x(t)$ be a non oscillatory solution of equation (E) and assume that $x(t) > 0$ *for all* $t \ge T_2 \ge t_0$.

Define

$$\omega(t) = \frac{\rho(t)r(t)\psi(x(t))\dot{x}(t)}{g(x(t))}, \quad t \ge T_2,$$

then, for every $t \ge T_2$, we obtain

$$\dot{\omega}(t) = \frac{-\rho(t)g_1(t,x(t))}{g(x(t))} + \frac{\dot{\rho}(t)r(t)\psi(x(t))\dot{x}(t)}{g(x(t))} - \frac{\rho(t)r(t)\psi(x(t))\dot{x}^2(t)g'(x(t))}{g^2(x(t))}.$$

Therefore, for all $t \ge T_2$, we have

$$\dot{\omega}(t) \le -\rho(t)q(t) + \frac{\dot{\rho}(t)}{\rho(t)}\omega(t) - H_1\frac{1}{\rho(t)r(t)}\omega^2(t), \qquad t \ge T_2.$$

Then, for all $t \ge T_2$, we obtain

$$\int_{T_2}^{t} H(t,s)\rho(s)q(s)ds \le -\int_{T_2}^{t} H(t,s)\dot{\omega}(s)ds + \int_{T_2}^{t}\frac{H(t,s)\dot{\rho}(s)}{\rho(s)}\omega(s)ds - k_3\int_{T_2}^{t}\frac{H(t,s)}{\rho(s)r(s)}\omega^2(s)ds,$$

where $k_3 = H_1$ is a positive constant.

Then, for all $t \ge T_2$, we have

$$\int_{T_2}^{t} H(t,s)\rho(s)q(s)ds \le -\left[H(t,s)\omega(s)\Big|_{T_2}^{t} - \int_{T_2}^{t}\frac{\partial H(t,s)}{\partial s}\omega(s)ds\right] + \int_{T_2}^{t}\frac{H(t,s)\dot{\rho}(s)}{\rho(s)}\omega(s)ds$$

$$-k_3\int_{T_2}^{t}\frac{H(t,s)}{\rho(s)r(s)}\omega^2(s)ds$$

$$\le H(t,T_2)\omega(T_2) - \int_{T_2}^{t}\left[h(t,s)\sqrt{H(t,s)} + \frac{H(t,s)\dot{\rho}(s)}{\rho(s)}\right]\omega(s)ds - k_3\int_{T_2}^{t}\frac{H(t,s)}{\rho(s)r(s)}\omega^2(s)ds$$

$$\leq H(t,T_2)\omega(T_2)-\int_{T_2}^{t}\left\{\frac{k_3H(t,s)}{\rho(s)r(s)}\omega^2(s)+\sqrt{H(t,s)}\left[h(t,s)-\frac{\dot\rho(s)}{\rho(s)}\sqrt{H(t,s)}\right]\omega(s)\right\}ds,$$

$$\leq H(t,T_2)\omega(T_2)+\int_{T_2}^{t}\frac{\rho(s)r(s)}{4k_3}\left[h(t,s)-\frac{\dot\rho(s)}{\rho(s)}\right]^2 ds$$

$$-\int_{T_2}^{t}\left\{\sqrt{\frac{k_3H(t,s)}{\rho(s)r(s)}}\omega(s)+\frac{1}{2}\sqrt{\frac{\rho(s)r(s)}{k_3}}\left[h(t,s)-\frac{\dot\rho(s)}{\rho(s)}\sqrt{H(t,s)}\right]\right\}^2 ds$$

Then, *for all* $T_2 \geq t_0$, we have

$$\leq H(t,T_2)\omega(T_2)+\frac{1}{4k_3}\int_{T_2}^{t}\rho(s)r(s)\left[h(t,s)-\frac{\dot\rho(s)}{\rho(s)}\sqrt{H(t,s)}\right]^2 ds \quad . \ \textit{for all } T_2 \geq t_0 \qquad (2\text{-}7)$$

Now if we divide (2-7) by $H(t,t_0)$, take the upper limit as $t\to\infty$, and apply (7).we obtain

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\rho(s)q(s)ds<\infty .$$

This contradicts to the condition (8); hence, the proof is completed.

Remark 2.4

Theorem 2.4 extends the results of Grace [12] and [23].

Example 2.4

Consider the differential equation

$$\left[\left(\frac{t^2+2}{t^2+3}\right)\left(\frac{x^4(t)+1}{x^4(t)+5}\right)\dot x(t)\right]^{\cdot}+x^5(t)\left[\frac{2}{t}+2\sin t+x^4(t)\right]=0, \quad t>0.$$

We note that

(1) $r(t)=\dfrac{t^2+2}{t^2+3}>0, \qquad \forall t \ge t_0>0,$

(2) $\psi(x)=\dfrac{x^4+1}{x^4+5}>0$ and $\dfrac{1}{\psi(x)}=\dfrac{x^4+5}{x^4+1}\ge 1 \qquad \forall x \in R,$

(3) $\dfrac{g_1(t,x(t))}{g(x(t))}=\dfrac{x^5(t)\left[\dfrac{2}{t}+2\sin t+x^4(t)\right]}{x^5(t)}=\dfrac{2}{t}+2\sin t+x^4(t)$

$$\ge \frac{2}{t}+2\sin t=q(t) \quad \textit{for all } x\neq 0 \text{ and } t\in[t_0,\infty),$$

and $xg(x)=x^6>0$ and $g'(x)=5x^4>0$ *for all* $x \neq 0$.

Let $H(t,s)=(t-s)^2\ge 0 \qquad , \forall t\ge s\ge t_0>0$

then $\dfrac{\partial H(t,s)}{\partial s}=-2(t-s)$ and then $h(t,s)=2$

and taking $\rho(t)=3>0$ *for* $t>0$, then $\dot{\rho}(t)=0$

(4) $\limsup\limits_{t\to\infty}\dfrac{1}{H(t,t_0)}\displaystyle\int_{t_0}^{t}\rho(s)r(s)\left[h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right]^2 ds=\limsup_{t\to\infty}\frac{12}{(t-t_0)^2}\int_{t_0}^{t}\frac{s^2+2}{s^2+3}\,ds$

$$=\limsup_{t\to\infty}\frac{12}{(t-t_0)^2}\int_{t_0}^{t}\left[1-\frac{1}{s^2+3}\right]ds$$

$$=\limsup_{t\to\infty}\frac{12}{(t-t_0)^2}\left[s-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{s}{\sqrt{3}}\right)\right]_{t_0}^{t}$$

$$= \limsup_{t\to\infty} \frac{12}{(t-t_0)^2}\left[t - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) - t_0 + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t_0}{\sqrt{3}}\right)\right] < \infty,$$

(5) $$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)\rho(s)q(s)ds = \limsup_{t\to\infty} \frac{3}{(t-t_0)^2}\int_{t_0}^{t}(t-s)^2\left[\frac{2}{s} + 2\sin s\right]ds$$

$$= \limsup_{t\to\infty} \frac{3}{(t-t_0)^2}\int_{t_0}^{t}\left[\frac{2t^2}{s} - 4t + 2s + 2t^2\sin s - 4ts\sin s + 2s^2\sin s\right]ds$$

$$= \limsup_{t\to\infty} \frac{3}{(t-t_0)^2}\Big[2t^2\ln s - 4ts + s^2 - 2t^2\cos s + 4ts\cos s - 4t\sin s$$

$$- 2s^2\cos s + 4s\sin s + 4\cos s\Big]_{t_0}^{t}$$

$$= \limsup_{t\to\infty} \frac{3}{(t-t_0)^2}\Big[2t^2\ln t - 3t^2 + 4\cos t - 2t^2\ln t_0 + 4tt_0 - t_0^2 + 2t^2\cos t_0$$

$$- 4tt_0\cos t_0 + 4t\sin t_0 + 2t_0^2\cos t_0 - 4t_0\sin t_0 - 4\cos t_0\Big] = \infty,$$

it follows from Theorem 2.4 that the given equation is oscillatory.

<u>**Theorem 2.5**</u>

Suppose that (1) holds. and

(9) $r(t) \le l_4$ on $[t_0, \infty)$.

assume that n be an integer with $n \ge 3$ and ρ be a positive continuously differentiable function on the interval $[t_0, \infty)$ such that

(10) $$\limsup_{t\to\infty} \frac{1}{t^{n-1}}\int_{t_0}^{t}\frac{(t-s)^{n-3}}{\rho(s)}[(n-1)\rho(s) - (t-s)\dot{\rho}(s)]^2 ds < \infty,$$

$$(11)\ \limsup_{t\to\infty}\frac{1}{t^{n-1}}\int_{t_0}^{t}(t-s)^{n-1}\rho(s)q(s)ds=\infty,$$

then all solution of equation (E) is oscillatory.

Proof

Let $x(t)$ be a non oscillatory solution of the differential equation (E) and that $x(t)\neq 0$

for $t\geq T\geq t_0>0$.

Define

$$\omega(t)=\frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))},\qquad T\geq t_0>0.$$

Then, for $t\geq T$, we have

$$\dot{\omega}(t)=\frac{\left(r(t)\psi(x(t))\dot{x}(t)\right)^{\bullet}}{g(x(t))}-\frac{r(t)\psi(x(t))\dot{x}^2(t)g'(x(t))}{g^2(x(t))},\qquad t\geq T.$$

Hence, for all $t\geq T$, we have

$$\dot{\omega}(t)=\frac{-g_1(t,x(t))}{g(x(t))}-\frac{r(t)\psi(x(t))\dot{x}^2(t)g'(x(t))}{g^2(x(t))}.$$

From the conditions (1) and (9), for all $t\geq T$, we have

$$\dot{\omega}(t)\leq -q(t)-\frac{H_1}{l_4}\omega^2(t).$$

Then, for all $t\geq T$, we obtain

$$\dot{\omega}(t)\leq -q(t)-A_1\omega^2(t),\qquad t\geq T,$$

where $A_1 = \dfrac{u_1}{l_4}$ is a positive constant.

Hence, for every $t \geq T$, we obtain

$$\int_T^t (t-s)^{n-1}\rho(s)q(s)\,ds \leq -\int_T^t (t-s)^{n-1}\rho(s)\dot{\omega}(s)\,ds - \int_T^t A_1 (t-s)^{n-1}\rho(s)\omega^2(s)\,ds$$

Then, for $t \geq T$, we get

$$\int_T^t (t-s)^{n-1}\rho(s)q(s)\,ds \leq (t-T)^{n-1}\rho(T)\omega(T) - \int_T^t A_1 (t-s)^{n-1}\rho(s)\omega^2(s)\,ds - \int_T^t \left[(n-1)(t-s)^{n-2}\rho(s) - (t-s)^{n-1}\dot{\rho}(s)\right]\omega(s)\,ds$$

$$\leq (t-T)^{n-1}\rho(T)\omega(T) + \int_T^t \frac{\left[(n-1)(t-s)^{n-2}\rho(s) - (t-s)^{n-1}\dot{\rho}(s)\right]^2}{4A_1 (t-s)^{n-1}\rho(s)}\,ds$$

$$-\int_T^t \left\{ \sqrt{A_1 (t-s)^{n-1}\rho(s)}\,\omega(s) + \frac{1}{2}\frac{\left[(n-1)(t-s)^{n-2}\rho(s) - (t-s)^{n-1}\dot{\rho}(s)\right]}{\sqrt{A_1 (t-s)^{n-1}\rho(s)}} \right\}^2 ds$$

$$\leq (t-T)^{n-1}\rho(T)\omega(T) + \frac{1}{4A_1}\int_T^t \frac{(t-s)^{n-3}}{\rho(s)}\left[(n-1)\rho(s) - (t-s)\dot{\rho}(s)\right]^2 ds.$$

Then, for all $t \geq T$, we get

$$\int_T^t (t-s)^{n-1}\rho(s)q(s)\,ds \leq (t-T)^{n-1}\rho(T)\omega(T) + \frac{1}{4A_1}\int_T^t \frac{(t-s)^{n-3}}{\rho(s)}\left[(n-1)\rho(s) - (t-s)\dot{\rho}(s)\right]^2 ds.$$

Now, we know that

$$\int_{t_0}^t (t-s)^{n-1}\rho(s)q(s)\,ds = \int_{t_0}^T (t-s)^{n-1}\rho(s)q(s)\,ds + \int_T^t (t-s)^{n-1}\rho(s)q(s)\,ds.$$

Dividing this inequality by t^{n-1} and taking the limit supremum on both sides, we obtain

$$\limsup_{t\to\infty}\frac{1}{t^{n-1}}\int_{t_0}^{t}(t-s)^{n-1}\rho(s)q(s)ds=\limsup_{t\to\infty}\frac{1}{t^{n-1}}\int_{t_0}^{T}(t-s)^{n-1}\rho(s)q(s)ds$$

$$+\limsup_{t\to\infty}\frac{1}{t^{n-1}}\int_{T}^{t}(t-s)^{n-1}\rho(s)q(s)ds$$

$$\le\limsup_{t\to\infty}\frac{1}{t^{n-1}}\int_{t_0}^{T}(t-s)^{n-1}\rho(s)q(s)ds+\limsup_{t\to\infty}\frac{1}{t^{n-1}}(t-T)^{n-1}\rho(T)\omega(T)$$

$$+\limsup_{t\to\infty}\frac{1}{4A_1t^{n-1}}\int_{T}^{t}\frac{(t-s)^{n-3}}{\rho(s)}\left[(n-1)\rho(s)-(t-s)\dot{\rho}(s)\right]^2ds<\infty.$$

This contradicts to the condition (11); hence, the proof is completed.

Remark 2.5

Theorem 2.5 extends the results of Philos's Criterion [26], Kameneve Criterion [17], the results of [35] when $\rho(t)=0$ and the results of [19], [20] and [23].

Example 2.5

Consider the differential equation

$$\left[\left(\frac{t^2}{t^2+1}\right)\left(\frac{2x^2(t)+3}{4x^2(t)+5}\right)\dot{x}(t)\right]^{\cdot}+x^3(t)\left[t^2+x^2(t)\right]=0\,,\quad t>0.$$

We note that

(1) $0<r(t)=\dfrac{t^2}{t^2+1}<1 \qquad \forall t\ge t_0>0,$

(2) $\psi(x) = \dfrac{2x^2+3}{4x^2+5} > 0$ and $\dfrac{1}{\psi(x)} = \dfrac{4x^2+5}{2x^2+3} \geq 1 \qquad \forall x \in R$,

(3) $\dfrac{g_1(t,x(t))}{g(x(t))} = \dfrac{x^3(t)\left[t^2 + x^2(t)\right]}{x^3(t)} = t^2 + x^2(t) \geq t^2 = q(t)$ *for all* $x \neq 0$ and $t \in [t_0, \infty)$.

and $xg(x) = x^4 > 0$ and $g'(x) = 3x^2 > 0$ *for all* $x \neq 0$.

Let $\rho(t) = \dfrac{1}{t^2} > 0$ and $\dot{\rho}(t) = \dfrac{-2}{t^3}$ for all $t > 0$.

where $n = 3$, then, we get

$$(4)\ \limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} \frac{(t-s)^{n-3}}{\rho(s)} \left[(n-1)\rho(s) - (t-s)\dot{\rho}(s)\right]^2 ds$$

$$= \limsup_{t\to\infty} \frac{1}{t^2} \int_{t_0}^{t} s^3 \left[\frac{4}{s^4} + \frac{8(t-s)}{s^5} + \frac{4(t-s)^2}{s^6}\right] ds = \limsup_{t\to\infty} \frac{1}{t^2} \int_{t_0}^{t} \frac{4t^2}{s^4} ds$$

$$= \limsup_{t\to\infty} \frac{1}{t^2} \left[\frac{-4t^2}{3s^3}\right]_{t_0}^{t}$$

$$= \limsup_{t\to\infty} \frac{1}{t^2} \left[\frac{-4}{3t} + \frac{4t^2}{3t_0^3}\right] < \infty .$$

$$(5)\ \limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} \rho(s) q(s) ds = \limsup_{t\to\infty} \frac{1}{t^2} \int_{t_0}^{t} (t-s)^2 ds = \limsup_{t\to\infty} \frac{1}{t^2} \left[\frac{-(t-s)^3}{3}\right]_{t_0}^{t}$$

$$= \limsup_{t\to\infty} \frac{1}{t^2} \left[\frac{(t-t_0)^3}{3}\right] = \infty ,$$

it follows from Theorem 2.5 that the given equation is oscillatory.

Theorem 2.6

Suppose that (8) holds. and

(12) $\dfrac{1}{\psi(x)} \geq C \qquad \forall x \neq 0,$

(13) $\displaystyle\int^{\infty} \frac{\psi(u)}{g(u)} du < \infty$ and $\displaystyle\int^{-\infty} \frac{\psi(u)}{g(u)} du < \infty,$

and moreover, assume that there exists a differentiable function

$$\rho : [t_0, \infty) \rightarrow (0, \infty)$$

such that $\rho(t) > 0, \dot{\rho}(t) \geq 0$ and $\left(r(t)\dot{\rho}(t)\right)^{\cdot} \leq 0$ for all $t \geq t_0$.

and the continuous functions

$$h, H : D \equiv \{(t,s) : t \geq s \geq t_0\} \rightarrow R,$$

where H has a continuous and non positive partial derivative on D with respect to the

second variable such that

$$H(t,t) = 0 \qquad \text{for } t \geq t_0 \quad H(t,s) > 0 \qquad \text{for } t > s \geq t_0,$$

and $\qquad \dfrac{-\partial H(t,s)}{\partial s} = h(t,s)\sqrt{H(t,s)} \qquad$ *for all* $(t,s) \in D$.

(14) $\displaystyle\limsup_{t \to \infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \rho(s) r(s) h^2(t,s) ds < \infty.$

Then equation (E) is oscillatory.

Proof

Let $x(t)$ be a non oscillatory solution of equation (E) and assume that $x(t) > 0$

for all $t \geq T_3 \geq t_0$.

Define

$$\omega(t) = \frac{\rho(t)r(t)\psi(x(t))\dot{x}(t)}{g(x(t))}, \qquad t \ge T_3.$$

Then, for every $t \ge T_3$, we get

$$\dot{\omega}(t) = \frac{-\rho(t)g_1(t,x(t))}{g(x(t))} + \frac{\dot{\rho}(t)r(t)\psi(x(t))\dot{x}(t)}{g(x(t))} - \frac{\rho(t)r(t)\psi(x(t))\dot{x}^2(t)g'(x(t))}{g^2(x(t))}.$$

From the condition (12) for all $t \ge T_3$, we have

$$\dot{\omega}(t) \le -\rho(t)q(t) + \frac{\dot{\rho}(t)}{\rho(t)}\omega(t) - lC\frac{1}{\rho(t)r(t)}\omega^2(t). \qquad T_3 \ge t_0. \tag{2-8}$$

thus, for every $t \ge T_3 \ge t_0$, we have

$$\int_{T_3}^{t} H(t,s)\rho(s)q(s)ds \le -\int_{T_3}^{t} H(t,s)\dot{\omega}(s)ds + \int_{T_3}^{t}\frac{H(t,s)\dot{\rho}(s)}{\rho(s)}\omega(s)ds - lC\int_{T_3}^{t}\frac{H(t,s)}{\rho(s)r(s)}\omega^2(s)ds$$

$$\le H(t,T_3)\omega(T_3) - \int_{T_3}^{t}\left(\frac{-\partial H(t,s)}{\partial s}\right)\omega(s)ds + \int_{T_3}^{t}\frac{H(t,s)\dot{\rho}(s)}{\rho(s)}\omega(s)ds$$

$$-A_2\int_{T_3}^{t}\frac{H(t,s)}{\rho(s)r(s)}\omega^2(s)ds. \tag{2-9}$$

where $A_2 = lC$ is a positive constant.

Now, we note that

$$\int_{T_3}^{t}\frac{H(t,s)\dot{\rho}(s)}{\rho(s)}\omega(s)ds = \int_{T_3}^{t} H(t,s)\dot{\rho}(s)\frac{r(s)\psi(x(t))\dot{x}(s)}{g(x(s))}ds$$

$$= \int_{T_3}^{t}\left[\left(\frac{-\partial H(t,s)}{\partial s}\right)\int_{T_3}^{s}(\dot{\rho}(u)r(u))\frac{\psi(x(u))\dot{x}(u)}{g(x(u))}du\right]ds .$$

By the Bonnet Theorem, for a fixed $s \geq T_3$ and for some $a_s \in [T_3, s]$

$$\int_{T_3}^{s}\dot{\rho}(u)r(u)\frac{\psi(x(u))\dot{x}(u)}{g(x(u))}du = \dot{\rho}(T_3)r(T_3)\int_{T_3}^{a_s}\frac{\psi(x(u))\dot{x}(u)}{g(x(u))}du$$

$$= \dot{\rho}(T_3)r(T_3)\int_{x(T_3)}^{x(a_s)}\frac{\psi(y)}{g(y)}dy,$$

and , since $\dot{\rho}(T_3)r(T_3) > 0$, and

$$\int_{x(T_3)}^{x(a_s)}\frac{\psi(y)}{g(y)}dy < \begin{cases} 0 & if\, x(a_s) \leq x(T_3) \\ \int_{x(T_3)}^{\infty}\frac{\psi(y)}{g(y)}dy & if\, x(a_s) \geq x(T_3). \end{cases}$$

then, we have

$$\int_{T_3}^{s}\dot{\rho}(u)r(u)\frac{\psi(x(u))\dot{x}(u)}{g(x(u))}du \leq k_4 ,$$

where $k_4 = \dot{\rho}(T_3)r(T_3)\int_{x(T_3)}^{\infty}\frac{\psi(y)}{g(y)}dy.$

Then, (2-9) becomes

$$\int_{T_3}^{t}H(t,s)\rho(s)q(s)ds \leq H(t,T_3)\omega(T_3) - \int_{T_3}^{t}h(t,s)\sqrt{H(t,s)}\omega(s)ds$$

$$+k_4\int_{T_3}^{t}\left(\frac{-\partial H(t,s)}{\partial s}\right)ds-A_2\int_{T_3}^{t}\frac{H(t,s)}{\rho(s)r(s)}\omega^2(s)ds$$

$$\leq H(t,T_3)[\omega(T_3)+k_4]-\int_{T_3}^{t}h(t,s)\sqrt{H(t,s)}\omega(s)ds-A_2\int_{T_3}^{t}\frac{H(t,s)}{\rho(s)r(s)}\omega^2(s)ds$$

$$\leq H(t,T_3)[\omega(T_3)+k_4]+\int_{T_3}^{t}\frac{h^2(t,s)\rho(s)r(s)}{4A_2}ds$$

$$-\int_{T_3}^{t}\left\{\sqrt{\frac{A_2H(t,s)}{\rho(s)r(s)}}\omega(s)+\frac{1}{2}\frac{\left[h(t,s)\sqrt{H(t,s)}\right]}{\sqrt{\frac{A_2H(t,s)}{\rho(s)r(s)}}}\right\}^2ds$$

$$\leq H(t,T_3)[\omega(T_3)+k_4]+\int_{T_3}^{t}\frac{h^2(t,s)\rho(s)r(s)}{4A_2}ds. \qquad (2\text{-}10)$$

Now if we divide (2-10) by $H(t,t_0)$, take the upper limit as $t\to\infty$, and apply (14),

we obtain

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\rho(s)q(s)ds<\infty .$$

This contradicts to the condition (8); hence, the proof is completed.

Remark 2.6

Theorem 2.6 extends the results of Grace [12] and [23].

Example 2.6

Consider the differential equation

$$\left[\left(\frac{t}{t+1}\right)\left(\frac{x^4(t)}{x^4(t)+1}\right)\dot{x}(t)\right]^{\cdot} + x^3(t)\left(t+x^2(t)\right)=0 \quad ,t>0.$$

We note that

(1) $r(t)=\dfrac{t}{t+1}>0 \qquad \forall t \ge t_0 > 0.$

(2) $\dfrac{1}{\psi(x)}=\dfrac{1+x^4}{x^4}\ge 1 \qquad \textit{for all } x \ne 0,$

(3) $\dfrac{g_1(t,x(t))}{g(x(t))}=\dfrac{x^3(t)\left(t+x^2(t)\right)}{x^3(t)}=t+x^2(t)\ge t=q(t) \qquad \textit{for } x \ne 0 \text{ and } [t_0,\infty).$

and $xg(x)=x^4>0$ and $g'(x)=3x^2>0$ *for all* $x \ne 0$,

let $\rho(t)=1$ we have $\dot{\rho}(t)=0$ and $\left(\dot{\rho}(t)r(t)\right)^{\cdot}=0$ *for* $t \ge t_0 > 0$.

let $H(t,s)=(t-s)^2$ *for* $t \ge s \ge t_0 > 0$ we get

(4) $$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\rho(s)q(s)ds=\limsup_{t\to\infty}\frac{1}{(t-t_0)^2}\int_{t_0}^{t}(t-s)^2 s\,ds$$

$$=\limsup_{t\to\infty}\frac{1}{(t-t_0)^2}\left[\frac{t^2s^2}{2}-\frac{2ts^3}{3}+\frac{s^4}{4}\right]_{t_0}^{t}$$

$$=\limsup_{t\to\infty}\frac{1}{(t-t_0)^2}\left[\frac{t^4}{12}-\frac{t^2t_0^2}{2}+\frac{2tt_0^3}{3}-\frac{t_0^4}{4}\right]=\infty,$$

(5) $\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \rho(s) r(s) h^2(t,s) ds = \limsup_{t\to\infty} \frac{4}{(t-t_0)^2} \int_{t_0}^{t} \frac{s}{s+1} ds$

$$= \limsup_{t\to\infty} \frac{4}{(t-t_0)^2} \left[s - \ln(s+1)\right]_{t_0}^{t}$$

$$= \limsup_{t\to\infty} \frac{4}{(t-t_0)^2} \left[t - \ln(t+1) - t_0 + \ln(t_0+1)\right] < \infty,$$

(6) $\int^{\infty} \frac{\psi(u)}{g(u)} du = \int^{\infty} \frac{u}{1+u^4} du = \frac{1}{2} \tan^{-1} u^2 \Big|^{\infty} < \infty$

and $\int^{-\infty} \frac{\psi(u)}{g(u)} du = \int^{-\infty} \frac{u}{1+u^4} du = \frac{1}{2} \tan^{-1} u^2 \Big|^{-\infty} < \infty$.

it follows from Theorem 2.6 that the given equation is oscillatory.

Now we need the following lemma which is an extension to that of Erbe [9], Wong [38] and Greaf and Spikes [14].

Lemma 2.1

Suppose that

(i) $\lim_{t\to\infty} \int_{t_0}^{t} \frac{ds}{r(s)} = \infty,$

(ii) $\liminf_{t\to\infty} \int_{T}^{t} q(s) ds \geq 0$ *for all* large T

(iii) $0 < l_2 \leq \psi(x(t)) \leq l_3$ *for all* $x \in R$

then every non oscillatory solution of equation (E) which is not eventually a constant, must satisfy $x(t)\dot{x}(t) > 0$ *for all* large t.

Proof

Suppose that $x(t) > 0 \quad for\ t \geq T_1 \geq t_0$

If the lemma is not true, then either $\dot{x}(t) < 0 \quad for\ all$ large t.

or $\dot{x}(t)$ oscillates *for all* large t

In the former case we may suppose that T_1 is sufficiently large

$$\therefore \int_{T_1}^{t} q(s)ds \geq 0 \quad for\ t \geq T_1 \text{ and } \dot{x}(t) < 0 \quad for\ t \geq T_1,$$

but

$$\left(r(t)\psi(x(t))\dot{x}(t)\right)^{\cdot} = -g_1(t, x(t))$$

$$\leq -g(x(t))q(t).$$

$$\therefore \left(r(t)\psi(x(t))\dot{x}(t)\right)^{\cdot} + g(x(t))q(t) \leq 0.$$

Now integrating the last inequality, we have

$$\left[r(t)\psi(x(t))\dot{x}(t)\right] - \left[r(T_1)\psi(x(T_1))\dot{x}(T_1)\right] + g(x(t))\int_{T_1}^{t} q(s)ds - \int_{T_1}^{t}\left[\dot{x}(s)g'(x(s))\int_{T_1}^{s} q(u)du\right]ds \leq 0,$$

$$\text{but} \qquad g(x(t))\int_{T_1}^{t} q(s)ds \geq 0 \quad \text{and} \quad -\int_{T_1}^{t}\dot{x}(s)g'(x(s))\int_{T_1}^{s} q(u)duds \geq 0.$$

Then, for every $t \geq T_1$, we get

$$\left[r(t)\psi(x(t))\dot{x}(t)\right]-\left[r(T_1)\psi(x(T_1))\dot{x}(T_1)\right]\leq 0.$$

or

$$\left[r(t)\psi(x(t))\dot{x}(t)\right]\leq\left[r(T_1)\psi(x(T_1))\dot{x}(T_1)\right].$$

Dividing by $r(t)\psi(x(t))$, we obtain

$$\dot{x}(t)\leq\frac{\left[r(T_1)\psi(x(T_1))\dot{x}(T_1)\right]}{r(t)\psi(x(t))}.$$

Then, *for all* $t \geq T_1$, we have

$$\dot{x}(t)\leq\frac{\left[r(T_1)\psi(x(T_1))\dot{x}(T_1)\right]}{l_3}\times\frac{1}{r(t)}.$$

Integrating, we get

$$x(t)\leq x(T_1)+\frac{\left[r(T_1)\psi(x(T_1))\dot{x}(T_1)\right]}{l_3}\times\int_{T_1}^{t}\frac{ds}{r(s)}.$$

then, we get $\quad x(t)\to-\infty \quad$ *as* $t \to \infty$.

This a contradiction to the assumption that $x(t)>0$ *for* $t \geq T_1$.

If $\dot{x}(t)$ oscillates, then there exists sequence $\{\tau_n\}\to\infty$ such that $\dot{x}(\tau_n)=0$ $(n=1,2,3,\dots)$

for all $t \geq T_1$.

Define $\qquad \omega(t)=\dfrac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))}, \qquad t \geq T_1$.

Then, for all $t \geq T_1$, we obtain

$$\dot{\omega}(t) = \frac{-g_1(t, x(t))}{g(x(t))} - \frac{r(t)\psi(x(t))\dot{x}^2(t)g'(x(t))}{g^2(x(t))}.$$

Hence, for $t \geq T_1$, we have

$$\dot{\omega}(t) \leq -q(t) - \frac{r(t)\psi(x(t))\dot{x}^2(t)g'(x(t))}{g^2(x(t))}.$$

Thus, *for all* $t \geq T_1$

$$\dot{\omega}(t) \leq -q(t). \qquad t \geq T_1.$$

Thus, for every $\tau_{n+1} \geq \tau_n$, we get

$$\int_{\tau_n}^{\tau_{n+1}} q(t)dt \leq -\int_{\tau_n}^{\tau_{n+1}} \dot{\omega}(t)dt$$

$$= \omega(\tau_n) - \omega(\tau_{n+1}) = 0.$$

$$\therefore \int_{\tau_n}^{\tau_{n+1}} q(t)dt \leq 0.$$

This a contradiction to the condition (ii); hence the proof is completed.

Theorem 2.7

Suppose that (5) holds, and

(15) $\liminf_{t\to\infty} \int_T^t q(s)ds \geq 0$ *for all* large T,

(16) $\lim_{t\to\infty}\int_{t_0}^{t}\frac{ds}{r(s)}=\infty$,

(17) $\limsup_{t\to\infty}\frac{1}{t^{\beta}}\int_{t_0}^{t}(t-s)^{\beta}q(s)ds=\infty$ *for some* $\beta \geq 0$.

Then equation (E) is oscillatory.

Proof

Let $x(t)$ be a non oscillatory solution of equation (E) and assume that $x(t)>0$ *for* $t \geq T_1 \geq t_0$. It follows from lemma (2-1), that $\dot{x}(t)>0$ on $[T_2,\infty)$, $\forall T_2 \geq T_1$.

Define

$$\omega(t)=\frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))} \qquad \textit{for } t \geq T_2 .$$

Then, for every $t \geq T_2$, we obtain

$$\dot{\omega}(t)=\frac{\left(r(t)\psi(x(t))\dot{x}(t)\right)^{\cdot}}{g(x(t))}-\frac{r(t)\psi(x(t))\dot{x}^2(t)g'(x(t))}{g^2(x(t))}.$$

Hence, *for* $t \geq T_2$, we have

$$\dot{\omega}(t)=\frac{-g_1(t,x(t))}{g(x(t))}-\frac{r(t)\psi(x(t))\dot{x}^2(t)g'(x(t))}{g^2(x(t))}.$$

Then, for all $t \geq T_2$, we obtain

$$\dot{\omega}(t)\leq -q(t), \qquad \forall t \geq T_2 .$$

Then, for every $t \geq T_2$, we obtain

$$\int_{T_2}^{t}(t-s)^{\beta}q(s)ds \leq -\int_{T_2}^{t}(t-s)^{\beta}\dot{\omega}(s)ds .$$

By the Bonnet Theorem, for a fixed $c_t \in [T_2, t]$

$$-\int_{T_2}^{t}(t-s)^{\beta}\dot{\omega}(s)ds = -(t-T_2)^{\beta}\int_{T_2}^{c_t}\dot{\omega}(s)ds$$

$$= -(t-T_2)^{\beta}\omega(c_t) + (t-T_2)^{\beta}\omega(T_2)$$

$$\leq (t-T_2)^{\beta}\omega(T_2) .$$

Hence, *for* $t \geq T_2 \geq t_0$, we have

$$-\int_{T_2}^{t}(t-s)^{\beta}\dot{\omega}(s)ds \leq (t-T_2)^{\beta}\omega(T_2) . \tag{2-11}$$

Now if we divide (2-11) by t^{β}, take the upper limit *as* $t \rightarrow \infty$, we get

$$\limsup_{t\to\infty}\frac{1}{t^{\beta}}\int_{t_0}^{t}(t-s)^{\beta}q(s)ds < \infty .$$

This a contradiction to the condition (17); hence, the proof is completed.

Remark 2.7

Theorem 2.7 extends the results of Wong and Yan [40] with $\rho(t) = 1$ and [23].

Example 2.7

Consider the differential equation

$$\left[\left(\frac{t^2+5}{t^2+3}\right)\left(5+\frac{x^6(t)}{x^6(t)+1}\right)\dot{x}(t)\right]^{\cdot}+x(t)\left(2+3\sin t+x^2(t)\right)=0,\quad t>0.$$

We note that

(1) $r(t)=\dfrac{t^2+5}{t^2+3}>0\ \ \forall t>0$ and $\lim\limits_{t\to\infty}\int_{t_0}^{t}\dfrac{ds}{r(s)}=\lim\limits_{t\to\infty}\int_{t_0}^{t}\dfrac{s^2+3}{s^2+5}ds=\lim\limits_{t\to\infty}\int_{t_0}^{t}\left[1-\dfrac{2}{s^2+5}\right]ds$

$$=\lim_{t\to\infty}\left[s-\frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{s}{\sqrt{5}}\right)\right]_{t_0}^{t}=\infty,$$

(2) $0<5\le\psi(x)=5+\dfrac{x^6}{x^6+1}<6$ *for all* $x\in R$.

(3) $\dfrac{g_1(t,x(t))}{g(x(t))}=\dfrac{x(t)\left(2+3\sin t+x^2(t)\right)}{x(t)}=2+3\sin t+x^2(t)\ge 2+3\sin t=q(t)$ *for* $x\neq 0$ and

$t\ge t_0>0$ and $xg(x)=x^2>0$ and $g'(x)=1>0$ *for* $x\neq 0$.

(4) $\liminf\limits_{t\to\infty}\int_{T}^{t}q(s)ds=\liminf\limits_{t\to\infty}\int_{T}^{t}[2+3\sin s]ds$

$=\liminf\limits_{t\to\infty}[2s-3\cos s]_T^t=\liminf\limits_{t\to\infty}[2t-3\cos t-2T+3\cos T]=\infty>0$.

By taking $\beta=1$, we get

(5) $\limsup\limits_{t\to\infty}\dfrac{1}{t^\beta}\int_{t_0}^{t}(t-s)^\beta q(s)ds=\limsup\limits_{t\to\infty}\dfrac{1}{t}\int_{t_0}^{t}(t-s)(2+3\sin s)ds$

$=\limsup\limits_{t\to\infty}\dfrac{1}{t}\left[2ts-s^2-3t\cos s+3s\cos s-3\sin s\right]_{t_0}^{t}$

$=\limsup\limits_{t\to\infty}\left[t-\dfrac{3\sin t}{t}+3\cos t_0-2t_0+\left(\dfrac{t_0^2-\cos t_0+3\sin t_0}{t}\right)\right]=\infty$,

it follows from Theorem 2.7 that the given equation is oscillatory.

Theorem 2.8

Suppose that (5), (15) and (16) holds, and

(18) $0 < l_5 \le r(t) \qquad \forall t \ge t_0,$

(19) $\limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\left[\frac{1}{r(s)}\int_{t_0}^{s} q(u)du\right]ds = \infty.$

then every solutions of equation (E) are oscillatory.

Proof

Let $x(t)$ be a non oscillatory solution of equation (E) and assume that $x(t) > 0$ *for* $t \ge T_1 \ge t_0$. It follows from lemma (2-1), that $\dot{x}(t) > 0$ on $[T_2, \infty)$, $\forall T_2 \ge T_1$.

Define

$$\omega(t) = \frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))}, \qquad \text{for } t \ge T_2 .$$

Then, for every $t \ge t_0$, we obtain

$$\dot{\omega}(t) = \frac{-g_1(t, x(t))}{g(x(t))} - \frac{r(t)\psi(x(t))\dot{x}^2(t)g'(x(t))}{g^2(x(t))}$$

$$\dot{\omega}(t) \le -q(t) - \frac{r(t)\psi(x(t))\dot{x}^2(t)g'(x(t))}{g^2(x(t))}.$$

Hence, *for all* $t \ge T_2$, we have

$$\dot{\omega}(t) \le -q(t), \qquad t \ge T_2 .$$

Then, for $t \ge T_2$, we get

$$\int_{T_2}^{t} \dot{\omega}(s)ds \le -\int_{T_2}^{t} q(s)ds.$$

$$\therefore \omega(t) \le \omega(T_2) - \int_{T_2}^{t} q(s)ds.$$

By the definition of ω, we get

$$\frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))} \le \omega(T_2) - \int_{T_2}^{t} q(s)ds.$$

$$\therefore \frac{\psi(x(t))\dot{x}(t)}{g(x(t))} \le \frac{\omega(T_2)}{r(t)} - \frac{1}{r(t)}\int_{T_2}^{t} q(s)ds$$

$$\therefore \frac{\psi(x(t))\dot{x}(t)}{g(x(t))} \le \frac{\omega(T_2)}{l_5} - \frac{1}{r(t)}\int_{T_2}^{t} q(s)ds.$$

Then, for every $t \ge T_2$, we have

$$\int_{T_2}^{t} \frac{\psi(x(s))\dot{x}(s)}{g(x(s))} \le \frac{\omega(T_2)}{l_5}(t - T_2) - \int_{T_2}^{t}\left[\frac{1}{r(s)}\int_{T_2}^{s} q(u)du\right]ds. \qquad (2\text{-}12)$$

Now if we divide (2-12) by t, take the upper limit as $t \to \infty$, we obtain

$$\limsup_{t\to\infty} \frac{1}{t}\int_{x(T_2)}^{x(t)} \frac{\psi(u)}{g(u)}du \le \limsup_{t\to\infty} \frac{1}{t}\left[\frac{\omega(T_2)}{l_5}(t - T_2)\right] - \limsup_{t\to\infty} \frac{1}{t}\int_{T_2}^{t}\left[\frac{1}{r(s)}\int_{T_2}^{s} q(u)du\right]ds = -\infty.$$

This contradicts; hence, the proof is completed

.

Remark 2.8

Theorem 2.8 extends the results of Philos [29] and [23].

Example 2.8

Consider the differential equation

$$\left[\left(10+\frac{x^8(t)}{x^8(t)+1}\right)\dot{x}(t)\right]' + \left(2t+x^2(t)\right)x^5(t)=0, \quad t>0.$$

We note that

(1) $r(t)=1>0, \ \forall t\geq t_0>0$ and $\lim\limits_{t\to\infty}\int\limits_{t_0}^{t}\frac{ds}{r(s)}=\lim\limits_{t\to\infty}\int\limits_{t_0}^{t}ds=\lim\limits_{t\to\infty}s\Big|_{t_0}^{t}=\infty.$

(2) $0<10\leq\psi(x)=10+\dfrac{x^8}{x^8+1}<11 \qquad \forall x\in R,$

(3) $\dfrac{g_1(t,x(t))}{g(x(t))}=\dfrac{\left(2t+x^2(t)\right)x^5(t)}{x^5(t)}=2t+x^2(t)\geq 2t=q(t)$ *for all* $x\neq 0$ and $t\in[t_0,\infty)$

and $xg(x)=x^6>0$ and $g'(x)=5x^4>0,$ *for all* $x\neq 0,$

(4) $\liminf\limits_{t\to\infty}\int\limits_{T}^{t}q(s)ds=\liminf\limits_{t\to\infty}\int\limits_{T}^{t}2s\,ds=\liminf\limits_{t\to\infty}\left(t^2-T^2\right)=\infty>0$ *for all* large T.

(5) $\limsup\limits_{t\to\infty}\dfrac{1}{t}\int\limits_{t_0}^{t}\left[\dfrac{1}{r(s)}\int\limits_{t_0}^{s}q(u)du\right]ds=\limsup\limits_{t\to\infty}\dfrac{1}{t}\int\limits_{t_0}^{t}\int\limits_{t_0}^{s}2u\,du\,ds$

$$=\limsup_{t\to\infty}\frac{1}{t}\int\limits_{t_0}^{t}\left(s^2-t_0^2\right)ds=\limsup_{t\to\infty}\left(\frac{t^2}{3}+\frac{2t_0^3}{3t}-t_0^2\right)=\infty.$$

it follows from Theorem 2.8 that the given equation is oscillatory.

Theorem 2.9

Suppose that (5), (15) and (16) holds, and moreover

assume that there exists a differentiable function

$$\rho:[t_0,\infty)\to(0,\infty),$$

such that $\rho(t)>0, \dot\rho(t)\geq 0$, $(r(t)\dot\rho(t))^{\cdot}\leq 0$ for all $t\geq t_0$.

(20) $\lim\limits_{t\to\infty}\int\limits_{t_0}^{t}\rho(s)q(s)ds=\infty$,

(21) $\lim\limits_{t\to\infty}\int\limits_{t_0}^{t}\left[\frac{1}{\rho(s)r(s)}\int\limits_{t_0}^{s}\rho(u)q(u)du\right]ds=\infty$,

then every solution of equation (E) is oscillatory.

Proof

Let $x(t)$ be a non oscillatory solution of equation (E) and assume that $x(t)>0$ *for* $t\geq T\geq t_0$. It follows from lemma (2-1), that $\dot x(t)>0$ on $[T_1,\infty)$, $\forall T_1\geq T$.

Multiplying equation (E) by $\frac{\rho(t)}{g(x(t))}$, we obtain

$$\frac{\rho(t)(r(t)\psi(x(t))\dot x(t))^{\cdot}}{g(x(t))}+\frac{\rho(t)g_1(t,x(t))}{g(x(t))}=0.$$

Then, for every $t\geq T_1$, we get

$$\int\limits_{T_1}^{t}\frac{\rho(s)(r(s)\psi(x(s))\dot x(s))^{\cdot}}{g(x(s))}ds+\int\limits_{T_1}^{t}\frac{\rho(s)g_1(s,x(s))}{g(x(s))}ds=0.$$

Thus, we get

$$\left.\frac{\rho(s)\left(r(s)\psi(x(s))\dot{x}(s)\right)}{g(x(s))}\right|_{T_1}^{t}-\int_{T_1}^{t}\frac{\dot{\rho}(s)\left[r(s)\psi(x(s))\dot{x}(s)\right]}{g(x(s))}ds+\int_{T_1}^{t}\frac{\rho(s)r(s)\psi(x(s))\dot{x}^2(s)}{g^2(x(s))}g'(x(s))ds$$

$$+\int_{T_1}^{t}\frac{\rho(s)g_1(s,x(s))}{g(x(s))}ds=0$$

$$\therefore\ \frac{\rho(t)r(t)\psi(x(t))\dot{x}(t)}{g(x(t))}-\frac{\rho(T_1)r(T_1)\psi(x(T_1))\dot{x}(T_1)}{g(x(T_1))}\le\int_{T_1}^{t}\frac{\dot{\rho}(s)r(s)\psi(x(s))\dot{x}(s)}{g(x(s))}ds$$

$$-\int_{T_1}^{t}\rho(s)q(s)ds.$$

Then, for all $t\ge T_1$, we have

$$\frac{\rho(t)r(t)\psi(x(t))\dot{x}(t)}{g(x(t))}\le\frac{\rho(T_1)r(T_1)\psi(x(T_1))\dot{x}(T_1)}{g(x(T_1))}+\int_{T_1}^{t}\frac{\dot{\rho}(s)r(s)\psi(x(s))\dot{x}(s)}{g(x(s))}ds$$

$$-\int_{T_1}^{t}\rho(s)q(s)ds.$$

By The Bonnet Theorem, for a fixed $\varepsilon_t\in[T_1,t]$ such that

$$\int_{T_1}^{t}\frac{\dot{\rho}(s)r(s)\psi(x(s))\dot{x}(s)}{g(x(s))}ds=\left(\dot{\rho}(T_1)r(T_1)\right)\int_{T_1}^{\varepsilon_t}\frac{\psi(x(s))\dot{x}(s)}{g(x(s))}ds$$

$$=\dot{\rho}(T_1)r(T_1)\int_{x(T_1)}^{x(\varepsilon_t)}\frac{\psi(u)}{g(u)}du=N<\infty.$$

Let $\quad b_1=\dfrac{\rho(T_1)r(T_1)\psi(x(T_1))\dot{x}(T_1)}{g(x(T_1))}+N.$

then, for all $t \geq T_1$, we have

$$\frac{\rho(t)r(t)\psi(x(t))\dot{x}(t)}{g(x(t))} \leq b_1 - \int_{T_1}^{t}\rho(s)q(s)ds. \qquad (2\text{-}13)$$

From the condition (20), there exists $t \geq T_2 \geq T_1$ such that

$$\int_{T_2}^{t}\rho(s)q(s)ds \geq 2b_1.$$

Implies that

$$b_1 \leq \frac{1}{2}\int_{T_2}^{t}\rho(s)q(s)ds,$$

then, *for all* $t \geq T_2$, we have

$$\frac{\rho(t)r(t)\psi(x(t))\dot{x}(t)}{g(x(t))} \leq -\frac{1}{2}\int_{T_2}^{t}\rho(s)q(s)ds,$$

then, for $t \geq T_2$, we have

$$\frac{\psi(x(t))\dot{x}(t)}{g(x(t))} \leq -\frac{1}{2}\frac{1}{\rho(s)r(s)}\int_{T_2}^{t}\rho(s)q(s)ds.$$

Thus, for every $t \geq T_2$, we obtain

$$\int_{T_2}^{t}\frac{\psi(x(s))\dot{x}(s)}{g(x(s))}ds \leq -\frac{1}{2}\int_{T_2}^{t}\left[\frac{1}{\rho(s)r(s)}\int_{T_2}^{s}\rho(u)q(u)du\right]ds.$$

Using the condition (21), we get

$$\int_{x(T_2)}^{x(t)}\frac{\psi(u)}{g(u)}du \to -\infty \qquad as\ t \to \infty.$$

This a contradiction ; hence, the proof is completed.

Remark 2.9

Theorem 2.9 extends the results of Grace and Lalli [11] and [23].

Example 2.9

Consider the differential equation

$$\left[\left(\frac{1}{t+1}\right)\left(3+\frac{x^{12}(t)}{x^{12}(t)+1}\right)\dot{x}(t)\right]^{\cdot}+x^{3}(t)\left(\frac{1}{t}+x^{2}(t)\right)=0 \quad , t>0.$$

We note that

(1) $r(t)=\frac{1}{t+1}>0 \quad \forall t\geq t_0>0$ and $\lim_{t\to\infty}\int_{t_0}^{t}\frac{ds}{r(s)}=\lim_{t\to\infty}\int_{t_0}^{t}(s+1)ds=\lim_{t\to\infty}\left[\frac{s^2}{2}+s\right]_{t_0}^{t}=\infty$.

(2) $0<3\leq\psi(x)=3+\frac{x^{12}}{x^{12}+1}<4$, *for all* $x\in R$,

(3) $\frac{g_1(t,x(t))}{g(x(t))}=\frac{x^3(t)\left[\frac{1}{t}+x^2(t)\right]}{x^3(t)}=\frac{1}{t}+x^2(t)\geq\frac{1}{t}=q(t)$ *for all* $x\neq 0$ and $t\in[t_0,\infty)$.

and $xg(x)=x^4>0$ and $g'(x)=3x^2>0$ *for all* $x\neq 0$,

(4) $\liminf_{t\to\infty}\int_{T}^{t}q(s)ds=\liminf_{t\to\infty}\int_{T}^{t}\frac{ds}{s}=\liminf_{t\to\infty}\ln s\big|_{T}^{t}=\infty$ *for all* large T.

Let $\rho(t)=t$ we have $\dot{\rho}(t)=1>0$ and $(\dot{\rho}(t)r(t))^{\cdot}=\frac{-1}{(t+1)^2}<0$ *for* $t\geq t_0>1$

(5) $\lim_{t\to\infty}\int_{t_0}^{t}\left[\frac{1}{\rho(s)r(s)}\int_{t_0}^{s}\rho(u)q(u)du\right]ds=\lim_{t\to\infty}\int_{t_0}^{t}\frac{s+1}{s}[s-t_0]ds$

$$= \lim_{t\to\infty}\left(\frac{s^2}{2}+s-t_0(s+\ln s)\right)\Bigg|_{t_0}^{t} = \infty,$$

it follows from Theorem 2.9 that the given equation is oscillatory.

Theorem 2.10

Suppose that (5), (15) and (16) holds, and moreover

assume that there exists a differentiable function

$$\rho : [t_0, \infty) \to (0, \infty)$$

such that $\rho(t) > 0, \dot{\rho}(t) \geq 0$ and $(r(t)\dot{\rho}(t))' \leq 0$ for all $t \geq t_0$,

(22) $0 < \int_{\varepsilon}^{\infty} \frac{du}{g(u)} < \infty$ and $\int_{-\varepsilon}^{-\infty} \frac{du}{g(u)} < \infty \quad \forall \varepsilon > 0$,

(23) $\limsup_{t\to\infty} \frac{1}{t^{\beta}} \int_{t_0}^{t} (t-s)^{\beta} \rho(s) q(s) ds = \infty$ *for some* $\beta \geq 0$,

then equation (E) is oscillatory.

Proof

Let $x(t)$ be a non oscillatory solution of equation (E) and assume that $x(t) > 0$ *for* $t \geq T_1 \geq t_0$. It follows from lemma (2-1), that $\dot{x}(t) > 0 \ \ t \geq T_2 \geq T_1$.

Define

$$\omega(t) = \frac{\rho(t) r(t) \psi(x(t)) \dot{x}(t)}{g(x(t))}, \qquad \text{for all } t \geq T_2 .$$

Then, for every $t \geq T_2$, we obtain

$$\dot{\omega}(t)=\frac{-\rho(t)g_1(t,x(t))}{g(x(t))}+\frac{\dot{\rho}(t)r(t)\psi(x(t))\dot{x}(t)}{g(x(t))}-\frac{\rho(t)r(t)\psi(x(t))\dot{x}^2(t)g'(x(t))}{g^2(x(t))}.$$

Then, *for all* $t \ge T_2$, we get

$$\dot{\omega}(t) \le -\rho(t)q(t)+\frac{\dot{\rho}(t)r(t)\psi(x(t))\dot{x}(t)}{g(x(t))}, \qquad \forall t \ge T_2 .$$

Hence, *for all* $t \ge T_2$, we obtain

$$\int_{T_2}^{t}(t-s)^{\beta}\rho(s)q(s)ds \le -\int_{T_2}^{t}(t-s)^{\beta}\dot{\omega}(s)ds + l_3\int_{T_2}^{t}(t-s)^{\beta}\frac{\dot{\rho}(s)r(s)\dot{x}(s)}{g(x(s))}ds. \tag{2-14}$$

By The Bonnet Theorem, for a fixed $\eta_t \in [T_2,t]$ such that

$$-\int_{T_2}^{t}(t-s)^{\beta}\dot{\omega}(s)ds = -(t-T_2)^{\beta}\int_{T_2}^{\eta_t}\dot{\omega}(s)ds$$

$$= -(t-T_2)^{\beta}\omega(\eta_t)+(t-T_2)^{\beta}\omega(T_2)$$

$$\le (t-T_2)^{\beta}\omega(T_2). \tag{2-15}$$

But,

$$\left[(t-s)^{\beta}\left(\dot{\rho}(s)r(s)\right)\right]^{\cdot} = (t-s)^{\beta}\left(\dot{\rho}(s)r(s)\right)^{\cdot} - \beta(t-s)^{\beta-1}\left(\dot{\rho}(s)r(s)\right) \le 0.$$

By the Bonnet Theorem, for a fixed $\zeta_t \in [T_2,t]$ such that

$$\int_{T_2}^{t}(t-s)^{\beta}\dot{\rho}(s)r(s)\frac{\dot{x}(s)}{g(x(s))}ds = (t-T_2)^{\beta}\dot{\rho}(T_2)r(T_2)\int_{T_2}^{\zeta_t}\frac{\dot{x}(s)}{g(x(s))}ds$$

$$= (t-T_2)^{\beta}\dot{\rho}(T_2)r(T_2)\int_{x(T_2)}^{x(\zeta_t)}\frac{du}{g(u)}. \tag{2-16}$$

From inequalities (2-15), (2-16) and (2-14), we get

$$\int_{T_2}^{t}(t-s)^{\beta}\rho(s)q(s)ds \le (t-T_2)^{\beta}\omega(T_2) + l_3(t-T_2)^{\beta}\dot{\rho}(T_2)r(T_2)\int_{x(T_2)}^{x(\xi_t)}\frac{du}{g(u)}.$$

Now dividing by t^{β} and take the upper limit $as\ t \to \infty$ we obtain

$$\limsup_{t\to\infty}\frac{1}{t^{\beta}}\int_{t_0}^{t}(t-s)^{\beta}\rho(s)q(s)ds < \infty .$$

This a contradiction to the condition (23); hence, the proof is completed.

Remark 2.10

Theorem 2.10 extends the results of Wong and Yeh [40] and [23].

Example 2.10

Consider the differential equation

$$\left[\left(\frac{t^2+3}{t^2+2}\right)\left(2+\frac{x^4(t)}{x^4(t)+1}\right)\dot{x}(t)\right]^{\cdot} + x^3(t)\left(\frac{1}{t} - \sin t + x^2(t)\right) = 0, \quad t>0.$$

We note that

(1) $r(t) = \dfrac{t^2+3}{t^2+2} > 0$ and $\displaystyle\lim_{t\to\infty}\int_{t_0}^{t}\frac{ds}{r(s)} = \lim_{t\to\infty}\int_{t_0}^{t}\frac{s^2+2}{s^2+3}ds = \lim_{t\to\infty}\left[s - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{s}{\sqrt{3}}\right)\right]_{t_0}^{t} = \infty,$

(2) $0 < 2 \le \psi(x) = 2 + \dfrac{x^4}{x^4+1} < 3$ *for all* $x \in R$,

$$(3)\ \frac{g_1(t,x(t))}{g(x(t))}=\frac{x^3(t)\left(\frac{1}{t}-\sin t+x^2(t)\right)}{x^3(t)}=\frac{1}{t}-\sin t+x^2(t)\geq\frac{1}{t}-\sin t=q(t)$$

for all $x\neq 0$ and $t\in[t_0,\infty)$ and $xg(x)=x^4>0$ and $g'(x)=3x^2>0$ *for all* $x\neq 0$.

$$(4)\ \liminf_{t\to\infty}\int_T^t q(s)ds=\liminf_{t\to\infty}\int_T^t\left[\frac{1}{s}-\sin s\right]ds$$

$$=\liminf_{t\to\infty}\left[\ln t+\cos t-\ln T-\cos T\right]=\infty>0 \qquad \textit{for all large } T.$$

$$(5)\ 0<\int_\varepsilon^\infty\frac{du}{g(u)}=\int_\varepsilon^\infty\frac{du}{u^3}=\frac{1}{2\varepsilon^2}<\infty \text{ and } \int_{-\varepsilon}^{-\infty}\frac{du}{g(u)}=\int_{-\varepsilon}^{-\infty}\frac{du}{u^3}=\frac{1}{2\varepsilon^2}<\infty \quad \forall\varepsilon>0.$$

Take $\beta=1$ and $\rho(t)=1>0$ then $\dot\rho(t)=0$ $\left(r(t)\dot\rho(t)\right)^{\cdot}=0$ we obtain

$$(6)\ \limsup_{t\to\infty}\frac{1}{t^\beta}\int_{t_0}^t(t-s)^\beta\rho(s)q(s)ds=\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^t(t-s)\left(s-\sin s\right)ds$$

$$=\limsup_{t\to\infty}\frac{1}{t}\left[\frac{ts^2}{2}+t\cos s-\frac{s^3}{3}-s\cos s+\sin s\right]_{t_0}^{t}=\infty,$$

it follows from Theorem 2.10 that the given equation is oscillatory.

CHAPTER 3

THE CONTINUABILITY AND THE OSCILLATION OF THE EQUATION

$$\left(r(t)\psi(x(t))\dot{x}(t)\right)^{\cdot} + g_1(t,x(t)) = H(t,\dot{x}(t),x(t)).$$

3.1 Introduction

In this chapter we shall study the continuability and the oscillation of regular solutions of the equation

$$\left(r(t)\psi(x(t))\dot{x}(t)\right)^{\cdot} + g_1(t,x(t)) = H(t,\dot{x}(t),x(t)), \tag{E}$$

where r is a positive continuous function on the interval $[t_0,\infty)$, $t_0 \geq 0$, ψ is a positive continuous function on the real line R. g_1 is a continuous function on $R \times R$ with $\dfrac{g_1(t,x(t))}{g(x(t))} \geq q(t)$ *for all* $x \neq 0$ and $t \in [t_0,\infty)$, where g is continuously differentiable function on the real line R except possible at 0 with $xg(x) > 0$ and $g'(x) \geq l > 0$ *for all* $x \neq 0$ and q is a continuous function on the interval $[t_0,\infty)$ $t_0 \geq 0$ and H is a continuous function on $[t_0,\infty) \times R \times R$.

We consider the Continuability of equation (E) in the case where $\psi(x) = 1$ and $g_1(t,x(t)) = q(t)g(x(t))$ *for all* $x \in R$ and $t \in [t_0,\infty)$.

i.e. the equation

$$\left(r(t)\dot{x}(t)\right)^{\cdot} + g_1(t,x(t)) = H(t,\dot{x}(t),x(t)). \tag{E1}$$

Throughout this section, the following notation is used

$$F(x) = \int_0^x g(u)du.$$

We write equation (E_1) in the following equivalent form:

$$\begin{cases} \dot{x}(t) = y, \\ \dot{y} = \dfrac{H(t,y,x) - \dot{r}(t)y - g_1(t,x(t))}{r(t)}. \end{cases} \qquad (1)$$

3.2 Continuability of solutions

In this section some results of the continuability of solutions of the equation (E_1) will be established.

Definition 3.1

The solution $x(t)$ of equation (E_1) is said to be a continuable solution of equation (E_1) if it is defined on some interval $[T,\infty)$, $T \geq 0$, where T depends on the solution $x(t)$.

We shall give two theorems concerning the continuability of the solutions of the equation (E_1) in the case where $|H(t,\dot{x}(t),x(t))| \leq |m(t)|$ *for all* $x, \dot{x} \in R$ and $t \geq t_0 \geq 0$.

Theorem 3.1

Suppose that

(1) $F(x)$ is bounded below and $F(x) \to \infty$ as $|x| \to \infty$,

(2) r and $q : [t_0, \infty) \to (0, \infty)$ are non decreasing functions on $[t_0, \infty)$ and $q(t)$ is bounded $\forall t \geq t_0$.

then all solutions of the equation (E) can be defined for all $t \geq t_0$.

Proof

Suppose that there is a solution $(x(t), y(t))$ of (I) and $T \geq t_0$ such that

$$\lim_{t \to T^-} (|x(t)| + |y(t)|) = \infty . \qquad (*)$$

Since $F(x)$ is bounded below and $F(x) \to \infty$ as $|x| \to \infty$, say $F(x) \geq -\alpha_1$ for some $\alpha_1 > 0$ and for all $x \in R$.

We define the function Z as

$$Z(t) = \frac{F(x) + \alpha_1}{r(t)} + \frac{y^2}{2q(t)}, \qquad t \geq t_0,$$

then, for all $t \geq t_0$, we have

$$\dot{Z}(t) = \frac{\dot{x}(t) g(x(t))}{r(t)} - \frac{(F(x) + \alpha_1)\dot{r}(t)}{r^2(t)} + \frac{y\dot{y}}{q(t)} - \frac{y^2 \dot{q}(t)}{2q^2(t)}, \qquad t \geq t_0.$$

By (1) we get

$$\dot{Z}(t)=\frac{y(t)g(x(t))}{r(t)}-\frac{(F(x(t))+a_1)\dot{r}(t)}{r^2(t)}+\frac{y(t)H(t,y(t),x(t)}{r(t)q(t)}-\frac{y^2(t)\dot{r}(t)}{r(t)q(t)}-\frac{y(t)g(x(t))}{r(t)}$$

$$-\frac{y^2\dot{q}(t)}{2q^2(t)}\ ,\quad t\geq t_0.$$

Then, taking into account the above conditions, we get

$$\dot{Z}(t)\leq\frac{y(t)H(t,y(t),x(t))}{r(t)q(t)}\ ,\qquad t\geq t_0.$$

Then, for $t\geq t_0$, we have

$$Z(t)\leq Z(t_0)+\int_{t_0}^{t}\frac{y(s)H(s,y(s),x(s))}{r(s)q(s)}ds.$$

Hence, for all $t\geq t_0$, we obtain

$$\frac{y^2(t)}{2q(t)}\leq Z(t_0)+\int_{t_0}^{t}\frac{y(s)H(s,y(s),x(s))}{r(s)q(s)}ds.$$

Since $y\leq\frac{1}{2}(y^2+1)$, we have

$$\frac{y(t)}{q(t)}\leq\frac{1}{2q(t)}+Z(t_0)+\int_{t_0}^{t}\frac{y(s)H(s,y(s),x(s))}{r(s)q(s)}ds,\quad t\geq t_0.$$

Since q is non decreasing function, we obtain

$$\left|\frac{y(t)}{q(t)}\right|\leq\frac{1}{2q(t_0)}+Z(t_0)+\int_{t_0}^{t}\left|\frac{y(s)H(s,y(s),x(s))}{r(s)q(s)}\right|ds.$$

Then, we have

$$\left|\frac{y(t)}{q(t)}\right| \le \beta_1 + \int_{t_0}^{t}\left|\frac{m(s)}{r(s)}\right|\left|\frac{y(s)}{q(s)}\right| ds, \qquad t \ge t_0,$$

where $\beta_1 = \dfrac{1}{2q(t_0)} + Z(t_0)$ is a positive constant.

By the Gronwall's inequality, we have

$$\left|\frac{y(t)}{q(t)}\right| \le \beta_1 \exp\left[\int_{t_0}^{t}\frac{|m(s)|}{r(s)} ds\right], \qquad t \ge t_0.$$

Thus $\left|\dfrac{y(t)}{q(t)}\right|$ is bounded on $[t_0, T]$ and since $q(t)$ is bounded, $y(t)$ is also bounded on $[t_0, T]$.

Integrating the first equation of (1), we obtain

$$x(t) = x(t_0) + \int_{t_0}^{t} y(s)ds \quad , t \in [t_0, T].$$

Since $y(t)$ is bounded on $[t_0, T]$, it follows that $x(t)$ is also bounded.

This is a contradiction to the assumption (*). This completes the proof.

Example 3.1

Consider the differential equation

$$\left[\left(\frac{t}{t+1}\right)\dot{x}(t)\right]^{\cdot} + 2t^2 x(t) = \frac{4\dot{x}^4(t)}{\dot{x}^8(t)+1} \times \frac{x^2(t)}{x^4(t)+1} \times \frac{t}{t+1} \times e^{-t}, \qquad t > 0.$$

We note that

(1) $0 < r(t) = \frac{t}{t+1}$ and $\dot{r}(t) = \frac{1}{(t+1)^2} > 0 \quad \forall t \ge t_0 > 0$,

(2) $g_1(t,x(t)) = 2t^2 x(t)$, then $q(t) = 2t^2 > 0$ and $\dot{q}(t) = 4t > 0 \quad \forall t \ge t_0 > 0$,

(3) $|H(t,\dot{x}(t),x(t))| \le e^{-t} = |m(t)| \quad \forall x,\dot{x} \in R$ and $t \ge t_0 > 0$,

and $xg(x) = x^2 > 0 \quad \forall x \ne 0$ and $F(x) = \int_0^x g(u)du = \int_0^x u\,du = \frac{x^2}{2} \ge 0 > -\alpha_1$

$\forall x \in R, \alpha_1 > 0$ and $F(x) \to \infty$ as $|x| \to \infty$,

it follows from theorem 3.1 that all solutions of the given equation are continuable.

Theorem 3.2

Suppose that (1) holds, and

(3) $r : [t_0,\infty) \to (0,\infty)$ is non decreasing function on $[t_0,\infty)$ and q(t) is a positive on $[t_0,\infty)$.

(4) $\gamma(t) = \frac{q(t)-1}{r(t)}$ is a positive and non increasing function on $[t_0,\infty)$.

Then all solutions of equation (E) are defined for all $t \ge t_0$.

Proof

Suppose that there is a solution $(x(t),y(t))$ of (I) and $T \ge t_0$ such that

$$\lim_{t \to T^-}(|x(t)| + |y(t)|) = \infty . \qquad (*)$$

Since $F(x)$ is bounded below and $F(x) \to \infty$ as $|x| \to \infty$, say $F(x) \geq -\alpha_1$ for some $\alpha_1 > 0$ and for all $x \in R$.

We define the function Z as

$$Z(t) = \frac{F(x) + \alpha_1}{r(t)} + \frac{y^2}{2}, \quad t \geq t_0.$$

Then, for all $t \geq t_0$, we obtain

$$\dot{Z}(t) = \frac{\dot{x}(t)g(x(t))}{r(t)} - \frac{(F(x) + \alpha_1)\dot{r}(t)}{r^2(t)} + y\dot{y}, \quad t \geq t_0.$$

By (1), we get

$$\dot{Z}(t) = \frac{y(t)g(x(t))}{r(t)} - \frac{(F(x(t)) + \alpha_1)\dot{r}(t)}{r^2(t)} + \frac{y(t)H(t, y(t), x(t))}{r(t)} - \frac{y^2(t)\dot{r}(t)}{r(t)} - \frac{y(t)q(t)g(x(t))}{r(t)}.$$

Then, taking into account the above conditions, we have

$$\dot{Z}(t) \leq \frac{y(t)H(t, y(t), x(t))}{r(t)} - \gamma(t)y(t)g(x(t)), \quad t \geq t_0.$$

Then, for every $t \geq t_0$, we obtain

$$Z(t) \leq Z(t_0) + \int_{t_0}^{t} \frac{y(s)H(s, y(s), x(s))}{r(s)} ds - \int_{t_0}^{t} \gamma(s)y(s)g(x(s))ds.$$

By using the Bonnet Theorem, there exists $\eta_1 \in [t_0, t]$ such that

$$-\int_{t_0}^{t} \gamma(s)y(s)g(x(s))ds = -\gamma(t_0)\int_{t_0}^{\eta_1} g(x(s))\dot{x}(s)ds = -\gamma(t_0)\int_{x(t_0)}^{x(\eta_1)} g(u)du$$

$$= -\gamma(t_0)\left[\int_{x(t_0)}^{0} g(u)du + \int_{0}^{x(\eta_1)} g(u)du\right]$$

$$= -\gamma(t_0)\left[-\int_{0}^{x(t_0)} g(u)du + \int_{0}^{x(\eta_1)} g(u)du\right] = \gamma(t_0)F(x(t_0)) - \gamma(t_0)F(x(\eta_1))$$

$$\leq \alpha_1\gamma(t_0) + \gamma(t_0)F(x(t_0))$$

Hence, $\forall t \geq t_0$, we obtain

$$Z(t) \leq Z(t_0) + \gamma(t_0)\alpha_1 + \gamma(t_0)F(x(t_0)) + \int_{t_0}^{t} \frac{y(s)H(s, y(s), x(s))}{r(s)} ds .$$

Then, for all $t \geq t_0$, we obtain

$$\frac{y^2}{2} \leq Z(t_0) + \gamma(t_0)\alpha_1 + \gamma(t_0)F(x(t_0)) + \int_{t_0}^{t} \frac{y(s)H(s, y(s), x(s))}{r(s)} ds \quad ,$$

and therefore, we have

$$\frac{y^2+1}{2} \leq \frac{1}{2} + Z(t_0) + \gamma(t_0)\alpha_1 + \gamma(t_0)F(x(t_0)) + \int_{t_0}^{t} \frac{y(s)H(s, y(s), x(s))}{r(s)} ds .$$

Since $y \leq \frac{1}{2}\left(y^2+1\right)$, we obtain

$$y(t) \leq \frac{1}{2} + Z(t_0) + \gamma(t_0)\alpha_1 + \gamma(t_0)F(x(t_0)) + \int_{t_0}^{t} \frac{y(s)H(s, y(s), x(s))}{r(s)} ds .$$

then, we have

$$y(t) \leq \beta_2 + \int_{t_0}^{t} \frac{y(s)H(s, y(s), x(s))}{r(s)} ds .$$

where $\beta_2 = \frac{1}{2} + Z(t_0) + +\gamma(t_0)\alpha_1 + \gamma(t_0)F(x(T))$ is a positive constant.

Thus, we obtain

$$|y(t)| \le \beta_2 + \int_{t_0}^{t} \frac{|m(s)|}{r(s)} |y(s)| ds .$$

By the Gronwll's inequality, we get

$$|y(t)| \le \beta_2 \exp\left[\int_{t_0}^{t} \frac{|m(s)|}{r(s)} ds\right].$$

$y(t)$ is bounded on $[t_0, T]$. Integrating the first equation of (I), we obtain

$$x(t) = x(t_0) + \int_{t_0}^{t} y(s) ds \quad , \ t \in [t_0, T]$$

Since $y(t)$ is bounded on $[t_0, T]$, it follows that $x(t)$ is also bounded.

This is a contradiction to the assumption (*). This completes the proof.

Example 3.2

Consider the differential equation

$$\left[\left(\frac{t^2}{1+t^2}\right)\dot{x}(t)\right]^{\bullet} + x^3(t)\left(3 + \frac{t^2}{1+t^2}\right) = \frac{\cos 2t}{t^4+1} \times \frac{x^2(t)}{2x^4(t)+4} \times \frac{\dot{x}^4(t)}{3\dot{x}^8(t)+8}, \quad t > 0.$$

We note that

(I) $r(t) = \frac{t^2}{1+t^2} > 0$ and $\dot{r}(t) = \frac{2t}{(1+t^2)^2} > 0 \quad \forall t \ge t_0 > 0$.

(2) $g_1(t,x(t))=x^3(t)\left(3+\frac{t^2}{1+t^2}\right)$, then $q(t)=3+\frac{t^2}{1+t^2}>0$ and $\dot{q}(t)=\frac{2t}{\left(1+t^2\right)^2}>0 \quad \forall t \geq t_0 > 0$.

and $xg(x)=x^4>0 \ \forall x \neq 0$ and $F(x)=\int_0^x g(u)du=\int_0^x u^3 du=\frac{x^4}{4}>0>-\alpha_1 \ \forall x \in R, \alpha_1>0$

$F(x) \to \infty$ as $|x| \to \infty$,

(3) $\gamma(t)=\frac{q(t)-1}{r(t)}=\frac{3t^2+2}{t^2}>0$ and $\dot{\gamma}(t)=\frac{-4t}{t^4}=\frac{-4}{t^3}\leq 0 \qquad \forall t \geq t_0>0$.

(4) $|H(t,\dot{x}(t),x(t))| \leq 1=|m(t)|$, $\forall \dot{x}, x \in R$ and $t \geq t_0 > 0$,

it follows from theorem 3.2 that all solutions of the given equation are continuable.

Remark 3.1

Theorem 3.1 and theorem 3.2 extend the results of Theorem 12.2 of Bushaw [5] and Theorem 2.1 of Graef [13] to more general equations.

3.3 Oscillation of solutions

In the present section we shall state and prove some sufficient oscillation criteria of the solutions of the equation (E) in the case where $\frac{H(t,\dot{x}(t),x(t))}{g(x(t))} \leq m_1(t)$ *for all* $\dot{x} \in R$, $x \neq 0$ and $t \in [t_0,\infty)$ and we restrict our attention only to the solutions of the differential equation (E) which exists on some interval $[t_0,\infty)$, $t_0 \geq 0$ may depend on a particular solution.

Theorem 3.3

Suppose that

(1) $r(t) \leq l_4 \quad \forall t \geq t_0$.

(2) $\frac{1}{\psi(x)} \geq l_1$ *for all* $x \in R$.

(3) $\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} [q(u) - m_1(u)] du ds = \infty$.

Then equation (E) is oscillatory.

__Proof__

Without loss of generality, we may assume that there exists a solution $x(t) > 0$ on $[t_0, \infty)$ for some $T_1 \geq t_0 > 0$.

Define

$$\omega(t) = \frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))}, \qquad t \geq T_1.$$

Then, for $t \geq T_1$, we get

$$\dot{\omega}(t) = \frac{(r(t)\psi(x(t))\dot{x}(t))^{\cdot}}{g(x(t))} - \frac{r(t)\psi(x(t))\dot{x}^2(t)g'(x(t))}{g^2(x(t))},$$

and so, by equation (E), we have

$$\dot{\omega}(t) = \frac{H(t,\dot{x}(t),x(t))}{g(x(t))} - \frac{g_1(t,x(t))}{g(x(t))} - \frac{r(t)\psi(x(t))\dot{x}^2(t)g'(x(t))}{g^2(x(t))}.$$

Then, for every $t \geq T_1$, we obtain

$$\dot{\omega}(t) \leq m_1(t) - q(t) - \frac{r(t)\psi(x(t))\dot{x}^2(t)g'(x(t))}{g^2(x(t))}.$$

Thus, for every $t \geq T_1$, we have

$$\omega(t) \leq \omega(T_1) - \int_{T_1}^{t} \frac{r(s)\psi(x(s))\dot{x}^2(s)g'(x(s))}{g^2(x(s))} ds - \int_{T_1}^{t} [q(s) - m_1(s)] ds .$$

Then, by definition of ω, we get

$$\frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))} \leq d - \int_{T_1}^{t} \frac{r(s)\psi(x(s))\dot{x}^2(s)g'(x(s))}{g^2(x(s))} ds - \int_{T_1}^{t} [q(s) - m_1(s)] ds , \quad t \geq T_1 ,$$

where d is a constant.

From the conditions (1) and (2), we have

$$\frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))} \leq d - \frac{H_1}{l_4} \int_{T_1}^{t} \left[\frac{r(s)\psi(x(s))\dot{x}(s)}{g(x(s))} \right]^2 ds - \int_{T_1}^{t} [q(s) - m_1(s)] ds$$

Integrating again and dividing by t, we obtain

$$\frac{1}{t} \int_{T_1}^{t} \frac{r(s)\psi(x(s))\dot{x}(s)}{g(x(s))} ds \leq d\left(1 - \frac{T_1}{t}\right) - \frac{d_1}{t} \int_{T_1}^{t} \int_{T_1}^{s} \left[\frac{r(u)\psi(x(u))\dot{x}(u)}{g(x(u))} \right]^2 du ds$$

$$- \frac{1}{t} \int_{T_1}^{t} \int_{T_1}^{s} [q(u) - m_1(u)] du ds , \qquad t \geq T_1 .$$

where $d_1 = \frac{H_1}{l_4}$ is a positive constant.

Then, for $t \geq T_1$, by condition (3), we obtain

$$\lim_{t \to \infty} \frac{1}{t} \int_{T_1}^{t} \frac{r(s)\psi(x(s))\dot{x}(s)}{g(x(s))} ds = -\infty .$$

Now, defining

$$V(t)=\left|\int_{T_1}^{t}\frac{r(s)\psi(x(s))\dot{x}(s)}{g(x(s))}ds\right|,$$

and applying Schwarz's inequality, we have

$$V^2(t)=\left|\int_{T_1}^{t}\frac{r(s)\psi(x(s))\dot{x}(s)}{g(x(s))}ds\right|^2\le\int_{T_1}^{t}\left|\frac{r(s)\psi(x(s))\dot{x}(s)}{g(x(s))}\right|^2 ds * \int_{T_1}^{t}|1|^2 ds$$

$$\le(t-T_1)\int_{T_1}^{t}\left|\frac{r(s)\psi(x(s))\dot{x}(s)}{g(x(s))}\right|^2 ds,\quad t\ge T_1.$$

Thus,

$$V^2(t)\le t\int_{T_1}^{t}\left|\frac{r(s)\psi(x(s))\dot{x}(s)}{g(x(s))}\right|^2 ds.$$

By condition (3) implies that for sufficiently large t say $t\ge T_2\ge T_1$, we obtain

$$\frac{-V(t)}{t}+\frac{d_1}{t}\int_{T_2}^{t}\int_{T_2}^{s}\left[\frac{r(u)\psi(x(u))\dot{x}(u)}{g(x(u))}\right]^2 du ds\le 0.$$

Then, for all $t\ge T_2$, we have

$$\frac{-V(t)}{t}+\frac{d_1}{t}\int_{T_2}^{t}\frac{V^2(s)}{s}ds\le 0,$$

it follows that

$$\frac{d_1}{t}\int_{T_2}^{t}\frac{V^2(s)}{s}ds\le\frac{V(t)}{t}\,, \tag{3-1}$$

Thus, for all $t \geq T_2$, inequality (3-1), becomes

$$\frac{d_1^2}{t^2}\left[\int_{T_2}^{t}\frac{V^2(s)}{s}ds\right]^2 \leq \frac{V^2(t)}{t^2}.$$

Then, for $t \geq T_2$, we define

$$\varphi(t) = \int_{T_2}^{t}\frac{V^2(s)}{s}ds.$$

Then, we get

$$\frac{d_1^2}{t} \leq \frac{\varphi'(t)}{\varphi^2(t)}.$$

For $t \geq T_2$, integrating the last inequality from T_2 *to* t, we obtain

$$d_1^2 \ln\left(\frac{t}{T_2}\right) \leq \frac{1}{\varphi(T_2)} - \frac{1}{\varphi(t)} \leq \frac{1}{\varphi(T_2)}.$$

This is a contradiction; hence; the proof is completed.

Remark 3.2

Theorem 3.3 includes theorem (4) of Greaf, Rankin and Spike [15] and extends the results of [21].

Example 3.3

Consider the differential equation

$$\left[\left(\frac{1}{t+1}\right)\left(\frac{x^2(t)+1}{x^2(t)+2}\right)\dot{x}(t)\right]^{\cdot} + x(t)\left(t^2 + x^2(t)\right) = \frac{x^3(t)\dot{x}^2(t)}{t^3(\dot{x}^2(t)+1)(x^2(t)+1)}, \quad t > 0.$$

We note that

(1) $0 < r(t) = \dfrac{1}{t+1} < 1 \quad \textit{for all } t > 0$,

(2) $\psi(x) = \dfrac{x^2+1}{x^2+2} > 0$ and $\dfrac{1}{\psi(x)} = \dfrac{x^2+2}{x^2+1} > 1 \ \forall x \in R$.

(3) $\dfrac{g_1(t,x(t))}{g(x(t))} = \dfrac{x(t)\left(t^2+x^2(t)\right)}{x(t)} = \left(t^2+x^2(t)\right) \geq t^2 = q(t) \quad \textit{for } x \neq 0$ and $t \in [t_0,\infty)$

and $xg(x) = x^2 > 0$ and $g'(x) = 1 > 0 \quad \textit{for } x \neq 0$.

(4) $\dfrac{H(t,\dot{x}(t),x(t))}{g(x(t))} = \dfrac{x^3(t)\dot{x}^2(t)}{x(t)t^3(\dot{x}^2+1)(x^2+1)} \leq \dfrac{1}{t^3} = m_1(t) \ \ \forall x \neq 0,\ \dot{x} \in R,\ t > 0.$

(5) $$\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}\left[q(u) - m_1(u)\right]duds = \lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}\left[u^2 - \frac{1}{u^3}\right]duds$$

$$= \lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\left[\frac{s^3}{3} - \frac{1}{2s^2} - \frac{t_0^3}{3} + \frac{1}{2t_0^2}\right]ds$$

$$= \lim_{t\to\infty}\frac{1}{t}\left[\frac{s^4}{12} + \frac{1}{2s} - \frac{t_0^3 s}{3} + \frac{s}{2t_0^2}\right]_{t_0}^{t} = \infty,$$

it follows that the given equation is oscillatory by theorem 3.3.

<u>**Theorem 3.4**</u>

Suppose that (2) hold. and

(4) $\displaystyle\int_{t_0}^{\infty}\frac{ds}{r(s)} = \infty$,

(5) $\int_{t_0}^{\infty}[q(s)-m_1(s)]ds=\infty$,

then every solution of equation (E) is oscillatory.

Proof

Let $x(t)$ be a non oscillatory solution of equation (E) say $x(t)>0$ *for* $t \geq T_0 \geq t_0$.

Define

$$\omega(t)=\frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))}.$$

Then, for $t \geq T_0$, we have

$$\dot{\omega}(t)=\frac{H(t,\dot{x}(t),x(t))}{g(x(t))}-\frac{g_1(t,x(t))}{g(x(t))}-\frac{r(t)\psi(x(t))\dot{x}^2(t)g'(x(t))}{g^2(x(t))}.$$

Then, for $t \geq T_0$, we get

$$\dot{\omega}(t)\leq m_1(t)-q(t).$$

Then, for every $t \geq T_0$, we obtain

$$\omega(t)\leq\omega(T_0)-\int_{T_0}^{t}[q(s)-m_1(s)]ds.$$

Then, by definition of ω, we get

$$\frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))}\leq\omega(T_0)-\int_{T_0}^{t}[q(s)-m_1(s)]ds.$$

By condition (5) there exists $T_1 \geq T_0$ such that

$$\dot{x}(t) < 0, \quad for\ t \geq T_1,$$

also, by condition (5) implies that there exists $T_2 \geq T_1$ such that

$$\int_{T_1}^{T_2} [q(s) - m_1(s)] ds = 0 \quad \text{and} \quad \int_{T_2}^{t} [q(s) - m_1(s)] ds \geq 0, \quad t \geq T_2.$$

Integrating equation (E) by parts, we get

$$r(t)\psi(x(t))\dot{x}(t) = r(T_2)\psi(x(T_2))\dot{x}(T_2) + \int_{T_2}^{t} H(s, \dot{x}(s), x(s)) ds - \int_{T_2}^{t} g_1(s, x(s)) ds.$$

Then, for $t \geq T_2$, we have

$$r(t)\psi(x(t))\dot{x}(t) \leq r(T_2)\psi(x(T_2))\dot{x}(T_2) - \int_{T_2}^{t} g(x(s))[q(s) - m_1(s)] ds$$

$$\leq r(T_2)\psi(x(T_2))\dot{x}(T_2) - g(x(t)) \int_{T_1}^{t} [q(s) - m_1(s)] ds$$

$$+ \int_{T_2}^{t} \dot{x}(s) g'(x(s)) \int_{T_1}^{s} [q(u) - m_1(u)] du ds.$$

Hence, for $t \geq T_2$, we obtain

$$r(t)\psi(x(t))\dot{x}(t) \leq r(T_2)\psi(x(T_2))\dot{x}(T_2).$$

Then, for every $t \geq T_2$, we have

$$r(t)\dot{x}(t) \leq r(T_2)\psi(x(T_2))\dot{x}(T_2) J_1, \qquad t \geq T_2.$$

Thus,

$$\dot{x}(t) \le l_1 r(T_2)\psi(x(T_2))\dot{x}(T_2)\frac{1}{r(t)}. \tag{3-2}$$

Integrating the inequality (3-2) from T_2 *to* t , we obtain

$$x(t) \le x(T_2) + l_1 r(T_2)\psi(x(T_2))\dot{x}(T_2)\int_{T_2}^{t}\frac{ds}{r(s)} \to -\infty \quad \text{as } t \to \infty,$$

which is a contradiction to the fact that $x(t) > 0$; hence, the proof is completed.

Remark 3.3

Theorem 3.4 extends the results of Grafe Rankin and Spikes [15] and extends the results of [21].

Example 3.4

Consider the differential equation

$$\left[\left(\frac{t^2}{t^2+1}\right)\left(\frac{2x^2(t)+3}{2x^2(t)+4}\right)\dot{x}(t)\right]^{\cdot} + x(t)\left(\frac{1}{t}+x^4(t)\right) = \frac{x^3(t)\dot{x}^4(t)(\sin t - 1)}{\left(1+x^2(t)\right)\left(\dot{x}^4(t)+1\right)}. \quad t>0.$$

We note that

(1) $0 < r(t) = \dfrac{t^2}{t^2+1} < 1$ and $\displaystyle\int_{t_0}^{\infty}\frac{ds}{r(s)} = \int_{t_0}^{\infty}(1+\frac{1}{s^2})ds = \infty$, $t>0$,

(2) $\psi(x) = \dfrac{2x^2+3}{2x^2+4} > 0$ and $\dfrac{1}{\psi(x)} = \dfrac{2x^2+4}{2x^2+3} > 1$ *for all* $x \in R$.

(3) $\dfrac{g_1(t,x(t))}{g(x(t))} = \dfrac{x(t)\left(\dfrac{1}{t}+x^4(t)\right)}{x(t)} = \left(\dfrac{1}{t}+x^4(t)\right) \geq \dfrac{1}{t} = q(t)$ *for all* $x \neq 0$ and $t \in [t_0,\infty)$

and $xg(x) = x^2 > 0$ and $g'(x) = 1 > 0$ *for* $x \neq 0$,

(4) $\dfrac{H(t,\dot{x}(t),x(t))}{g(x(t))} = \dfrac{x^3\dot{x}^4(\sin t - 1)}{(1+x^2)(\dot{x}^4+1)} \leq (\sin t - 1) = m_1(t)$ $\forall \dot{x} \in R$, $x \in R$ and $t \in [t_0,\infty)$,

(5) $\int\limits_{t_0}^{\infty}[q(s)-m_1(s)]ds = \int\limits_{t_0}^{\infty}\left[\dfrac{1}{s} - \sin s + 1\right]ds = lns + \cos s + s\Big|_{t_0}^{\infty} = \infty$,

it follows that the given equation is oscillatory by theorem 3.4.

Theorem 3.5

Suppose that (4) holds, and

(6) $\psi(x) \leq l_1'$ $\forall x \in R$,

(7) $\int\limits_{t_0}^{\infty}[q(s)-m_1(s)]ds < \infty$,

(8) $\liminf\limits_{t\to\infty} \int\limits_{t_0}^{t}[q(s)-m_1(s)]ds \geq 0$ for all large T,

(9) $\lim\limits_{t\to\infty} \int\limits_{t_0}^{t}\dfrac{1}{r(s)}\int\limits_{s}^{\infty}[q(u)-m_1(u)]duds = \infty$,

then the super linear differential equation (E) is oscillatory.

Proof

Let $x(t)$ be a non oscillatory solution of equation (E) say $x(t) > 0$ *for* $t \geq T_1 \geq t_0$.

Define

$$\omega(t) = \frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))}, \qquad t \geq T_1 .$$

Then, for $t \geq T_1$, we have

$$\dot{\omega}(t) = \frac{H(t,\dot{x}(t),x(t))}{g(x(t))} - \frac{g_1(t,x(t))}{g(x(t))} - \frac{r(t)\psi(x(t))\dot{x}^2(t)g'(x(t))}{g^2(x(t))} .$$

Then, we get

$$\dot{\omega}(t) \leq m_1(t) - q(t), \qquad t \geq T_1 .$$

Then, for every $T_2 \geq T_1$, we have

$$\omega(t) \leq \omega(T_2) - \int_{T_2}^{t} [q(s) - m_1(s)] ds .$$

Then, by definition of ω, we get

$$\frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))} \leq \omega(T_2) - \int_{T_2}^{t} [q(s) - m_1(s)] ds .$$

Now, if $\dot{x}(t) > 0$ for all $t \geq T_2'$. Then, by condition (7), we obtain

$$0 \leq \frac{r(T_2)\psi(x(T_2))\dot{x}(T_2)}{g(x(T_2))} - \int_{T_2}^{\infty} [q(s) - m_1(s)] ds .$$

Hence, for all $t \geq T_2$, we have

$$\int_{t}^{\infty} [q(s) - m_1(s)] ds \leq \frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))} \leq \frac{r(t)\dot{x}(t)}{g(x(t))} l_1' . \tag{3-3}$$

Now, by integrating the inequality (3-3), we obtain

$$\int_{T_1}^{t}\frac{1}{r(s)}\int_{s}^{\infty}[q(u)-m_1(u)]du\,ds \le l_1'\int_{T_2}^{t}\frac{\dot{x}(s)}{g(x(s))}ds = l_1'\int_{x(T_2)}^{x(t)}\frac{du}{g(u)}.$$

Hence, for $t \ge T_2$, we get

$$\int_{T_2}^{t}\frac{1}{r(s)}\int_{s}^{\infty}[q(u)-m_1(u)]du\,ds \le l_1'\int_{x(T_2)}^{x(t)}\frac{du}{g(u)} < \infty \quad \text{as} \quad t \to \infty .$$

This is a contradiction to the condition (9).

If $\dot{x}(t)$ changes signs, then there exists a sequence $\{a_n\} \to \infty$ such that $\dot{x}(a_n) < 0$ choose N large enough so that (8) holds, we obtain

$$\frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))} \le \frac{r(a_N)\psi(x(a_N))\dot{x}(a_N)}{g(x(a_N))} - \int_{a_N}^{t}[q(s)-m_1(s)]ds .$$

Thus, we get

$$\lim_{t\to\infty}\frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))} \le \frac{r(a_N)\psi(x(a_N))\dot{x}(a_N)}{g(x(a_N))} + \lim_{t\to\infty}\left\{-\int_{a_N}^{t}[q(s)-m_1(s)]ds\right\} < 0 ,$$

which contradicts the fact that $\dot{x}(t)$ oscillates.

Then, there exists $t_3 \ge T_1$ such that $\dot{x}(t) < 0$ for $t \ge t_3$, by condition (8) there exists $T_4 \ge T_1$, such that

$$\int_{T_4}^{t}[q(s)-m_1(s)]ds \ge 0 , \qquad t \ge T_4 .$$

Choosing $T_4 \geq t_3$ as indicated, and then integrating (E), we have

$$r(t)\psi(x(t))\dot{x}(t) \leq r(T_4)\psi(x(T_4))\dot{x}(T_4) - g(x(t))\int_{T_4}^{t}[q(s)-m_1(s)]ds$$

$$+\int_{T_4}^{t}\dot{x}(s)g'(x(s))\int_{T_4}^{s}[q(u)-m_1(u)]duds .$$

Hence, for every $t \geq T_4$, we get

$$r(t)\psi(x(t))\dot{x}(t) \leq r(T_4)\psi(x(T_4))\dot{x}(T_4)$$

Hence, for $t \geq T_4$, we have

$$x(t) \leq x(T_4) + \frac{1}{L_1'}r(T_4)\psi(x(T_4))\dot{x}(T_4)\int_{T_4}^{t}\frac{ds}{r(s)} \to -\infty \quad \text{as} \quad t \to \infty,$$

this is a contradiction to the fact that $x(t) > 0$ for $t \geq T_1$; hence, the proof is completed.

Remark 3.4

Theorem 3.5 extends the results of Grafe, Rankin and Spikes [15] and extends the results of [21].

Example 3.5

Consider the differential equation

$$\left[\left(\frac{1}{t}\right)\left(\frac{x^2(t)+1}{x^2(t)+2}\right)\dot{x}(t)\right]^{\cdot} + x^5(t)\left(\frac{2}{t^2}+x^2(t)\right) = \frac{x^5(t)\dot{x}^2(t)}{t^5(\dot{x}^2(t)+1)(1+x^2(t))}, \quad t>0.$$

We note that

(1) $0 < r(t) = \frac{1}{t}$ and $\int_{t_0}^{\infty}\frac{ds}{r(s)} = \int_{t_0}^{\infty}s\,ds = \frac{s^2}{2}\bigg|_{t_0}^{\infty} = \infty$ for t >0 ,

(2) $0 < \psi(x(t)) = \dfrac{x^2+1}{x^2+2} \le 1 \quad \forall x \in R$

(3) $\dfrac{g_1(t,x(t))}{g(x(t))} = \dfrac{x^5(t)\left(\dfrac{2}{t^2}+x^2(t)\right)}{x^5(t)} = \left(\dfrac{2}{t^2}+x^2(t)\right) \ge \dfrac{2}{t^2} = q(t)$ *for* $x \ne 0$ and $t \in [t_0,\infty)$

and $xg(x) = x^6 > 0$ and $g'(x) = 5x^4 > 0$ *for all* $x \ne 0$.

(4) $\dfrac{H(t,\dot{x}(t),x(t))}{g(x(t))} \le \dfrac{1}{t^5} = m_1(t) \quad \forall \dot{x} \in R,\ x \in R$ and $t \in [t_0,\infty)$,

(5) $\liminf\limits_{t\to\infty} \int_T^t [q(s)-m_1(s)]ds = \liminf\limits_{t\to\infty} \int_T^t \left[\dfrac{2}{s^2}-\dfrac{1}{s^5}\right]ds = \liminf\limits_{t\to\infty}\left[\dfrac{-2}{s}+\dfrac{1}{4s^4}\right]_T^t = \dfrac{2}{T}-\dfrac{1}{4T^4} > 0$,

(6) $\int_{t_0}^{\infty}[q(s)-m_1(s)]ds = \int_{t_0}^{\infty}\left[\dfrac{2}{s^2}-\dfrac{1}{s^5}\right]ds = \dfrac{-2}{s}+\dfrac{1}{4s^4}\Big|_{t_0}^{\infty} < \infty,$

(7) $\lim\limits_{t\to\infty}\int_{t_0}^{t}\dfrac{1}{r(s)}\int_s^{\infty}[q(u)-m_1(u)]duds = \lim\limits_{t\to\infty}\int_{t_0}^{t} s\int_s^{\infty}\left[\dfrac{2}{u^2}-\dfrac{1}{u^5}\right]duds$

$$= \lim_{t\to\infty}\int_{t_0}^{t} s\left[\frac{-2}{u}+\frac{1}{4u^4}\right]_s^{\infty} = \lim_{t\to\infty}\int_{t_0}^{t} s\left(\frac{2}{s}-\frac{1}{4s^4}\right)ds$$

$$= \lim_{t\to\infty}\int_{t_0}^{t}\left[2-\frac{1}{4s^3}\right] = \lim_{t\to\infty}\left[2s+\frac{1}{8s^2}\right]_{t_0}^{t}$$

$$= \lim_{t\to\infty}\left[2t+\frac{1}{8t^2}-2t_0-\frac{1}{8t_0^2}\right] = \infty ,$$

(8) $\int_{\varepsilon}^{\infty}\dfrac{du}{g(u)} = \int_{\varepsilon}^{\infty}\dfrac{du}{u^4} = \dfrac{1}{4\varepsilon^4} < \infty$ and $\int_{-\varepsilon}^{-\infty}\dfrac{du}{g(u)} = \int_{-\varepsilon}^{-\infty}\dfrac{du}{u^4} = \dfrac{1}{4\varepsilon^4} < \infty \quad \forall \varepsilon > 0$.

it follows that the given equation is oscillatory by theorem 3.5.

Theorem 3.6

Suppose that (8) holds, and

(10) $\int_{t_0}^{\infty} \frac{ds}{r(s)} = C \ , \quad C > 0 \, ,$

(11) $\psi(x) > 0 \quad \forall x \neq 0 \, ,$

(12) $\int_{0}^{\varepsilon} \frac{\psi(u)}{g(u)} du < \infty$ and $\int_{0}^{-\varepsilon} \frac{\psi(u)}{g(u)} du < \infty \quad \forall \varepsilon > 0 \, ,$

(13) $\int_{t_0}^{\infty} \frac{1}{r(s)} \int_{t_0}^{s} [q(u) - m_1(u)] du ds = \infty \, .$

Then all solutions of equation (E) are oscillatory.

Proof

Let $x(t)$ be a non oscillatory solution of equation (E) say $x(t) > 0$ *for* $t \geq t_1 \geq t_0$.

Define

$$\omega(t) = \frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))} \ , \quad t \geq t_1 .$$

Then, for every $t \geq t_1$, we have

$$\dot{\omega}(t) = \frac{H(t, \dot{x}(t), x(t))}{g(x(t))} - \frac{g_1(t, x(t))}{g(x(t))} - \frac{r(t)\psi(x(t))\dot{x}^2(t)g'(x(t))}{g^2(x(t))} .$$

Then, we get

$$\dot{\omega}(t) \leq m_1(t) - q(t)$$

Then, for every $t \geq t_1$, we obtain

$$\omega(t) \leq \omega(t_1) - \int_{t_1}^{t} [q(s) - m_1(s)] ds .$$

Then, by the definition of ω, we get

$$\frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))} \leq \omega(t_1) - \int_{t_1}^{t} [q(s) - m_1(s)] ds .$$

If $\dot{x}(t) > 0$, for $t \geq t_1$, we have

$$\frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))} \leq d_2 - \int_{t_1}^{t} [q(s) - m_1(s)] ds . \tag{3-4}$$

where d_2 is a constant.

Now, multiplying the inequality (3-4) by $\frac{1}{r(t)}$, we obtain

$$\frac{\psi(x(t))\dot{x}(t)}{g(x(t))} \leq \frac{d_2}{r(t)} - \frac{1}{r(t)} \int_{t_1}^{t} [q(s) - m_1(s)] ds .$$

Then, for every $t \geq t_1$, we have

$$\int_{t_1}^{t} \frac{\psi(x(s))\dot{x}(s)}{g(x(s))} ds \leq \int_{t_1}^{t} \frac{d_2}{r(s)} ds - \int_{t_1}^{t} \frac{1}{r(s)} \int_{t_1}^{s} [q(u) - m_1(u)] du ds .$$

Then, we get

$$\int_{x(t_1)}^{x(t)} \frac{\psi(u)}{g(u)} du \leq \int_{t_1}^{t} \frac{d_2}{r(s)} ds - \int_{t_1}^{t} \frac{1}{r(s)} \int_{t_1}^{s} [q(u) - m_1(u)] du ds .$$

From the conditions (10) and (13), we have

$$J(t)=\int_{x(t_1)}^{x(t)}\frac{\psi(u)}{g(u)}du.$$

and if $x(t)\geq x(t_1)$ *for large t*, $J(t)>0$.

Then, $\forall t\geq t_1$ we obtain

$$\int_{x(t_1)}^{x(t)}\frac{\psi(u)}{g(u)}du\to-\infty \qquad \text{as} \quad t\to\infty.$$

This a contradiction.

Now, if $\dot{x}(t)<0$, we obtain

$$\frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))}\leq\omega(t_1)-\int_{t_1}^{t}[q(s)-m_1(s)]ds.$$

Then, we get

$$\frac{\psi(x(t))\dot{x}(t)}{g(x(t))}\leq\frac{d_2}{r(t)}-\frac{1}{r(t)}\int_{t_1}^{t}[q(s)-m_1(s)]ds,$$

where d_2 is a constant

For all $t\geq t_1$, we have

$$\int_{t_1}^{t}\frac{\psi(x(s))\dot{x}(s)}{g(x(s))}ds\leq\int_{t_1}^{t}\frac{d_2}{r(s)}ds-\int_{t_1}^{t}\frac{1}{r(s)}\int_{t_1}^{s}[q(u)-m_1(u)]duds.$$

From the conditions (10) and (13), we obtain

$$J(t)=\int_{t_1}^{t}\frac{\psi(x(s))\dot{x}(s)}{g(x(s))}ds\to-\infty \qquad \text{as} \quad t\to\infty.$$

Since $x(t) < x(t_1)$, then for all $t \geq t_1$, we have

$$J(t) = \int_{x(t_1)}^{x(t)} \frac{\psi(u)}{g(u)} du = -\int_{x(t)}^{x(t_1)} \frac{\psi(u)}{g(u)} du = \left[\int_0^{x(t)} \frac{\psi(u)}{g(u)} du - \int_0^{x(t_1)} \frac{\psi(u)}{g(u)} du\right] \geq -\int_0^{x(t_1)} \frac{\psi(u)}{g(u)} du > -\infty .$$

This a contradiction.

Now, if $\dot{x}(t)$ changes signs, the there exists a sequence $\{a_n\} \to \infty$ such that $\dot{x}(a_n) < 0$ choose N large enough so that (8) holds, we obtain

$$\frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))} \leq \frac{r(a_N)\psi(x(a_N))\dot{x}(a_N)}{g(x(a_N))} - \int_{a_N}^{t} [q(s) - m_1(s)] ds .$$

Then, we get

$$\lim_{t \to \infty} \frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))} \leq \frac{r(a_N)\psi(x(a_N))\dot{x}(a_N)}{g(x(a_N))} + \lim_{t \to \infty}\left[-\int_{a_N}^{t} [q(s) - m_1(s)] ds\right] < 0 .$$

Hence,

$$\lim_{t \to \infty} \dot{x}(t) < 0 .$$

This is a contradiction to the fact that $\dot{x}(t)$ oscillates; hence, the proof is completed.

Remark 3.5

Theorem 3.6 is an extension of theorem (B) of Graf, Rankin Spikes [15] and extends the results of [21].

Example 3.6

Consider the differential equation

$$\left[t^2\left(\frac{x^2(t)}{1+x^2(t)}\right)\dot{x}(t)\right]^{\bullet} + x(t)\left(\frac{1}{t^2} + x^2(t)\right) = \frac{x^3(t)\cos x(t)\sin 2\dot{x}(t)}{t^3(1+x^4(t))(1+\dot{x}^2(t))}, \quad t>0.$$

We note that

(1) $0 < r(t) = t^2$ and $\int_{t_0}^{\infty} \frac{ds}{r(s)} = \int_{t_0}^{\infty} \frac{ds}{s^2} = \frac{-1}{s}\Big|_{t_0}^{\infty} = \frac{1}{t_0} \quad t > 0.$

(2) $\psi(x) = \frac{x^2}{1+x^2} > 0 \quad \forall x \neq 0.$

(3) $\frac{g_1(t, x(t))}{g(x(t))} = \frac{x(t)\left(\frac{1}{t^2} + x^2(t)\right)}{x(t)} = \left(\frac{1}{t^2} + x^2(t)\right) \geq \frac{1}{t^2} = q(t) \quad \forall x \neq 0$ and $t \in [t_0, \infty)$

and $xg(x) = x^2 > 0$ and $g'(x) = 1 > 0 \quad \forall x \neq 0,$

(4) $\frac{H(t, \dot{x}(t), x(t))}{g(x(t))} = \frac{x^3(t)\cos x(t)\sin \dot{x}(t)}{t^3(1+x^4(t))(1+\dot{x}^2(t))} \times \frac{1}{x(t)} \leq \frac{1}{t^3} = m_1(t) \; \forall \dot{x} \in R, x \in R$ and $t \in [t_0, \infty).$

(5) $\int_0^{\varepsilon} \frac{\psi(u)}{g(u)} du = \int_0^{\varepsilon} \frac{u}{(1+u^2)} du = \frac{1}{2}\ln(1+u^2)\Big]_0^{\varepsilon} = \frac{1}{2}\ln\left(1+(\varepsilon)^2\right) < \infty$ and

$$\int_0^{-\varepsilon} \frac{\psi(u)}{g(u)} du = \int_0^{-\varepsilon} \frac{u}{(1+u^2)} du = \frac{1}{2}\ln(1+u^2)\Big]_0^{-\varepsilon} = \frac{1}{2}\ln\left(1+(-\varepsilon)^2\right) < \infty \quad \forall \varepsilon > 0.$$

(6) $\liminf_{t\to\infty} \int_T^t [q(s) - m_1(s)] ds = \liminf_{t\to\infty} \int_T^t \left(\frac{1}{s^2} - \frac{1}{s^3}\right) ds = \liminf_{t\to\infty} \left[\frac{-1}{s} + \frac{1}{2s^2}\right]_T^t > 0.$

(7) $\int_{t_0}^{\infty} \frac{1}{r(s)} \int_{t_0}^{s} [q(u) - m_1(u)] du ds = \int_{t_0}^{\infty} s^2 \int_{t_0}^{s} \left[\frac{1}{u^2} - \frac{1}{u^3}\right] du ds$

$$= \int_{t_0}^{\infty} s^2 \left[\frac{-1}{s} + \frac{1}{2s^2} + \frac{1}{t_0} - \frac{1}{2t_0^2}\right] ds$$

$$= \left[\frac{-s^2}{2} + \frac{s}{2} + \frac{s^3}{3t_0} - \frac{s^3}{6t_0^2}\right]_{t_0}^{\infty} = \infty,$$

it follows that the given equation is oscillatory by theorem 3.6.

Theorem 3.7

Suppose that

(14) $l_2 \le \psi(x(t)) \le l_3$ *for all* $x \in R$,

(15) $g_1(t,x(t)) = q(t)g(x(t))$ *for all* $x \neq 0$,

(16) $H(t,\dot{x}(t),x(t)) = -m_1(t)\psi(x(t))\dot{x}(t) \quad \forall x,\dot{x} \in R$ and $t \in [t_0,\infty)$,

(17) $0 < \int_{\varepsilon}^{\infty} \frac{\psi(u)}{g(u)}du < \infty$ and $\int_{-\varepsilon}^{-\infty} \frac{\psi(u)}{g(u)}du < \infty \quad \forall \varepsilon > 0$.

Furthermore suppose that there exists a function

$$\rho : [t_0,\infty) \to (0,\infty)$$

such that $\dot{\rho}(t) \ge 0$, $(r(t)\rho(t))^{\cdot} \ge 0$, $(r(t)\rho(t))^{\cdot\cdot} \le 0$, $[r(t)\dot{\rho}(t) - \rho(t)m_1(t)]^{\cdot} \le 0$,

(18) $\liminf_{t\to\infty} \int_{t_0}^{t} \rho(s)q(s)ds > -\infty$,

(19) $\limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\left[\int_{t_0}^{s} \rho(u)q(u)du\right]^2 ds = \infty$,

then all solutions of equation (E) are oscillatory.

Proof

Let $x(t)$ be a non oscillatory solution of equation (E) say $x(t) > 0$ *for* $t \ge T \ge t_0$.

Define

$$\omega(t) = \frac{\rho(t)r(t)\psi(x(t))\dot{x}(t)}{g(x(t))}, \qquad t \ge T.$$

Then, for every $t \geq T$, we have

$$\dot{\omega}(t) = \frac{\dot{\rho}(t)r(t)\psi(x(t))\dot{x}(t)}{g(x(t))} + \frac{\rho(t)\left(r(t)\psi(x(t))\dot{x}(t)\right)^{\bullet}}{g(x(t))} - \frac{\rho(t)r(t)\psi(x(t))\dot{x}^2(t)g'(x(t))}{g^2(x(t))}$$

$$\therefore\ \dot{\omega}(t) = \frac{\rho(t)H(t,\dot{x}(t),x(t))}{g(x(t))} - \frac{\rho(t)g_1(t,x(t))}{g(x(t))} + \frac{\dot{\rho}(t)r(t)\psi(x(t))\dot{x}(t)}{g(x(t))} - \frac{\rho(t)r(t)\psi(x(t))\dot{x}^2(t)g'(x(t))}{g^2(x(t))}.$$

Then, taking into account the above conditions, we have

$$\dot{\omega}(t) = \frac{-\rho(t)m_1(t)\psi(x(t))\dot{x}(t)}{g(x(t))} - \rho(t)q(t) + \frac{\dot{\rho}(t)r(t)\psi(x(t))\dot{x}(t)}{g(x(t))} - \frac{\omega^2(t)g'(x(t))}{\rho(t)r(t)\psi(x(t))}.$$

Then, for $t \geq T$, we obtain

$$\dot{\omega}(t) = -\rho(t)q(t) + \frac{\psi(x(t))\dot{x}(t)}{g(x(t))}\left[r(t)\dot{\rho}(t) - \rho(t)m_1(t)\right] - \frac{\omega^2(t)g'(x(t))}{\rho(t)r(t)\psi(x(t))}, \qquad t \geq T.$$

Then, for all $t \geq T$, we have

$$\int_T^t \rho(s)q(s)ds = -\int_T^t \dot{\omega}(s)ds + \int_T^t\left[r(s)\dot{\rho}(s) - \rho(s)m_1(s)\right]\frac{\psi(x(s))\dot{x}(s)}{g(x(s))}ds - \int_T^t \frac{\omega^2(s)g'(x(s))}{\rho(s)r(s)\psi(x(s))}ds.$$

Then,

$$\int_T^t \rho(s)q(s)ds = -\omega(t) + \omega(T) + \int_T^t\left[r(s)\dot{\rho}(s) - \rho(s)m_1(s)\right]\frac{\psi(x(s))\dot{x}(s)}{g(x(s))}ds - \int_T^t \frac{\omega^2(s)g'(x(s))}{\rho(s)r(s)\psi(x(s))}ds. \qquad (3\text{-}5)$$

Now, we consider the following two cases:

Case I

The integral $\displaystyle\int_T^\infty \frac{\omega^2(s)g'(x(s))}{\rho(s)r(s)\psi(x(s))}ds$ is finite.

then there exists a positive constant M such that

$$\int_T^t \frac{\omega^2(s)g'(x(s))}{\rho(s)r(s)\psi(x(s))}ds \le M, \quad \text{for all } t \ge T. \tag{3-6}$$

Now we know that

$$\int_{t_0}^t \rho(s)q(s)ds = \int_{t_0}^T \rho(s)q(s)ds + \int_T^t \rho(s)q(s)ds$$

$$= \int_{t_0}^T \rho(s)q(s)ds - \omega(t) + \omega(T) + \int_T^t \left[r(s)\dot{\rho}(s) - \rho(s)m_1(s)\right]\frac{\psi(x(s))\dot{x}(s)}{g(x(s))}ds - \int_T^t \frac{\omega^2(s)g'(x(s))}{\rho(s)r(s)\psi(x(s))}ds$$

$$= -\omega(t) + C_1 + \int_T^t \left[r(s)\dot{\rho}(s) - \rho(s)m_1(s)\right]\frac{\psi(x(s))\dot{x}(s)}{g(x(s))}ds - \int_T^t \frac{\omega^2(s)g'(x(s))}{\rho(s)r(s)\psi(x(s))}ds. \tag{3-7}$$

where $C_1 = \omega(T) + \int_{t_0}^T \rho(s)q(s)ds$.

By using the Bonnet Theorem for a fixed $t \ge T$ there exists $b_t \in [T, t]$ such that

$$\int_T^t \left[r(s)\dot{\rho}(s) - \rho(s)m_1(s)\right]\frac{\psi(x(s))\dot{x}(s)}{g(x(s))}ds = \left[r(T)\dot{\rho}(T) - \rho(T)m_1(T)\right]\int_T^{b_t}\frac{\psi(x(s))\dot{x}(s)}{g(x(s))}ds$$

$$= \left[r(T)\dot{\rho}(T) - \rho(T)m_1(T)\right]\int_{x(T)}^{x(b_t)}\frac{\psi(u)}{g(u)}du.$$

Since $\left[r(T)\dot{\rho}(T) - \rho(T)m_1(T)\right] \ge 0$ and $\int_{x(T)}^{x(b_t)}\frac{\psi(u)}{g(u)}du < \begin{cases} 0 & if\, x(b_t) < x(T) \\ \int_{x(T)}^{\infty}\frac{\psi(u)}{g(u)}du & if\, x(b_t) > x(T). \end{cases}$

Then, for $t \geq T$, we have

$$-\infty < \int_T^t \left[r(s)\dot{\rho}(s) - \rho(s)m_1(s) \right] \frac{\psi(x(s))\dot{x}(s)}{g(x(s))} ds \leq k, \qquad (3\text{-}8)$$

where $k = \left[r(T)\dot{\rho}(T) - \rho(T)m_1(T) \right] \int_{x(T)}^{\infty} \frac{\psi(u)}{g(u)} du$.

Then, for $t \geq T$, we obtain

$$\left[\int_{t_0}^t \rho(s)q(s)ds \right]^2 = \left[C_1 - \omega(t) + \int_T^t \left[r(s)\dot{\rho}(s) - \rho(s)m_1(s) \right] \frac{\psi(x(s))\dot{x}(s)}{g(x(s))} ds - \int_T^t \frac{\omega^2(s)g'(x(s))}{\rho(s)r(s)\psi(x(s))} ds \right]^2$$

$$\leq 4C_1^2 + 4(\omega(t))^2 + 4\left[\int_T^t \left[r(s)\dot{\rho}(s) - \rho(s)m_1(s) \right] \frac{\psi(x(s))\dot{x}(s)}{g(x(s))} ds \right]^2 + 4\left[\int_T^t \frac{\omega^2(s)g'(x(s))}{\rho(s)r(s)\psi(x(s))} ds \right]^2,$$

therefore, taking into account (3-6) and (3-8), we have

$$\left[\int_{t_0}^t \rho(s)q(s)ds \right]^2 \leq C_2 + 4\omega^2(t),$$

where $C_2 = 4C_1^2 + 4\left[\int_T^t \left[r(s)\dot{\rho}(s) - \rho(s)m_1(s) \right] \frac{\psi(x(s))\dot{x}(s)}{g(x(s))} ds \right]^2 + 4\left[\int_T^t \frac{\omega^2(s)g'(x(s))}{\rho(s)r(s)\psi(x(s))} ds \right]^2$.

Thus, for every $t \geq T$, we obtain

$$\int_{t_0}^t \left[\int_{t_0}^s \rho(u)q(u)du \right]^2 ds = \int_{t_0}^T \left[\int_{t_0}^s \rho(u)q(u)du \right]^2 ds + \int_T^t \left[\int_{t_0}^s \rho(u)q(u)du \right]^2 ds$$

$$= C_3 + \int_T^t \left[\int_{t_0}^s \rho(u)q(u)du \right]^2 ds$$

$$\le C_3 + C_2(t-T) + 4\int_T^t \omega^2(s)ds$$

$$= C_3 + C_2(t-T) + 4\int_T^t \frac{\omega^2(s)g'(x(s))}{\rho(s)r(s)\psi(x(s))g'(x(s))}\rho(s)r(s)\psi(x(s))ds$$

$$\le C_3 + C_2(t-T) + \frac{4l_3}{l}\int_T^t \frac{\omega^2(s)g'(x(s))}{\rho(s)r(s)\psi(x(s))}\rho(s)r(s)ds.$$

Then, for $t \ge T$, we obtain

$$\int_{t_0}^t \left[\int_{t_0}^s \rho(u)q(u)du \right]^2 ds \le C_3 + C_2(t-T) + \frac{4l_3}{l}\int_T^t \frac{\omega^2(s)g'(x(s))}{\rho(s)r(s)\psi(x(s))}\rho(s)r(s)ds.$$

By using the Bonnet Theorem for a fixed $t \ge T$ there exists $\delta_t \in [T,t]$ such that

$$\int_T^t \frac{\omega^2(s)g'(x(s))}{\rho(s)r(s)\psi(x(s))}\rho(s)r(s)ds = \rho(t)r(t)\int_{\delta_t}^t \frac{\omega^2(s)g'(x(s))}{\rho(s)r(s)\psi(x(s))}ds.$$

also, since $\rho(t)r(t)$ is a positive on $[t_0,\infty)$ and $(r(t)\rho(t))'$ is non negative and bounded above, it follows that $\rho(t)r(t) \le \beta t$ for all large t where $\beta > 0$ is constant and this implies that

$$\int_{t_0}^\infty \frac{ds}{\rho(s)r(s)} = \infty.$$

Then, for $t \ge T$, we obtain

$$\int_{t_0}^t \left[\int_{t_0}^s \rho(u)q(u)du \right]^2 ds \le C_3 + C_2(t-T) + \frac{4l_3\beta t}{l}\int_{\delta_t}^t \frac{\omega^2(s)g'(x(s))}{\rho(s)r(s)\psi(x(s))}ds.$$

Dividing the last inequality by t and taking the limit supremum on both sides, we have

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\int_{t_0}^{s} \rho(u) q(u) du \right]^2 ds \le C_2 + \frac{4 l_3 \beta}{l} M < \infty .$$

this is a contradiction to the condition (19).

Case 2

The integral $\int_T^\infty \frac{\omega^2(s) g'(x(s))}{\rho(s) r(s) \psi(x(s))} ds$ is infinite.

From (3-5) taking into account (3-7) and (3-8) for every $t \ge T$, we obtain

$$\int_{t_0}^{t} \rho(s) q(s) ds \le -\omega(t) + A - \int_T^t \frac{\omega^2(s) g'(x(s))}{\rho(s) r(s) \psi(x(s))} ds , \tag{3-9}$$

where $A = C_1 + k$.

Then, by condition (18) and from (3-9), it follows that for some constant B for every $t \ge T$, we obtain

$$-\omega(t) \ge B + \int_T^t \frac{\omega^2(s) g'(x(s))}{\rho(s) r(s) \psi(x(s))} ds . \tag{3-10}$$

Then, for $t \ge T_1 \ge T$, we have

$$\theta = B + \int_T^{T_1} \frac{\omega^2(s) g'(x(s))}{\rho(s) r(s) \psi(x(s))} ds > 0 .$$

Then, (3-10) ensures that $\omega(t)$ is negative on $[T_1,\infty)$. Now (3-10) gives

$$\left(\frac{\omega^2(s)g'(x(t))}{\rho(t)r(t)\psi(x(t))}\right)\left[B+\int_T^t \frac{\omega^2(s)g'(x(s))}{\rho(s)r(s)\psi(x(s))}ds\right]^{-1} \geq \frac{\omega^2(t)g'(x(t))}{\rho(t)r(t)\psi(x(t))\omega(t)}$$

$$\geq \frac{g'(x(t))\rho(t)r(t)\psi(x(t))\dot{x}(t)}{\rho(t)r(t)\psi(x(t))g(x(t))}.$$

Then, for every $t \geq T_1$, we have

$$\left(\frac{\omega^2(t)g'(x(t))}{\rho(t)r(t)\psi(x(t))}\right)\left[B+\int_T^t \frac{\omega^2(s)g'(x(s))}{\rho(s)r(s)\psi(x(s))}ds\right]^{-1} \geq \frac{g'(x(t))\dot{x}(t)}{g(x(t))}, \quad t \geq T_1,$$

and consequently for all $t \geq T_1$, we obtain

$$\ln\frac{1}{\theta}\left[B+\int_T^t \frac{\omega^2(s)g'(x(s))}{\rho(s)r(s)\psi(x(s))}ds\right] \geq \ln\frac{g(x(T_1))}{g(x(t))}.$$

Hence for $t \geq T_1$, we have

$$B+\int_T^t \frac{\omega^2(s)g'(x(s))}{\rho(s)r(s)\psi(x(s))}ds \geq \theta\frac{g(x(T_1))}{g(x(t))}, \qquad t \geq T_1.$$

So, inequality (3-10) yields

$$\omega(t) \leq -\left[B+\int_T^t \frac{\omega^2(s)g'(x(s))}{\rho(s)r(s)\psi(x(s))}ds\right]$$

$$\frac{\rho(t)r(t)\psi(x(t))\dot{x}(t)}{g(x(t))} \leq -\theta\frac{g(x(T_1))}{g(x(t))}.$$

Then, for $t \geq T_1$, we have

$$\dot{x}(t) \leq -\theta\frac{g(x(T_1))g(x(t))}{\rho(t)r(t)\psi(x(t))g(x(t))}.$$

Then, for every $t \geq T_1$, we obtain

$$\dot{x}(t) \leq \frac{-b}{\rho(t)r(t)}, \qquad t \geq T_1,$$

where $b = \theta\dfrac{g(x(T_1))}{l_3} > 0$.

Thus, for $t \geq T_1$, we have

$$x(t) \leq x(T_1) - b\int_{T_1}^{t}\frac{ds}{\rho(s)r(s)}, \qquad t \geq T_1.$$

This leads to $\lim_{t\to\infty} x(t) = -\infty$.

This contradicts the assumption that $x(t) > 0$; hence, the proof is completed.

Remark 3.6

Theorem 3.7 is an extension of theorem 2.1 of E.M.EL.Abbasy, T.S.Hassan and S.H.Saker [8].

Example 3.7

Consider the differential equation

$$\left[\left(\frac{1}{t}\right)\left(\frac{x^2(t)}{1+x^2(t)}\right)\dot{x}(t)\right]^{\cdot} + \frac{x^5(t)}{t(1+x^2(t))} = -\left[\frac{x^2(t)\dot{x}(t)}{t(1+x^2(t))}\right], \quad t > 1.$$

We note that

(1) $r(t) = \dfrac{1}{t} > 0$ for t >0,

(2) $0 < l_2 \leq \psi(x) = \dfrac{x^2}{1+x^2} < 1$ *for all* $x \in R$,

(3) $g_1(t,x(t)) = \dfrac{x^5}{t(1+x^2)}$ then $q(t) = \dfrac{1}{t}$ *for all* $x \neq 0$ and $t \in [t_0,\infty)$

and $xg(x) = \dfrac{x^6}{1+x^2} > 0$ and $g'(x) = \dfrac{5x^4+3x^6}{(1+x^2)^2} > 0$ *for all* $x \neq 0$.

(4) $H(t,\dot{x}(t),x(t)) = -\left[\dfrac{x^2\dot{x}}{t(1+x^2)}\right]$ then $m_1(t) = \dfrac{1}{t}$ $\forall x,\dot{x} \in R$ and $t \in [t_0,\infty)$.

(5) $0 < \int_{\varepsilon}^{\infty}\dfrac{\psi(u)}{g(u)}du = \int_{\varepsilon}^{\infty}\dfrac{u^2}{1+u^2}\times\dfrac{1+u^2}{u^5}du = \int_{\varepsilon}^{\infty}\dfrac{du}{u^3} = \dfrac{1}{2\varepsilon^2} < \infty$ and

$$\int_{-\varepsilon}^{-\infty}\frac{\psi(u)}{g(u)}du = \int_{-\varepsilon}^{-\infty}\frac{u^2}{1+u^2}\times\frac{1+u^2}{u^5}du = \int_{-\varepsilon}^{-\infty}\frac{du}{u^3} = \frac{1}{2(-\varepsilon^2)} < \infty \qquad \forall\varepsilon > 0.$$

Let $\rho(t) = t$ then $\dot{\rho}(t) = 1 > 0$, $(\rho(t)r(t))^{\cdot} = 0$, $(\rho(t)r(t))^{\cdot\cdot} = 0$ and $\left[r(t)\dot{\rho}(t) - \rho(t)m_1(t)\right] = \dfrac{-1}{t^2} \le 0$.

(6) $\liminf\limits_{t\to\infty}\int_{t_0}^{t}\rho(s)q(s)ds = \liminf\limits_{t\to\infty}\int_{t_0}^{t}ds = \liminf\limits_{t\to\infty} s\Big|_{t_0}^{t} > -\infty$,

(7) $$\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\left[\int_{t_0}^{s}\rho(u)q(u)du\right]^2 ds = \limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\left[\int_{t_0}^{s}du\right]^2 ds = \limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}[s-t_0]^2 ds$$

$$= \limsup_{t\to\infty}\frac{1}{t}\left[\frac{s^3}{3} - s^2t_0 + t_0^2 s\right]_{t_0}^{t}$$

$$= \limsup_{t\to\infty}\frac{1}{t}\left[\frac{t^3}{3} - t^2t_0 + tt_0^2 - \frac{t_0^3}{3}\right]$$

$$= \limsup_{t\to\infty}\left[\frac{t^2}{3} - tt_0 + t_0^2 - \frac{t_0^3}{3t}\right] = \infty,$$

it follows that the given equation is oscillatory by theorem 3.7.

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ملخــص الرسالــة

لقد تم في هذه الرسالة دراسة المعادلات التفاضلية العادية ذات الرتبة الثانية و الدرجة الأولى التي على الصورة:

$$\left(r(t)\psi(x(t))\dot{x}(t)\right)^{\cdot} + g_1(t,x(t)) = H(t,\dot{x}(t),x(t))$$

كما تمت مقارنة النتائج التي تم الحصول عليها بالنتائج السابقة التي قد حصل عليها بعض الباحثين في حالات خاصة من هذه المعادلة التي تم تقديم بعض الأمثلة التي توضح ذلك.

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

﴿ لَا يُكَلِّفُ اللَّهُ نَفْساً إِلاَّ وُسْعَهَا لَهَا مَا كَسَبَتْ
وَعَلَيْهَا مَا اكْتَسَبَتْ رَبَّنَا لاَ تُؤَاخِذْنَا إِن نَّسِينَا أَوْ
أَخْطَأْنَا رَبَّنَا وَلاَ تَحْمِلْ عَلَيْنَا إِصْراً كَمَا حَمَلْتَهُ
عَلَى الَّذِينَ مِن قَبْلِنَا رَبَّنَا وَلاَ تُحَمِّلْنَا مَا لاَ طَاقَةَ
لَنَا بِهِ وَاعْفُ عَنَّا وَاغْفِرْ لَنَا وَارْحَمْنَا أَنتَ
مَوْلاَنَا فَانصُرْنَا عَلَى الْقَوْمِ الْكَافِرِينَ ﴾

صدق الله العظيم

سورة البقرة الآية 286

الجماهيرية العربية الليبية الشعبية الاشتراكية العظمى
جامعة التحدي
سرت

إن الدراسة ليست غاية في حد ذاتها
وإنما الغاية هي خلق الإنسان النموذجي الجديد

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كليـــــة العلـــــوم

قسم الرياضيـــات

عنـــوان البحـــث

دراســة ســلــوك حلــول المعــادلات التفاضليــة العاديــة ذات الرتبــة الثانيــة و الدرجــة الأولـــى

مقدمــة مـــن الطـــالبــة

جفالـــة جمعـــة علـــي

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الجماهيرية العربية الليبية الشعبية الاشتراكية العظمى

جامعة التحدي – سرت

كلية العلوم – قسم الرياضيات

بحث بعنوان:

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استكمالا لمتطلبات الإجازة العالية الماجستير في علوم الرياضيات

مقدمة من الطالبة:

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