AL-TAHADI UNIVERSITY
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS

((INTEGRAL AVERAGES AND OSCILLATION OF SECOND ORDER SUPERLINEAR DIFFERENTIAL EQUATION))

A THESIS SUBMITTED FOR PARTIAL FULFILLMEMENT FOR THE REQUIRMENTS OF THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS BY:

EMBARKA ABDALLH SALHEEN AGROOSH

SUPERVISED BY:

DR. NABEEL ZAKI FREED

SIRT – LIBYA
(2005 – 2006)

جامعة التحدي

سرت

التاريخ :

الموافق : 10 / 7 / 2007

الرقم الاشاري : ك ع / 16 / 1 / 7

Faculty of Science

Department of Mathematics

Title of Thesis

((Integral Averages and Oscillation of Second Order Superlinear Differential Equation))

By

Embarka Abdallh Salheen Agroosh

Approved by:

Dr. Nabil Zaki Farid ....
(Supervisor)

Dr.Mustafa Sharif Shaieb ....
(External examiner)

Dr. Fathalla S. Mestiri ....
(Internal examiner)

Countersigned by:

Dr. Mohamed Ali Salem

(Dean of Faculty of Science)

بسم الله الرحمن الرحيم

((وقل اعملوا

فسيرى الله عملكم ورسوله والمؤمنون))

صدق الله العظيم

من سورة التوبة

Contents

INTRODUCTION

The second – order ordinary differential equations are frequently encountered as mathematical models of most dynamic processes in electromechanical systems. Since these equations are only of the second order,we would naturally be inclined to compute their solutions explicitly or numerically . However, as we know from practice, there are very few such equations, e.g., linear equations with constant coefficients, for which this can be effectively done. In most instances, this can be accomplished only under very restrictive conditions.

Therefore, the problem there for is to find those techniques that will be useful in obtaining some qualitative information about the elusive solutions of these equations.

The qualitative study concerning the oscillation problem of second – order linear differential equations originated in the classic paper of Sturm [35] . However, many authors use some different techniques in studying the oscillatory behavior of the second – order linear differential equations especially what so –called averaging techniques that dates back to works of Wintner [37] and its generalization by Hartman [15] .

The problem of determining oscillation criteria for second order nonlinear differential equations has received a great deal of attention in the twenty years following the publication of the new–classic paper by Atkinson [2]. The study of the oscillation of second – order nonlinear ordinary differential equations with alternating coefficients is of special interest because of the fact that many physical systems are modeled by second-order nonlinear ordinary differential equations. for example, the so –called Emden – Fowler equation arises in the study of gas dynamics and fluid mechanics. This equation appears also in the study of relativistic mechanics, nuclear physics and in the study of chemically reacting systems. The averaging techniques are used, also, in study of the nonlinear oscillations.

In this these we consider the problem of determining oscillation and boundness criteria for the second order nonlinear perturbed differential equation of the form:

$$\left(r(t)\Psi(x(t))f(\dot{x}(t))\right)^{\cdot} + q(t)g(x(t)) = H(t,\dot{x}(t),x(t)) \qquad \text{(E)}$$

Our results are fairly sharp will be illustrated by some examples. We apply the averaging technique to discuss the oscillation of this equation.

In many instances our result will include, as special cases, known oscillation theorems for less general equations.

We divided the thesis into three chapters as follows:

Chapter (1) This chapter is an introductory chapter and it deals with the problem of oscillation of second – order ordinary differential equations. It contains some basic definitions, elementary results that will be used throughout the next chapters and most of the main results of oscillation for the second order ordinary differential equations that can be found in the literatures.

Chapter (2) This chapter is devoted to the study of the oscillation of the equation (E) with $H(t,\dot{x}(t),x(t))=0$.

The oscillation criteria that obtained in this chapter contains as special case some of the earlier results that can be found in the literature for the equation (E)with $H(t,\dot{x}(t),x(t))=0$ and /or particular forms and this results will be illustrated by some examples.

Chapter (3) This chapter is devoted to the study of the boundness and the oscillation of the second – order ordinary differential equation (E). Some oscillations criteria for solutions of (E) with alternating coefficient are given. The results extend and improve some results of oscillation that obtained before and these results will be illustrated by some examples.

CHAPTER (1)

OSCILLATION OF THE EQUATION

1.1 Introduction

In this chapter we study the oscillation of the solutions of the second order ordinary differential equations of the form :

$$\ddot{x}(t) + q(t)x(t) = 0 \qquad (E_1)$$

$$\left(r(t)\dot{x}(t)\right)^{\cdot} + q(t)x(t) = 0 \qquad (E_2)$$

$$\ddot{x}(t) + q(t)g(x(t)) = 0 \qquad (E_3)$$

$$\left(r(t)\dot{x}(t)\right)^{\cdot} + q(t)g(x(t)) = 0 \qquad (E_4)$$

$$\left(r(t)\Psi(x(t))\dot{x}(t)\right)^{\cdot} + q(t)g(x(t)) = 0 \qquad (E_5)$$

$$\ddot{x}(t) + q(t)g(x(t)) = p(t) \qquad (E_6)$$

Where r, q and p are continuous functions on $[t_0,\infty)$ and Ψ and g are continuous functions on R.

Section 1.2: This section contains basic definitions and elementary results that will be used throughout the next chapters.

Section 1.3: This section is devoted to the oscillation of equation (E_1).

Section 1.4: This section is devoted to the oscillation of equation (E_2).

Section 1.5: This section is devoted to the oscillation of equation (E_3).

Section 1.6: This section is devoted to the oscillation of equation (E_4).

Section 1.7: This section is devoted to the oscillation of equation (E_5).

Section 1.8: This section is devoted to the oscillation of equation (E_6).

The oscillation properties of the second order differential equation have been the subject of interest of many authors since the early paper by Fite [10].

The investigation of the oscillation of (E) may be done by following many directions, many of these, often consider the way of determining "integral tests " involving function q in order to obtain oscillation criteria ,whenever $\int^{\infty}$ is written ,it is to be assume that

$$\int^{\infty} = \lim_{t\to\infty} \int^{t} .$$

and that this limit exists in the extended real number $[-\infty,\infty]$.

1.2 PRELIMINARY RESULTS AND DEFINITIONS

Definition 1.2.1

A point t = $\tau \geq 0$ is called a zero of the solution $x(t)$ if $x(\tau) = 0$.

Definition 1.2.2

A solution $x(t)$ will be called non oscillatory if there exists a point $t_0 \geq 0$ such that $x(t) \neq 0 \ \forall t \geq t_0$,and $x(t)$ will be called oscillatory if it has no last zero.

Definition 1.2.3

(a) If all the solutions of a differential equation are oscillatory, then, equation is called oscillatory.

(b) The equation is called non oscillatory if (a) is false.

Some important elementary oscillation criteria are derived using the following famous Sturm's Comparison.

Theorem 1.2.1(Sturm's Comparison theorem) [33]

Let $r(t)>0$ and $q_1(t), q_2(t)$ and $r(t)$ be continuous functions on (a,b)

Assume that $x(t), y(t)$ are real solutions of

$$\left(r(t)\dot{x}(t)\right)^{\cdot} + q_1(t)x(t) = 0 \qquad (1\text{-}1)$$

$$\left(r(t)\dot{y}(t)\right)^{\cdot} + q_2(t)y(t) = 0 \qquad (1\text{-}2)$$

Respectively on (a,b). Further, let $q_2(t) \geq q_1(t)$ *for* $t \in (a,b)$.then between any two consecutive zeros t_1, t_2 of $x(t)$ in (a,b) there exists at least one zero of $y(t)$ unless $q_1(t) = q_2(t)$ on $[t_1, t_2]$. Moreover, in this case the conclusion is still true if the solution y(t) is linearly independent of $x(t)$.

Theorem 1.2.1 states, as a special case, that if $q_1(t) = q_2(t) = q(t)$ and $x(t), y(t)$ are two linearly independent solutions of (E_2), then between any two consecutive zeros of $y(t)$ (or $x(t)$) there exists at least one zero of $x(t)$ (or $y(t)$). So the zeros of $x(t)$ and $y(t)$ separate each other. Therefore, the following result is directly satisfied.

Corollary 1.2.1

The equation of the form (E_2) is either oscillatory or non oscillatory that is, any one of these equations can not have both oscillatory and non oscillatory solutions.

An immediate consequence of Theorem 1.2.1 is the following result.

Corollary 1.2.2

Assuming the same conditions of theorem 1.2.1then,

(a) If equation (1-1) is oscillatory, then equation (1-2) is also oscillatory.

(b) If equation (1-2) is non oscillatory, then equation (1-1) is also non oscillatory.

Since the equation $\ddot{x} + m^2 x = 0$ is oscillatory, using the above corollary, we have

Corollary 1.2.3

If $q(t) \geq m^2 > 0$, then $\ddot{x} + q(t)x = 0$ is oscillatory.

Definition 1.2.4

The differential equation $(E_i, i \geq 3)$ is called:

(I) Sub linear if the function g satisfies that

$$0 < \int_0^{\varepsilon} \frac{du}{g(u)} < \infty \quad and \quad 0 < \int_0^{-\varepsilon} \frac{du}{g(u)} < \infty \;\; \forall \varepsilon > 0$$

(2) Super linear if the function g satisfies that

$$0 < \int_{\varepsilon}^{\infty} \frac{du}{g(u)} < \infty \quad and \; 0 < \int_{-\varepsilon}^{-\infty} \frac{du}{g(u)} < \infty \quad \forall \varepsilon > 0$$

The following theorem (1.2.2) is quit useful element in our study in the following chapters.

Theorem 1.2.2(The Bonnet theorem) [3]:

Suppose that h is continuous function on $[a,b]$ *and* ρ is non negative function on $[a,b]$:

If ρ is increasing function on $[a,b]$, then there exists a point $c \in [a,b]$ such that

$$\int_a^b \rho(s)h(s)ds = \rho(b)\int_c^b h(s)\,ds$$

If ρ is decreasing function on $[a,b]$, then there exists a point $c \in [a,b]$ such that

$$\int_a^b \rho(s)h(s)\,ds = \rho(a)\int_a^c h(s)\,ds$$

This theorem is a part of the second mean value theorem for integrals [3] .

1.3 The oscillation of (E_1)

This section is devoted to the oscillation criteria for the second order linear differential equation of the form (E_1), the oscillation of equation (E_1) has brought the attention of many authors since the early paper by Fite [10].

Among the numerous papers dealing with this subject we refer in particular to the following:

Theorem 1-3-1 Fite [10] Suppose that q is a positive function on $[t_0,\infty)$ and

$$\lim_{t\to\infty}\int_{t_0}^{t} q(s)\,ds=\infty,$$

Then equation (E_1) is oscillatory.

The following theorem extends the results of Fite to an equation in which q is of arbitrary sign.

Theorem 1-3-2 Wintener [37]

$$\text{If}\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}(t-s)\,q(s)\,ds=\infty\ ,$$

Then every solutions of equation (E_1) are oscillatory.

Example 1.3.1 consider the following differential equation

$$\ddot{x}(t)+(7+5\cos t)x(t)=0 \qquad ,\ t\geq 0$$

Theorem 1-3-2 ensure that the given equation is oscillatory. how ever Theorem 1-3-1 we can not use it here.

In the following, Kamenev [16] as proved a new integral criterion for the oscillation of the differential equation (E_1) based on the use of the n th primitive of the coefficient, *q(t)*, which has Wintner's result [37] as a particular case.

Theorem 1-3-3 Kamenev[16]

If $\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s)\, ds = \infty$, *for some* $n \geq 3$,

then equation (E_1) is oscillatory.

Philos [26] improves the bove Kamenv's result.

Theorem 1-3-4 Phios[26]

Let ρ be a positive continuously differentiable function on the interval $[t_0, \infty)$ such that

(i) $\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} \frac{(t-s)^{n-3}}{\rho(s)} \left[(n-1)\rho(s) - (t-s)\dot{\rho}(s) \right] ds < \infty$ for some integer $n \geq 3$.

(ii) $\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} \rho(s) q(s)\, ds = \infty$,

then equation (E_1) is oscillatory .

Remark 1-3-1

By setting $\rho(t) = 1$ in the above theorem 1-3-4, theorem 1-3-4 leads to Kamenev's result [16] (theorem 1-3-3) for equation (E_1).

Yan [41] resented another new oscillation theorem for equation (E_1).

Theorem 1-3-5 Yan[41]

Suppose that

$$\limsup_{t\to\infty}\frac{1}{t^{n-1}}\int_{t_0}^{t}(t-s)^{n-1}q(s)ds<\infty \text{ , for some integer } n\ge 3$$

Let $\Omega(t)$ be a continuous function on $[t_0,\infty)$ with

$$\liminf_{t\to\infty}\frac{1}{t^{n-1}}\int_{T}^{t}(t-s)^{n-1}q(s)\,ds\ge\Omega(T) \quad \text{for every } T\ge t_0 .$$

$$\lim_{t\to\infty}\int_{t_0}^{t}\Omega_+(s)ds=\infty \text{ , where } \Omega_+(t)=\max\{\Omega(t),0\},\ t\ge t_0 .$$

then equation (E_1) is oscillatory .

Philos [30] extended the Kamenev's result [16] as follows

Theorem 1-3-6 Philos[30]

Let H and h be continuous functions,

$$h,H:D\equiv\{(t,s):t\ge s\ge t_0\}\to R \text{ ,}$$

and H has a continuous and non positive partial derivative on with respect to the second variable such that

$$H(t,t)=0 \quad for\ t\ge t_0 \quad ,H(t,s)>0 \quad for\ t>s\ge t_0 .$$

and

$$\frac{-\partial}{\partial s}H(t,s)=h(t,s)\sqrt{H(t,s)} \quad for\ all\ (t,s)\in D.$$

then equation (E_1) is oscillatory. If

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left(H(t,s)q(s)-\frac{1}{4}h^2(t,s)\right)ds=\infty.$$

Also, Philos [30] extended and improved Yan's result [41]in the following theorem

Theorem1-3-7 Philos[30]

Let H and h as in Theorem 1-3-6. Moreover, suppose that

$$0<\inf_{s\ge t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right]\le\infty \quad \text{and}$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}h^2(t,s)\,ds<\infty$$

Assume that $\Omega(t)$ as in Theorem 1-3-5 with

$$\lim_{t\to\infty}\int_{t_0}^{t}\Omega_+^2(s)ds=\infty,$$

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left[H(t,s)q(s)-\frac{1}{4}h^2(t,s)\right]ds\ge\Omega(T) \qquad \textit{for every}\,T\ge t_0.$$

then equation (E_1) is oscillatory.

Remark 1-3-2

An though there is an extensive literature on the topic of oscillation criteria of (E_1) no completely .satisfactory answer has yet been obtained because, as for as we know, necessary and sufficient conditions ensuring the oscillation of (E_1) ,in which only the function q is involved do not appear in the literature.

1.4 the oscillation of (E_2)

This section is devoted to the study of the oscillation of the equation (E_2). It is of interest to discuss conditions on the alternating coefficient $q(t)$ which are sufficient for all solution of (E_2) to be oscillation.

An interesting case is that of finding oscillations criteria of (E_2) which involve the average behavior of the integral of q. This problem has received the attention of many authors in recent years.

Among numerous papers dealing with such averaging thechniques in the oscillation of (E_2), we choose to mention to the following :

Moore [21] gives the following oscillation criteria for (E_2).

Theorem 1-4-1 Moore [21]

If the function ρ satisfies $\rho \in c^2[t_0,\infty), \rho(t) > 0$,

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{ds}{r(s)\rho^2(s)} = \infty$$

and

$$\lim_{t\to\infty} \int_{t_0}^{t} \left[\rho(s)(r(s)\dot{\rho}(s))' + \rho(s)q(s) \right] ds = \infty .$$

then equation (E_2) is oscillatory.

In fact, Emil Popa [32] extends Kamenev's oscillation criterion to apply on equation of the form (E_2), he proved the following two theorems:

Theorem 1-4-2 Popa [32]

Suppose that r(t) is bounded above and

$$\limsup_{t\to\infty}\frac{1}{t^{n-1}}\int_{t_0}^{t}(t-s)^{n-1}q(s)ds=\infty \quad , n \text{ is an integer, } n>2 ,$$

then equation (E_2) is oscillatory.

He proved also that :

Theorem 1-4-3Popa [32]

Let $\frac{\dot{r}(t)}{r(t)}$ be bounded and

$$\limsup_{t\to\infty}\frac{1}{t^{n}}\int_{t_0}^{t}(t-s)^{n-1}\frac{q(s)}{r(s)}ds=\infty \text{ , n is an integer, } n>2 ,$$

then equation (E_2) is oscillatory.

1.5 the oscillation of (E_3)

This section is devoted to the oscillation criteria for the second order non linear differential equation of the form (E_3).

The oscillation of equation (E_3) has brought the attention of many authors since the early paper by Atkinson [2]. The prototype of equation (E_3) is so called the Emden-Fowler equation:

$$\ddot{x}(t)+q(t)|x(t)|^{\gamma}\operatorname{sgn}x(t)=0 \qquad ,\gamma>0 \qquad\qquad (EF)$$

Clearly equation (EF) is sub linear if $\gamma < 1$ and super linear if $\gamma > 1$.

The oscillation problem for second order nonlinear differential equation is of particular interest. Many physical systems are modeled by second order nonlinear ordinary differential equations. For example, equation (EF) arises in the study of gas dynamics and fluid mechanics, nuclear physics and chemically reacting systems.

The study of Emden –Fowler equation originates from earlier theorems concerning gaseous dynamics in as trophyics around the turn of the century. For more details for the equation we refer to the paper by Sevelo [34] or a detailed account of second order non - linear oscillation and its physical motivation.

There has recently been an increasing in studying the oscillation for equation (E_1) and (EF). We list some of more important oscillation criteria as follows.

The following theorem gives the necessary and sufficient conditions for oscillation of (E_3) with $g(x)=x^{2n+1}$, $n=1,2,3,...$

Theorem 1-5-1 Atkinson [2]

If q(t) >0 on $[t_0,\infty)$ and

$g(x)=x^{2n+1}$, $n=1,2,....$ then equation (E_3) is oscillatory if

$$\int_{t_0}^{\infty} s q(s) ds = \infty .$$

Waltman [36] extended Wintner's result [37](which presented to (E_1)) for the equation which considered by Atkinson [2]without any restriction on the sign of $q(t)$.

Theorem 1-5-2 Waltman [36]

If $g(x)=x^{2n+1}$, $n=1,2,...$

And

$$\lim_{t\to\infty}\int_{t_0}^{t} q(s)\,ds = \infty, \text{ then equation } (E_3) \text{ is oscillatory.}$$

Kiguradze [17] established the following theorem for the Emden-Fowler equation (EF).

Theorem 1-5-3 Kiguradze [17]

If $\int^{\infty} \rho(t)q(t)\,dt = \infty$

then equation (EF) is oscillatory for $\gamma > 1$. For a continuous, positive and concave function $\rho(t)$.

Wong [38]extended Wintner's oscillation criteria [37] to apply on the equation of the form(EF).

Theorem 1-5-4 Wong [38]

If $\gamma > 1$.Equation (EF) is oscillatory if

$$\liminf_{t\to\infty} \int_{t_0}^{t} q(s)\,ds > -\infty \text{ and } \limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} (t-s)\,q(s)\,ds = \infty .$$

Onose [25]proved a theorem of Wong's (Theorem 1-3-4) for the sub linear Emden-Fowler differential equation and also study the extension of Wong's result [39]to the more general super linear differential equation of the form (E_3) as in the following three theorems.

Theorem 1-5-5 Onose[25]

If

$$\text{(i)}\ \liminf_{t\to\infty}\int_{t_0}^{t} q(s)\,ds > -\lambda > -\infty \quad , \lambda > 0.$$

$$\text{(ii)}\ \limsup_{t\to\infty}\int_{t_0}^{t} q(s)\,ds = \infty\ ,$$

$$\text{(iii)}\ \limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(u)\,du\,ds = \infty.$$

then (EF) is oscillatory for $0<\gamma<1$.

Theorem 1-5-6 Onose [25]

Suppose that

$$\text{(1)}\ \liminf_{t\to\infty}\int_{t_0}^{t} q(s)\,ds \geq 0$$

$$\text{(2)}\ \limsup_{t\to\infty}\int_{t_0}^{t} q(s)\,ds = \infty\ ,$$

then equation (E_3) is oscillatory.

Theorem 1-5-7 Onose [25]

Let

$$\text{(1)}\ \liminf_{t\to\infty}\int_{t_0}^{t} q(s)\,ds > -\lambda > -\infty\ , \ \lambda > 0$$

(2) $\limsup_{t\to\infty} \int_{t_0}^{t}\int_{t_0}^{s} q(u)\, du\, ds = \infty$.

then equation (E_3) is oscillatory.

C.C.Yeh[42]stablished a new integral criteria for the equation (E_3) which has Wintner's result [37]as a particular case.

Theorem 1-5-8 Yeh [42]

If $\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s)\, ds = \infty$,for some integer n>2.

then equation (E_3) is oscillatory.

Philos [27]gave a new oscillation criteria for the differential equation (EF) with $0<\gamma<1$.

Theorem 1-5-9 Philos[27]

Let ρ be a positive continuous differential function on the interval $[t_0,\infty)$ such that

$$\gamma\rho(t)\ddot{\rho}(t)+(1-\gamma)\dot{\rho}^2(t)\le 0 \quad \forall t\ge t_0 \text{ and}$$

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} \rho(s) q(s)\, ds = \infty \text{ for some integer } n\ge 2,$$

then (EF) is oscillatory .

Philos [28]improved Onos 's result [25]which presented for the equation (E_3).

Theorem 1-5-10 Philos[28]

Suppose that

ρ is a positive twice continuously differentiable function on $[t_0,\infty)$ such that

$\dot{\rho}(t) \geq 0 \quad and \quad \ddot{\rho}(t) \leq 0 \quad on[t_0,\infty)$,

$$\lim_{t\to\infty} \int_{t_0}^{t} \rho(s)\, q(s)\, ds > -\infty \text{ and}$$

$$\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} (t-s)\, \rho(s) q(s)\, ds = \infty \ ,$$

then, equation (E_3) is oscillatory .

F.H. Wong and C.C.Yeh [39] proved an analogous result of Wong's result for (EF) to the more nction ρ on general equation (E_3).

Theorem 1-5-11 Wong and Yeh [39]

If $$\liminf_{t\to\infty} \int_{T}^{t} q(s)\, ds \geq 0$$

for all large T and there exists a positive concave function ρ on $[t_0,\infty)$ such that

$$\limsup_{t\to\infty} \frac{1}{t^{\beta}} \int_{t_0}^{t} (t-s)^{\beta} \rho(s) q(s)\, ds = \infty, \text{ for some } \beta \geq 0,$$

Then, the super linear differential equation is oscillatory (E_3) .

Theorem 1-5-12 Philos and Purnaras[31]

Suppose tha

(i) $\liminf_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s)\, ds > -\infty$ .for some integer n≥2

(ii) $\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \left(\int_{t_0}^{s} q(u)\, du \right)^2 ds = \infty$,

then sub linear differential equation (E_3) is oscillatory.

1.6 the oscillation of (E_4)

This section is devoted to the oscillation criteria for the second order non linear differential equation of the form (E_4).

Bhatia [4] presented the following oscillation criteria for the general equation (E_4) which contains as a special case and Wltman's result[36] for the non linear case.

Theorem 1-6-1 Bhatia [4]

If

1- $\lim_{t\to\infty} \int_{t_0}^{t} \frac{ds}{r(s)} = \infty$,

2- $\lim_{t\to\infty} \int_{t_0}^{t} q(s)\, ds = \infty$,

then equation (E_4) is oscillatory.

El-Abbasy [8] improved and extended the results of Philos [31] to the equation (E_4).

Theorem 1-6-2 El-Abbasy [8]:

Suppose that

(1) $\liminf_{t\to\infty} \int_{t_0}^{t} \rho(s)\, q(s)\,ds > -\infty.$

(2) $\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \left(\int_{t_0}^{s} \rho(u)q(u)\,du\right)^2 ds = \infty$

where $\rho : [t_0, \infty) \to (0, \infty)$ is continuously differentiable function such that

$\dot{\rho}(t) \geq 0$, $(r(t)\rho(t))^{\cdot} \geq 0$, $(r(t)\rho(t))^{\cdot\cdot} \leq 0$ and $\left(r(t)\dot{\rho}(t)\right)^{\cdot} \leq 0$,

then equation (E_4) is oscillatory.

1.7 the oscillation of (E_5)

This section is devoted to the study of the oscillation of the equation (E_5).

Theorem 1-7-1 Grace [12]

Suppose that

(1) $\frac{g'(x)}{\Psi(x)} \geq k > 0$ *for* $x \neq 0$,

(2) $\int^{\infty} \frac{du}{g(u)} < \infty$ *and* $\int^{-\infty} \frac{du}{g(u)} < \infty$. Moreover, assume that there exist a differentiable function $\rho : [t_0, \infty) \to (0, \infty)$ and the continuous functions

h,H : D $= \{(t, s) : t \geq s \geq t_0\} \to$ R,

and H has a continuous and non positive partial derivative with respect to the second variable such that

H(t,t) = 0 for $t \geq t_0$ H(t,s) > 0 for t > s $\geq t_0$,and

$$\frac{-\partial}{\partial s}H(t,s)=h(t,s)\sqrt{H(t,s)} \ \ for\ all\,(t,s)\in D.$$

If

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\rho(s)q(s)-\frac{r(s)\rho(s)}{4k}\left(h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right)^2\right]ds=\infty$$

then equation (E_5) is oscillatory.

__**Theorem 1-7-2 Grace [12]**__

Let condition (1) from theorem (1-7-1) holds and let the functions H,h, and ρ be defined as in theorem (1-7-1) and moreover, suppose that

$$0<\inf_{s\ge t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right]\le\infty,$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}r(s)\rho(s)\left(h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right)ds<\infty,$$

If there exists a continuous function Ω on $[t_0,\infty)$ such that

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}H(t,s)\rho(s)q(s)-\frac{r(s)\rho(s)}{4k}\left(h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right)^2 ds\ge\Omega(T)$$

for every $T\ge t_0$.

and

$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{\Omega_+^2(s)}{r(s)\rho(s)}ds=\infty, \text{ where } \Omega_+(t)=\max\{\Omega(t),0\},$$

then every solutions of (E_5) are oscillatory.

Theorem 1-7-3 Grace [12]

Suppose that the condition (1) from theorem (1-7-1) holds and the functions H.h, and ρ is defined as in theorem (1-7-1) and

$$\dot{\rho}(t)\geq 0 \ \ and\left(r(t)\,\dot{\rho}(t)\right)^{\bullet}\leq 0 \ \ for\ t\geq t_0.$$

and moreover ,suppose that

$$\liminf_{t\to\infty}\int_{t_0}^{t}\rho(s)q(s)ds>-\infty ,$$

And

$$\int_{t_0}^{\infty}\frac{1}{r(s)\rho(s)}ds=\infty ,$$

then equation (E_5) is oscillatory if there exists a continuous function Ω on $[t_0,\infty)$ such

that

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left[H(t,s)\rho(s)q(s)-\frac{r(s)\rho(s)}{4k}\left(h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right)^2\right]ds\geq\Omega(T)$$

for every $T\geq t_0$, and

$$\int_{t_0}^{\infty}\frac{\Omega_+^2(s)}{r(s)\rho(s)}ds=\infty \text{ where } \Omega_+(t)=\max\{\Omega(t),0\}.$$

1.8 the oscillation of (E_6)

This section is devoted to the study of the oscillation equation (E_6). Many authors are concerned with the oscillation criteria of solutions of the homogeneous second order nonlinear differential equations, however, for the nonhomogeneous equations, little are known.

Greaf, Rankin and Spikes [14] give the following theorem for the non homogeneous equation (E_6).

Theorem 1-8-1 Greaf, Rankin and Spikes [14]

If

$$\text{(i) } \lim_{t\to\infty}\int_{t_0}^{t}\frac{ds}{r(s)}=\infty,$$

$$\text{(ii) } \lim_{t\to\infty}\int_{t_0}^{t}(q(s)-p(s))=\infty,$$

then equation (E_6) is oscillatory.

Example 1-8-1 :

Consider the differential equation

$$\left(t\dot{x}(t)\right)^{\bullet}+(7+\cos t)x^{7}(t)=\frac{1}{t^{3}} \quad , t>0$$

Theorem 1-8-1 ensure that the given equation is oscillatory

Theorem 1-8-2 Greaf, Rankin and Spikes [14]

Suppose that

1- $\int_{t_0}^{\infty} \frac{ds}{r(s)} = \infty$,

2- $\lim_{t \to \infty} \int_{t_0}^{t} (q(s) - p(s))\, ds < \infty$,

3- $\liminf_{t \to \infty} \int_{t_0}^{t} \frac{1}{r(s)} \int_{t_0}^{s} (q(u) - p(u))\, du\, ds = \infty$.

then the super linear differential equation (E_6) is oscillatory.

Theorem 1-8-3 Greaf, Rankin and Spikes [14]

If

$$\int_{t_0}^{\infty} \frac{M}{r(s)} ds - \int_{t_0}^{\infty} \frac{1}{r(s)} \int_{t_0}^{s} (q(u) - p(u))\, du\, ds = -\infty \text{, for every constant } M,$$

then the sub linear differential equation (E_6) is oscillatory.

Theorem 1-8-4 Greaf, Rankin and Spikes [14]

Suppose that

(1) $r(t) \le r_1$,

(2) $\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}(q(u)-p(u))\,du\,ds=\infty$,

then all solutions of (E_6') are oscillatory.

Theorem 1-8-5 Greaf, Rankin and Spikes [14]

Suppose that

(i) $r(t)\le r_1$,

(ii) $\liminf_{t\to\infty}\int_{T}^{t}(q(s)-p(s))\,ds>-\lambda$ for all large T,

(iii) $\limsup_{t\to\infty}\frac{1}{t}\int_{T}^{t}\int_{T}^{s}(q(u)-p(u))\,du\,ds=\infty$,

then equation (E_6) is oscillatory.

Theorem 1-8-6 Greaf, Rankin and Spikes [14]

Suppose that

1) $r(t)=1$,

2) $\lim_{t\to\infty}\int_{t_0}^{t}s(q(s)-p(s))\,ds=\infty$,

then, the super linear differential equation (E_6) is oscillatory.

Example 1-8-2

Consider the differential equation

$$\left(t^4 \dot{x}(t)\right)^{\bullet} + (11+\cos t)x^{\frac{1}{5}}(t) = \frac{x^{\frac{1}{5}}}{7(|x|+1)} \quad ,t>0$$

Then theorem 1-8-6 ensure that the given equation is oscillatory.

El -Abbasy [7] gives the following theorem for nonhomogeneous equation (E_6).

Theorem 1-8-7 El-Abbasy [7]

If

1- $r(t)=1$,

2- $G(x)=\int_0^x g(u)\,du \to \infty \quad as \ |x| \to \infty$,

3- $p(t)$ is continuous real-valued function in every finite interval,

4- $q(t)>0$ for $t \geq t_0 > 0$,

Let $\rho(t)>0$ such that

5- $\lim_{t\to\infty} \int_{t_0}^{t} (\rho(s)|p(s)| - \frac{\dot{q}(s)}{q(s)})\,ds < \infty$,

6- $\lim_{t\to\infty} \int_{t_0}^{t} \frac{|p(s)|^2}{\rho(s)q(s)}\,ds < \infty$, 7- $\lim_{t\to\infty} \int_{t_0}^{t} \frac{|p(s)'|}{\rho(s)} ds < \infty$,

8- $\left(\frac{1}{\rho(t)}\right)'$ is positive and decreasing for $t \geq t_0 > 0$,

9- $\lim_{|x|\to\infty} \left|\frac{g(x)}{x}\right| = \infty$,

10- $\liminf_{t\to\infty} \int_{t_0}^{t} \frac{q(s)}{\rho(s)} ds > -\infty$,

11- $\limsup_{t\to\infty} \int_{t_0}^{t} (t-s)\frac{q(s)}{\rho(s)} ds = \infty$, Then, all bounded solutions of equation (E_6) are oscillatory.

CHAPTER (2)

THE OSCILLATION OF THE EQUATION

$$\left(r(t)\Psi(x(t))f(\dot{x}(t))\right)^{\bullet} + q(t)g(x(t)) = 0$$

2.1 Introduction

In this chapter we shall study the oscillatory behavior of the solutions of the differential equation is given by:

$$\left(r(t)\Psi(x(t))f(\dot{x}(t))\right)^{\bullet} + q(t)g(x(t)) = 0 \qquad (2\text{-}1)$$

where q and r are continuous functions on the interval $[t_0,\infty), t_0 \geq 0$, r(t) is a positive function. Ψ is a continuous function on the real line R. f is a continuous function on the real line R with $yf(y) > 0$ for $y \neq 0$ and g is continuously differentiable function on the real line R except possible at 0 with x g(x) >0 and $g'(x) \geq K > 0$ for all $x \neq 0$.

Throughout this study, we restrict our attention only to the solutions of the differential equation $(2-1)$ which exist on some ray $[t_0,\infty), t_0 \geq 0$ may depend on a particular solution.

2.2 OSILLATION OF THE SOLUTIONS

In this section we shall state and prove some sufficient oscillation criteria of the solutions of the equation $(2-1)$.

Theorem 2-1:

Suppose that

O_1 $r(t) \leq k_1$ on $[t_0,\infty)$,

O_2 $0 < k_2 \leq \Psi(x(t)) \leq k_3$ for all $x \in R$,

O_3 $y\, f(y) \geq k_4\, (f(y))^2$ for all $y \in R$ and $k_4 > 0$,

Assume that n be an integer with $n \leq 3$ and ρ be a positive continuously differentiable function on the interval $[t_0,\infty)$ such that

O_4 $$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} \frac{(t-s)^{n-3}}{\rho(s)} \left((n-1)\rho(s) - (t-s)\dot{\rho}(s) \right)^2 ds < \infty,$$

O_5 $$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} \rho(s) q(s) ds = \infty,$$

Then all solutions of (2-1) are oscillatory.

Proof:

Let $x(t)$ be a non oscillatory solution of the differential equation (2-1) and that

$x(t) \neq 0$ For $t \geq T > 0$

Define:

$$\omega(t) = \frac{r(t)\Psi(x(t)) f(\dot{x}(t))}{g(x(t))} \quad on\ [T,\infty), T \geq t_0 > 0$$

Then , for $t \geq T$, we have

$$\dot{\omega}(t)=\frac{\left[r(t)\Psi(x(t))f(\dot{x}(t))\right]^{\bullet}}{g(x(t))}-\frac{r(t)\Psi(x(t))f(\dot{x}(t))g'(x(t))\dot{x}(t)}{g^{2}(x(t))} \qquad \text{for all } t\geq T$$

Hence, for all $t\geq T$, we have

$$\dot{\omega}(t)=-q(t)-\frac{r^{2}(t)\Psi^{2}(x(t))\dot{x}(t)f(\dot{x}(t))g'(x(t))}{r(t)\Psi(x(t))g^{2}(x(t))}$$

From the conditions we have that

$$\dot{\omega}(t)\leq -q(t)-\frac{k_{4}k}{k_{3}k_{1}}\frac{r^{2}(t)\Psi^{2}(x(t))f^{2}(\dot{x}(t))}{g^{2}(x(t))}$$

Then, for all $t\geq T$, we obtain

$$\dot{\omega}(t)\leq -q(t)-A\,\omega^{2}(t) \qquad , for\ t\geq T$$

Where $A=\dfrac{kk_{4}}{k_{1}k_{3}}$ is a positive constant

Hence, for every $t\geq T$, we obtain

$$\int_{T}^{t}(t-s)^{n-1}\rho(s)q(s)\,ds\leq -\int_{T}^{t}(t-s)^{n-1}\rho(s)\dot{\omega}(s)\,ds-A\int_{T}^{t}(t-s)^{n-1}\rho(s)\,\omega^{2}(s)ds$$

$$=(t-T)^{n-1}\rho(T)\omega(T)-\int_{T}^{t}A(t-s)^{n-1}\rho(s)\omega^{2}(s)\,ds$$

$$-\int_{T}^{t}\left((n-1)(t-s)^{n-2}\rho(s)-(t-s)^{n-1}\dot{\rho}(s)\right)\omega(s)\,ds$$

$$= -\int_T^t \left[\sqrt{A(t-s)^{n-1}\rho(s)}\,\omega(s) + \frac{(n-1)(t-s)^{n-2}\rho(s) - (t-s)^{n-1}\dot{\rho}(s)}{2\sqrt{A(t-s)^{n-1}\rho(s)}} \right]^2 ds$$

$$+ \int_T^t \frac{\left[(n-1)(t-s)^{n-2}\rho(s) - (t-s)^{n-1}\dot{\rho}(s)\right]^2}{4A(t-s)^{n-1}\rho(s)} + (t-T)^{n-1}\rho(T)\omega(T)$$

$$\le (t-T)^{n-1}\rho(T)\,\omega(T) + \frac{1}{4A}\int_T^t \frac{(t-s)^{n-3}}{\rho(s)} [(n-1)\rho(s) - (t-s)\dot{\rho}(s)]^2\, ds$$

then ,for all $t \ge T$,we get .

$$\int_T^t (t-s)^{n-1}\rho(s)\, q(s)\, ds \le (t-T)^{n-1}\rho(T)\,\omega(T)$$

$$+ \frac{1}{4A}\int_T^t \frac{(t-s)^{n-3}}{\rho(s)} [(n-1)\rho(s) - (t-s)\dot{\rho}(s)]^2\, ds$$

Now ,we know that .

$$\int_{t_0}^t (t-s)^{n-1}\rho(s)q(s)\,ds = \int_{t_0}^T (t-s)^{n-1}\rho(s)q(s)\,ds + \int_T^t (t-s)^{n-1}\rho(s)q(s)\,ds$$

Dividing this equality by t^{n-1} and taking the limit superior of both sides , we obtain

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^t (t-s)^{n-1}\rho(s)q(s)\,ds = \limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^T (t-s)^{n-1}\rho(s)\,q(s)\,ds$$

$$+ \limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_T^t (t-s)^{n-1}\rho(s)q(s)\,ds$$

$$\le \limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^T (t-s)^{n-1}\rho(s)q(s)\,ds$$

$$+\limsup_{t\to\infty}\frac{1}{t^{n-1}}(t-T)\rho(T)\omega(T)$$

$$+\limsup_{t\to\infty}\frac{1}{4At^{n-1}}\int_T^t\frac{(t-s)^{n-3}}{\rho(s)}[(n-1)\rho(s)-(t-s)\dot{\rho}(s)]^2\,ds$$

$$<\infty\ ,$$

which contradicts O_5. Hence, the proof is completed.

Example (2-1) :

Consider the differential equation

$$\left[\left(\frac{t^2}{t^2+1}\right)\left(\frac{x^2(t)+2}{x^2(t)+1}\right)\left(\dot{x}(t)+\frac{\dot{x}(t)}{(\dot{x}(t))^2+1}\right)\right]^{\cdot}+\left(2x(t)+3x^3(t)+x^5(t)\right)=0. \qquad for\ t>0$$

We note that

$1-0<r(t)=\dfrac{t^2}{t^2+1}<1 \qquad for\ t\geq t_0>0$,

$2-0<1\leq\Psi(x(t))=\dfrac{x^2(t)+2}{x^2(t)+1}\leq 2 \quad for\ all\ x\in R$,

$3-x\,g(x)=x(2x+3x^3+x^5)=2x^2+3x^4+x^6>0$ and

$g'(x)=2+9x^2+5x^4>0\ \ for\ all\ x\neq 0$.

$4-yf(y)=y(y+\dfrac{y}{y^2+1})=y^2+\dfrac{y^2}{y^2+1}>0 \qquad for\ all\ y\neq 0\ and$

$$f^2(y) = y^2 + \frac{2y^2}{1+y^2} + \frac{y^2}{(y^2+1)^2} \le 4y^2$$

Then we have

$$\frac{1}{4} f^2(y) \le y^2 \le y^2 + \frac{y^2}{y^2+1} = yf(y) \qquad \textit{for all } y \in R.$$

By taking $\rho(t)=1$, $t \ge t_0 > 0$, we have

$$5 - \lim_{t\to\infty} \sup \frac{1}{t^{n-1}} \int_{t_0}^{t} \frac{(t-s)^{n-3}}{\rho(s)} \left[(n-1)\rho(s) - (t-s)\,\dot{\rho}(s) \right]^2 ds =$$

$$\lim_{t\to\infty} \sup \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-3}(n-1)^2 \; ds = \lim_{t\to\infty} \frac{(n-1)^2}{n-2} \; \frac{(t-t_0)^{n-2}}{t^{n-1}} = 0 < \infty \; .$$

$$6 - \lim_{t\to\infty} \sup \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} \;\; \rho(s)\; q(s) \;\; ds = \lim_{t\to\infty} \sup \frac{(t-t_0)^n}{n\,t^{n-1}} = \infty \; ,$$

it follows from theorem 2-1 that the equation is oscillatory.

Remark 2-1 :

Theorem 2-1 extends results of Philos's criterion [25],extends the results of Kameneve criterion [16], extends the results of [1] when $p(t)=0$ and the results of [18],[19].[22] and [23] .

Theorem(2-2) :

Suppose that O_2,O_3 hold and moreover , assume that there exists a differentiable function

$$\rho : [t_0, \infty) \to (0, \infty)$$

and the continuous functions

$$h, H : D \equiv \{(t,s) : t \geq s \geq t_0\} \rightarrow R,$$

where H has a continuous and non positive partial derivative on D with respect to the second variable such that

$$H(t,t)=0 \quad for\ t \geq t_0, \quad H(t,s)>0 \quad for\ t > s \geq t_0$$

and

$$\frac{-\partial}{\partial s} H(t,s) = h(t,s)\sqrt{H(t,s)} \qquad for\ all\ \ (t,s) \in D.$$

O_6 $\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} r(s)\rho(s)\left(h(t,s) - \frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)} \right)^2 ds < \infty,$

O_7 $\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)q(s)\,ds = \infty,$

then equation (2-1)is oscillatory.

Proof :

Let x(t) be a non oscillatory solution of equation (2-1),and assume that $x(t)>0$ $for\ t \geq T_1 \geq t_0$.

Define:

$$\omega(t) = \frac{\rho(t) r(t) \Psi(x(t)) f(\dot{x}(t))}{g(x(t))}, \quad t \geq T_1$$

then, for every $t \geq T_1$ we obtain

$$\dot{\omega}(t) = -\rho(t)q(t) + \frac{r(t)\dot{\rho}(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))} - \frac{\rho(t)r(t)\Psi(x(t))f(\dot{x}(t))g'(x(t))\dot{x}(t)}{g^2(x(t))}$$

Hence, for all $t \geq T_1$, we have

$$\dot{\omega}(t) \leq -\rho(t)q(t) + \frac{\dot{\rho}(t)}{\rho(t)}\omega(t) - \frac{kk_4}{k_3}\frac{1}{r(t)\rho(t)}\omega^2(t) \qquad \textit{for all } t \geq T_1$$

Then for all $t \geq T_1 \geq t_0$ we obtain

$$\int_{T_1}^{t} H(t,s)\rho(s)q(s)\,ds \leq -\int_{T_1}^{t} H(t,s)\dot{\omega}(s)\,ds + \int_{T_1}^{t}\frac{\dot{\rho}(s)}{\rho(s)}H(t,s)\omega(s)\,ds$$
$$-B\int_{T_1}^{t}\frac{H(t,s)}{\rho(s)r(s)}\omega^2(s)\,ds$$

where $B = \dfrac{kk_4}{k_3}$ is a positive constant .

$$\int_{T_1}^{t} H(t,s)\rho(s)q(s)\,ds \leq -\left[H(t,s)\omega(s)\Big|_{T_1}^{t} - \int_{T_1}^{t}\frac{\partial}{\partial s}H(t,s)\omega(s)\,ds\right] + \int_{T_1}^{t}\frac{\dot{\rho}(s)}{\rho(s)}H(t,s)\omega(s)\,ds$$
$$-B\int_{T_1}^{t}\frac{H(t,s)}{r(s)\rho(s)}\omega^2(s)\,ds$$

$$= H(t,T_1)\omega(T_1) - \int_{T_1}^{t}\left[h(t,s)\sqrt{H(t,s)} + \frac{\dot{\rho}(s)}{\rho(s)}H(t,s)\right]\omega(s)\,ds - B\int_{T_1}^{t}\frac{H(t,s)}{r(s)\rho(s)}\omega^2(s)\,ds$$

$$= H(t,T_1)\omega(T_1) - \int_{T_1}^{t}\left[\frac{B\,H(t,s)}{r(s)\,\rho(s)}\omega^2(s) + \sqrt{H(t,s)}\left(h(t,s) - \frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right)\omega(s)\right]ds$$

$$= H(t,T_1)\omega(T_1) - \int_{T_1}^{t}\left(\sqrt{B\frac{H(t,s)}{r(s)\rho(s)}}\,\omega(s) + \frac{\sqrt{H(t,s)}\left[h(t,s) - (\frac{\dot{\rho}(s)}{\rho(s)})\sqrt{H(t,s)}\right]}{\frac{2\sqrt{B\,H(t,s)}}{\sqrt{r(s)\rho(s)}}}\right)^2 ds +$$

$$\int_{T_1}^{t} H(t,s)\frac{\left[h(t,s) - \left(\dot{\rho}(s)/\rho(s)\right)\sqrt{H(t,s)}\right]^2}{\frac{4BH(t,s)}{r(s)\rho(s)}}ds$$

$$= H(t,T_1)\omega(T)_1 + \int_{T_1}^{t}\frac{r(s)\rho(s)}{4B}\left[h(t,s) - \dot{\rho}(s)/\rho(s)\right]^2 ds$$

$$- \int_{T_1}^{t}\left[\sqrt{\frac{BH(t,s)}{r(s)\rho(s)}}\,\omega(s) + \frac{1}{2}\sqrt{\frac{r(s)\rho(s)}{B}}\left[h(t,s) - \frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right]\right]^2 ds$$

for all $T_1 \geq t_0$

$$\leq H(t,T_1)\omega(T_1) + \frac{1}{4B}\int_{T_1}^{t} r(s)\rho(s)\left[h(t,s) - \frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right]^2 ds \quad \textit{for all } T_1 \geq t_0 \qquad (2\text{-}1)$$

Now ,dividing by $H(t,t_0)$ and take the upper limit as $t \to \infty$ and from inequality

(2-1) and apply the O_6 , we obtain

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)\rho(s)q(s)\,ds < \infty ,$$

which contradicts O_7 ,Hence the proof is completed .

Example(2-2) :

Consider the following differential equation

$$\left[\left(\frac{t^2+2}{t^2+3}\right)\left(8+\frac{x^{14}(t)}{x^{14}(t)+1}\right)\left(3\dot{x}(t)+\frac{\dot{x}(t)}{\dot{x}^2(t)+2}\right)\right]^{\cdot}+\left(\frac{3}{t}-2\sin t\right)2x^5(t)=0 \quad for\ t\geq t_0>0$$

We note that

(1) $r(t)=\dfrac{t^2+2}{t^2+3}>0 \qquad for\ t\geq t_0>0.$

(2) $0<8\leq\Psi(x(t))=8+\dfrac{x^{14}(t)}{x^{14}(t)+1}<9 \qquad for\ all\ x\in R,$

(3) $yf(y)=3y^2+\dfrac{y^2}{y^2+2}>0 \qquad for\ all\ y\neq 0\ and$

$$f^2(y)=9y^2+\frac{6y^2}{y^2+2}+\frac{y^2}{(y^2+2)^2}\leq 16y^2$$

Then we have

$$\frac{1}{16}f^2(y)\leq y^2\leq 3y^2+\frac{y^2}{y^2+2}=yf(y) \text{ for all } y\in R,$$

(4) $xg(x)=2x^6>0\ \ and\ \ g'(x)=10x^4>0 \qquad \forall x\neq 0,$

let $\rho(t)=2>0$ for $t \geq t_0 > 0$

and $H(t,s)=(t-s)^2$ we get $\frac{\partial}{\partial S}H(t,s)=-2(t-s)$ and

(5) $$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t} r(s)\rho(s)\left[h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right]^2 ds$$

$$=\limsup_{t\to\infty}\frac{1}{(t-t_0)^2}\int_{t_0}^{t}\frac{s^2+2}{s^2+3}(2)(4)\,ds$$

$$=\limsup_{t\to\infty}\frac{8}{(t-t_0)^2}\left[t-\frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}-t_0+\frac{1}{\sqrt{3}}\tan^{-1}\frac{t_0}{\sqrt{3}}\right]$$

$$=0<\infty,$$

(6) $$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\rho(s)q(s)\,ds=\limsup_{t\to\infty}\frac{1}{(t-t_0)^2}\int_{t_0}^{t}(t-s)^2(2)\left(\frac{3}{s}-2\sin s\right)ds$$

$$=\limsup_{t\to\infty}\frac{2}{(t-t_0)}\left[3t^2\ln s-6st+\frac{3}{2}s^2+2t^2\cos s+4t\cos s-2s^2\cos s+4s\sin s+4\cos s\right]_{t_0}^{t}$$

$$=\limsup_{t\to\infty}\frac{2}{(t-t_0)^2}\left[\begin{array}{l}3t^2\ln t-\frac{9}{2}t^2+2t^2\cos t+4t\cos t-2t^2\cos t+4t\sin t+4\cos t\\ -3t^2\ln t_0+6tt_0-\frac{3}{2}t_0-2t^2\cos t_0-4t\cos t_0+2t_0^2\cos t_0-4t_0\sin t_0\\ -4\cos t_0\end{array}\right]$$

$$=\infty,$$

it follows from Theorem (2-2) that the equation is oscillatory.

Remark 2-2 : Theorem (2-2) extends the results of Grace [12] and [23].

We need the following lemma which is extension to lemmas of Erbe [9] ,Wong [37] and Graef and Spikes [13]

Lemma 2-1 :

Suppose that

$i-\lim_{t\to\infty}\frac{1}{r(t)}=A \quad ;A>0$,

$ii-\liminf_{t\to\infty}\int_T^t q(s)\,ds\geq 0 \;\; for\ all\ large\ T$

$iii-0<k_2\leq\psi(x(t))\leq k_3 \;\; for\ all\ x\in R,$

then every non oscillatory solution of equation (2-1) which is not eventually a constant must satisfy $x(t)\,\dot{x}(t)>0 \;\; for\ all\ large\ t.$

proof :

suppose that x(t) > 0 for $t\geq T_1\geq t_0$. If the lemma is not true , then either $\dot{x}(t)<0$ for all large t or $\dot{x}(t)$ oscillates for all large t .

In the former case we may suppose that T1 is sufficiently large such that

$$\int_{T_1}^t q(s)\,ds\geq 0 \;\; for\ t\geq T_1 \;\; and\ \dot{x}(t)<0 \;\; for\ t\geq T_1.$$

Now, integrating the equation (2-1) , we have ,

$$\left(r(t)\Psi(x(t))\,f(\dot{x}(t))\right)-\left(r(T_1)\Psi(x(T_1)\,f(\dot{x}(T_1))\right)+g(x(t))\int_{T_1}^t q(s)\,ds$$
$$-\int_{T_1}^t[\dot{x}(s)g'(x(s))\int_{T_1}^s q(u)\,du\,]ds=0$$

But

$$g(x(t))\int_{T_1}^{t} q(s)\,ds \geq 0 \quad and \quad -\int_{T_1}^{t} \dot{x}(s)\,g'(x(s))\int_{T_1}^{s} q(u)\,du\,ds \geq 0$$

Thus, for every $t \geq T_1$ we get

$$\left(r(t)\Psi(x(t))f(\dot{x}(t))\right) - \left(r(T_1)\Psi(x(T_1))f(\dot{x}(T_1))\right) \leq 0$$

Or

$$\left(r(t)\Psi(x(t))f(\dot{x}(t))\right) \leq \left(r(T_1)\Psi(x(T_1))f(\dot{x}(T_1))\right)$$

Dividing by $r(t)\Psi(x(t))$ we obtain

$$f(\dot{x}(t)) \leq \frac{r(T_1)\Psi(x(T_1))f(\dot{x}(T_1))}{r(t)\Psi(x(t))}$$

Then, for all $t \geq T_1$, we have

$$f(\dot{x}(t)) \leq \frac{r(T_1)\Psi(x(T_1))f(\dot{x}(T_1))}{k_3} \times \frac{1}{r(t)}.$$

Since $\frac{1}{r(t)} \to A$ $(A > 0)$ as $t \to \infty$ and

$$\frac{r(T_1)\Psi(x(T_1))f(\dot{x}(T_1))}{k_3} < 0,$$

One can find a constant $M' > 0$ such that $f(\dot{x}(T_1)) < -M'$, $t \geq T_1 \geq t_0$.

Therefore, there exists a constant $M_1' > 0$ such that $\dot{x}(t) \le -M_1'$, $t \ge T_1$.

Integrating, we get that $x(t) \to -\infty$ as $t \to \infty$,

which contradicts the assumption that $x(t) > 0$ for $t \ge T_1$.

If $\dot{x}(t)$ oscillates ,then there exists sequence $\{\tau_n\} \to \infty$ such that $\dot{x}(\tau_n) = 0$ $(n = 1,2,3,......)$ *for all* $t \ge T_1$.

Define

$$\omega(t) = \frac{r(t)\Psi(x(t))\,f(\dot{x}(t))}{g(x(t))}, \quad t \ge T_1 .$$

Then, we obtain

$$\dot{\omega}(t) = -q(t) - \frac{r(t)\Psi(x(t))\,g'(x(t))\,f(\dot{x}(t))\,\dot{x}(t)}{g^2(x(t))}$$

$$\dot{\omega}(t) \le -q(t) \qquad \textit{for all } t \ge T_1$$

Thus for every $\tau_{n+1} \ge \tau_n$ we get

$$\int_{\tau_n}^{\tau_{n+1}} q(t)\,dt \le -\int_{\tau_n}^{\tau_{n+1}} \dot{\omega}(t)\,dt = \omega(\tau_n) - \omega(\tau_{n+1})$$

$$= \frac{r(\tau_n)\Psi(x(\tau_n))\,f(\dot{x}(\tau_n))}{g(x(\tau_n))} - \frac{r(\tau_{n+1})\Psi(x(\tau_{n+1}))\,f(\dot{x}(\tau_{n+1}))}{g(x(\tau_{n+1}))}$$

$$= 0$$

$$\int_{\tau_n}^{\tau_{n+1}} q(t)\, dt \le 0 ,$$

which contradicts to (ii) ,Hence , the proof is completed.

Theorem (2-3) :

Suppose that O_2 hold and

O_8 $\liminf_{t\to\infty} \int_T^t q(s)ds \ge 0$ *for all large T*

O_9 $\lim_{t\to\infty} \frac{1}{r(t)} = A$, $A > 0$

O_{10} $\limsup_{t\to\infty} \frac{1}{t^\beta} \int_{t_0}^t (t-s)^\beta q(s)ds = \infty$ *for some* $\beta \ge 0$.

then equation (2-1) is oscillatory.

Proof :

Let $x(t)$ be a nonoscillatory solution of equation (2-1) and assume that $x(t) > 0$ $for\ t \ge T_1 \ge t_0$. It follows from Lemma 2-1, that $\dot{x}(t) > 0$ $on [T_2, \infty)$, $\forall\ T_2 \ge T_1$.

Define

$$\omega(t) = \frac{r(t)\Psi(x(t)) f(\dot{x}(t))}{g(x(t))} \qquad for\ t \ge T_2$$

Then, for every $t \geq T_2$, we obtain

$$\dot{\omega}(t) = \frac{\left(r(t)\Psi(x(t))f(\dot{x}(t))\right)^{\cdot}}{g(x(t))} - \frac{r(t)\Psi(x(t))f(\dot{x}(t))g'(x(t))\dot{x}(t)}{g^2(x(t))}$$

Hence, for $t \geq T_2$, we have

$$\dot{\omega}(t) \leq -q(t) \qquad \forall t \geq T_2$$

Then for every $t \geq T_2$

$$\int_{T_2}^{t} (t-s)^{\beta} q(s)ds \leq -\int_{T_2}^{t} (t-s)^{\beta} \dot{\omega}(s)ds$$

By the Bonnet theorem , for a fixed $c_t \in [T_2, t]$ such that

$$-\int_{T_2}^{t} (t-s)^{\beta} \dot{\omega}(s)ds = -(t-T_2)^{\beta} \int_{T_2}^{c_t} \dot{\omega}(s)ds = -(t-T_2)^{\beta} \omega(c_t) + (t-T_2)^{\beta} \omega(T_2)$$

$$\leq (t-T_2)^{\beta} \omega(T_2) \qquad (2\text{-}2)$$

Hence, $t \geq T_2' \geq t_0$, we have

$$\int_{T_2}^{t} (t-s)^{\beta} \omega(s)ds \leq (t-T_2')^{\beta} \omega(T_2) \qquad (2\text{-}3)$$

Now ,dividing by t^{β} and take the upper limit as $t \to \infty$ by taking into account (2-2), we derive

$$\limsup_{t\to\infty} \frac{1}{t^{\beta}} \int_{t_0}^{t} (t-s)^{\beta} q(s)ds < \infty \quad .$$

which contradicts O_8, hence the proof is completed.

Example (2-3) :

Consider the differential equation

$$\left[\left(\frac{t^2+1}{t^2+2}\right)\left(1+\frac{x^4(t)}{x^4(t)+1}\right)\left(\dot{x}(t)-\frac{\dot{x}(t)}{\dot{x}^2(t)+1}\right)\right]^{\cdot}+\left(\frac{1}{t}+\cos t\right)x^5(t)=0 \qquad , t\geq t_0>0$$

We note that

$$(1)\ 0<r(t)=\frac{t^2+1}{t^2+1}\ \ and\ \ \lim_{t\to\infty}\frac{1}{r(t)}=\lim_{t\to\infty}\frac{t^2+2}{t^2+1}=1>0,\quad for\ all\ t\geq t_0>0,$$

$$(2)\ 0<1\leq\Psi(x(t))=1+\frac{x^4(t)}{x^4(t)+1}<2\ \ \forall x\in IR,$$

$$(3)\ yf(y)=y^2-\frac{y^2}{y^2+1}=\frac{y^4}{y^2+1}>0\quad\forall y\neq 0,$$

$$(4)\ xg(x)=x.x^5=x^6>0\ and\ \ g'(x)=5x^4>0\ \ \forall x\neq 0,$$

$$(5)\ \liminf_{t\to\infty}\int_T^t q(s)ds=\liminf_{t\to\infty}\int_T^t\left(\frac{1}{s}+\cos s\right)ds=\liminf_{t\to\infty}\left[\ln t+\sin t-\ln T-\sin T\right]=\infty>0,$$

By taking $\beta=1$,

(6) $\limsup_{t\to\infty}\frac{1}{t^{\beta}}\int_{t_0}^{t}(t-s)^{\beta}q(s)ds=\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}(t-s)\left(\frac{1}{s}+\cos s\right)ds$

$$=\limsup_{t\to\infty}\left[\ln t+\sin t-1-\sin t-\frac{\cos t}{t}-\ln t_0-\sin t_0+\frac{t_0}{t}\sin t_0+\frac{\cos t_0}{t}+t_0\right]=\infty\ ,s$$

it follows from theorem (2-3) that the equation is oscillatory.

Remark 2-3: Theorem (2-3) extends the results of Wong and Yeh [40] with $\rho(t)=1$ and [23].

Theorem (2-4) :

Suppose that O_7, O_8 and O_9 hold and moreover, assume that there exists a differentiable function

$$\rho:[t_0,\infty)\to(0,\infty)$$

such that

$$\rho(t)>0 \text{ and } \left(r(t)\dot{\rho}(t)\right)^{\cdot}\le 0 \text{ for all } t\le t_0 \text{ and}$$

O_{11} $0<\frac{f(y)}{y}\le k_s \quad \forall y\neq 0$

O_{12} $0<\int_{\varepsilon}^{\infty}\frac{du}{g(u)}<\infty \text{ and } \int_{-\varepsilon}^{-\infty}\frac{du}{g(u)}<\infty \quad \forall\varepsilon>0$

O_{13} $\limsup_{t\to\infty}\frac{1}{t^{\beta}}\int_{t_0}^{t}(t-s)^{\beta}\rho(s)q(s)ds=\infty \quad \text{for some } \beta\ge 0,$

then equation (2-1) is oscillatory.

Proof:

Let x(t) a nonoscillatory solution of equation (2-1) and assume that $x(t) > 0$ $for\ t \geq T_1 \geq t_0$, it follows from Lemma 2-1 that $\dot{x}(t) > 0\ on\ [T_2, \infty)\ \forall T_2 \geq T_1$.

Define

$$\omega(t) = \rho(t)\frac{r(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))} \qquad \forall t \geq T_2$$

Then, for every $t \geq T_2$, we obtain

$$\dot{\omega}(t) = \frac{\rho(t)\left(r(t)\Psi(x(t))f(\dot{x}(t))\right)^{\bullet}}{g(x(t))} + \frac{\dot{\rho}(t)\left(r(t)\Psi(x(t))f(\dot{x}(t))\right)}{g(x(t))}$$

$$-\frac{\left[\rho(t)r(t)\Psi(x(t))f(\dot{x}(t))g'(x(t))\dot{x}(t)\right]}{g^2(x(t))}$$

Then, for all $t \geq T_2$, we get,

$$\dot{\omega}(t) \leq \frac{\dot{\rho}(t)\left(r(t)\Psi(x(t))f(\dot{x}(t))\right)}{g(x(t))} - \rho(t)q(t) \qquad \forall t \geq T_2$$

By O_9 we have

$$\dot{\omega}(t) \leq k_5\frac{\dot{\rho}(t)\left(r(t)\Psi(x(t))\dot{x}(t)\right)}{g(x(t))} - \rho(t)q(t) \qquad \forall t \geq T_2$$

Hence, for all t.T_2 with $t \geq T_2$, we obtain,

$$\int_{T_2}^{t}(t-s)^{\beta}\rho(s)q(s)ds \le -\int_{T_2}^{t}(t-s)^{\beta}\dot{\omega}(s)ds + k_3k_5\int_{T_2}^{t}(t-s)^{\beta}\dot{\rho}(s)r(s)\frac{\dot{x}(s)}{g(x(s))}ds \qquad (2\text{-}4)$$

By the Bonnet theorem, for a fixed $a_t \in [T_2, t]$ such that

$$-\int_{T_2}^{t}(t-s)^{\beta}\dot{\omega}(s)ds = -(t-T_2)^{\beta}\int_{T_2}^{a_t}\dot{\omega}(s)ds = -(t-T_2)^{\beta}\omega(a_t) + (t-T_2)^{\beta}\omega(T_2)$$

$$\le (t-T_2)^{\beta}\omega(T_2) \qquad (2\text{-}5)$$

But

$$\left[(t-s)^{\beta}\left(\dot{\rho}(s)r(s)\right)\right]^{\cdot} = (t-s)^{\beta}\left(\dot{\rho}(s)r(s)\right)^{\cdot} - \beta(t-s)^{\beta-1}\left(\dot{\rho}(s)r(s)\right) \le 0$$

By the Bonnet theorem, for a fixed $b_t \in [T_2, t]$ such that

$$\int_{T_2}^{t}(t-s)^{\beta}\left(\dot{\rho}(s)r(s)\frac{\dot{x}(s)}{g(x(s))}\right)ds = (t-T_2)^{\beta}\dot{\rho}(T_2)r(T_2)\int_{T_2}^{b_t}\frac{\dot{x}(s)}{g(x(s))}ds$$

$$= (t-T_2)^{\beta}\dot{\rho}(T_2)r(T_2)\int_{x(T_2)}^{x(b_t)}\frac{du}{g(u)} \qquad (2\text{-}6)$$

From inequalities (2-4),(2-5)and (2-6), we have

$$\int_{T_2}^{t}(t-s)^{\beta}\rho(s)q(s)ds \le (t-T_2)^{\beta}\omega(T_2) + k_3k_5\dot{\rho}(T_2)r(T_2)(t-T_2)^{\beta}\int_{x(T_2)}^{x(b_t)}\frac{du}{g(u)} \qquad (2\text{-}7)$$

Now dividing by t^{β} and take the upper limit as $t \to \infty$, and from inequality (2-7), we have

$$\limsup_{t\to\infty}\frac{1}{t^{\beta}}\int_{t_0}^{t}(t-s)^{\beta}\,p(s)q(s)ds<\infty \quad ,$$

which contradicts O_{13} hence, the proof is completed.

Example (2-4) :

Consider the differential equation

$$\left[\left(\frac{8t^2+3t-5}{8t^2+3t-1}\right)\left(1+\frac{x^2(t)}{x^2(t)+1}\right)\left(\dot{x}(t)-\frac{\dot{x}^{8}(t)}{\dot{x}^{9}(t)+1}\right)\right]'+\left(\frac{1}{t}-\sin t\right)x^3=0\ ,t\geq t_0(t)=0\ \ \forall t>0$$

We note that

$$(1)\ 0<\lim_{t\to\infty}\frac{1}{r(t)}=\lim_{t\to\infty}\frac{8t^2+3t-1}{8t^2+3t-5}=1\quad \forall t\geq t_0>0,$$

$$(2)\ 0<1\leq\Psi(x(t))=1+\frac{x^2(t)}{x^2+1}<2\quad \forall x\in R\ ,$$

$$(3)\ 0<\frac{f(y)}{y}=1+\frac{y^8}{y^8(t)+1}<2\ \forall y\neq 0,$$

$$(4)\ xg(x)=x.x^3=x^4>0\ \ and\ \ g'(x)=3x^2>0\ \ \forall x\neq 0\ ,$$

$$(5)\ 0<\int_{\varepsilon}^{\infty}\frac{du}{g(u)}=\int_{\varepsilon}^{\infty}\frac{du}{u^3}=\frac{1}{2\varepsilon^2}<\infty\quad and\quad \int_{-\varepsilon}^{-\infty}\frac{du}{g(u)}=\int_{-\varepsilon}^{-\infty}\frac{du}{u^3}=\frac{1}{2\varepsilon^2}<\infty\quad \forall\varepsilon>0$$

$$(6)\ \liminf_{t\to\infty}\int_{T}^{t}q(s)ds=\liminf_{t\to\infty}\int_{T}^{t}\left(\frac{1}{s}-\sin s\right)ds=\liminf_{t\to\infty}\left[\ln t+\cos t-\ln T-\cos T\right]=\infty>0$$

$$(7)\ \limsup_{t\to\infty}\frac{1}{t^{\beta}}\int_{t_0}^{t}(t-s)^{\beta}\,p(s)q(s)ds$$

Let $\beta = 1$ *and* $\rho(t) = 1 > 0$, we have $\dot{\rho}(t) = 0$ *and* $\left(r(t)\dot{\rho}(t)\right)^{\cdot} = 0$, we obtain.

$$\limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}(t-s)\left(\frac{1}{s} - \sin s\right)ds$$

$$= \limsup_{t\to\infty} \frac{1}{t}\left[t\ln t + t\cos t - t - t\cos t + \sin t - t\ln t_0 - t\cos t_0 + t_0 + t_0\cos t_0 - \sin t_0\right]$$

$$= \limsup_{t\to\infty}\left[\ln t + \cos t - 1 - \cos t + \frac{\sin t}{t} - \ln t_0 - \cos t_0 + \frac{t_0}{t} + \frac{t_0\cos t_0}{t} - \frac{\sin t_0}{t}\right]$$

$$= \limsup_{t\to\infty}\left[\ln t - 1 + \frac{\sin t}{t} - \ln t_0 - \cos t_0 + \frac{t_0}{t} + \frac{t_0\cos t_0}{t} - \frac{\sin t_0}{t}\right]$$

$$= \limsup_{t\to\infty}\ln t - 1 + \limsup_{t\to\infty}\frac{\sin t}{t} - \ln t_0 - \cos t_0 + t_0\limsup_{t\to\infty}\frac{1}{t}$$

$$+ t_0\cos t_0 \limsup_{t\to\infty}\frac{1}{t} - \sin t_0 \limsup_{t\to\infty}\frac{1}{t} = \infty,$$

It follows from Theorem (2-4) that the equation is oscillatory.

__Remark 2-4__: **Theorem** (2-4) extends the results of Wong and Yeh [40] and [23].

__**Theorem (2-5)**__:

Suppose that O_2,O_8 and O_9 hold. And moreover, suppose that

O_{14} $0 < k_6 \le r(t)$.

O_{15} $0 < k_7 \le \dfrac{f(y)}{y} \quad \forall y \ne 0,$

O_{16} $\limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\left(\frac{1}{r(s)}\int_{t_0}^{s} q(u)du\right)ds = \infty$,

then every solutions of superlinear equation (2-1) are oscillatory .

Proof :

Let x(t) be a non oscillatory solution of equation (2-1) and assume that x(t) > 0 for $t \ge T_1 \ge t_0$. It follows from lemma 2-1, we obtain $\dot{x}(t) > 0$ on $[T_2,\infty), T_2 \ge T_1$

Define :

$$\omega(t) = \frac{r(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))} \qquad for\ t \ge T_2$$

Thus , for every $t \ge t_0$, we obtain

$$\dot{\omega}(t) = -q(t) - \frac{r(t)\Psi(x(t))f(\dot{x}(t))g'(x(t))\dot{x}(t)}{g^2(x(t))}$$

Hence , for all $t \ge T_2$,we have

$$\dot{\omega}(t) \le -q(t) \quad for\ all\ t \ge T_2$$

Thus ,

$$\int_{T_2}^{t}\dot{\omega}(s)\,ds \le -\int_{T_2}^{t} q(s)\,ds$$

$$\omega(t) \le \omega(T_2) - \int_{T_2}^{t} q(s)\,ds$$

By the definition of ω ,we get .

$$\frac{f(\dot{x}(t))}{g(x(t))} \leq \frac{\omega\,(T_2)}{r(t)\Psi(x(t))} - \frac{1}{r(t)\Psi(x(t))}\int_{T_2}^{t} q(s)\,ds$$

$$\frac{k_7\,\dot{x}(t)}{g(x(t))} \leq \frac{1}{k_2 k_6}\omega(T_2) - \frac{1}{k_3 r(t)}\int_{T_1}^{t} q(s)\,ds$$

then , for every $t \geq T_2$, we have .

$$\int_{T_2}^{t}\frac{k_7\,\dot{x}(s)}{g(x(s)}\,ds \leq \frac{\omega(T_2)}{k_2 k_6}(t-T_2) - \frac{1}{k_3}\int_{T_2}^{t}\left(\frac{1}{r(s)}\int_{T_2}^{s} q(u)\,du\right)ds \qquad (2\text{-}8)$$

Dividing (2-8) by t and take the upper limit as $t \to \infty$, we have

$$\limsup_{t\to\infty}\frac{k_7}{t}\int_{x(T_2)}^{x(t)}\frac{du}{g(u)} \leq \limsup_{t\to\infty}\frac{1}{t}\left(\frac{\omega(T_2)}{k_2 k_6}(t-T_2)\right) - \frac{1}{k_3}\limsup_{t\to\infty}\frac{1}{t}\int_{T_2}^{t}\left(\frac{1}{r(s)}\int_{T_2}^{s} q(u)\,du\right)ds$$

$$< -\infty \;,$$

which contradicts , Hence , the proof is completed .

<u>Example (2-5)</u> :

Consider the differential equation

$$\left[\left(\frac{t^2+1}{t^2}\right)\left(4+\frac{x^4(t)}{x^4(t)+1}\right)\left(2\dot{x}(t)-\frac{\dot{x}^7(t)}{\dot{x}^6(t)+1}\right)\right]' + t^3\left(2x(t)+7x^3(t)\right) = 0\;,\; for\; t \geq t_0 > 0$$

We note that

1) $0 < 1 \le r(t) = \dfrac{t^2+1}{t^2}$ *and* $\lim_{t\to\infty} \dfrac{1}{r(t)} = \lim_{t\to\infty} \dfrac{t^2}{t^2+1} = 1 > 0$ *for* $t \ge t_0 > 0$,

2) $0 < 4 \le \Psi(x,t) = 4 + \dfrac{x^4(t)}{x^4(t)+1} \le 5 \quad \forall x \in R$,

3) $\dfrac{f(y)}{y} = 2 + \dfrac{y^6}{y^6+1} > 2 \qquad , \forall y \ne 0$,

4) $xg(x) = x(2x+7x^3) = 2x^2 + 7x^4 > 0$ *and* $g'(x) = 2 + 21x^2 > 0 \;\; \forall x \ne 0$,

5) $\liminf_{t\to\infty} \int_T^t q(s)ds = \liminf_{t\to\infty} \int_T^t s^3 ds = \liminf_{t\to\infty} \left[\dfrac{t^4}{4} - \dfrac{T^4}{4}\right] = \infty > 0$ for all large T .

6) $\limsup_{t\to\infty} \dfrac{1}{t} \int_{t_0}^{t} \left(\dfrac{1}{r(s)} \int_{t_0}^{s} q(u)du \right) ds$

$$= \limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \left(\frac{s^2}{s^2+1} \int_{t_0}^{s} u^3 du \right) ds$$

$$= \limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \frac{s^2}{s^2+1} \left(\frac{s^4}{4} - \frac{t_0^4}{4} \right) ds$$

$$= \limsup_{t\to\infty} \left[\frac{1}{4t} \int_{t_0}^{t} \frac{s^6}{s^2+1} ds - \frac{t_0^4}{4t} \int_{t_0}^{t} \frac{s^2}{s^2+1} ds \right]$$

$$= \limsup_{t\to\infty} \left[\frac{1}{4t} \left(\frac{t^5}{5} - \frac{t^3}{3} + t - \tan^{-1} t - \frac{t_0^5}{5} + \frac{t_0^3}{3} - t_0 + \tan^{-1} t_0 \right) - \frac{t_0^4}{4t} \left(t - \tan^{-1} t - t_0 + \tan^{-1} t_0 \right) \right]$$

$$= \limsup_{t\to\infty}\left[\frac{t^5}{4t}\left(\begin{array}{l}\frac{1}{5}-\frac{1}{3t^2}+\frac{1}{t^4}-\frac{\tan^{-1}t}{t^5}-\frac{t_0^5}{5t^5}+\frac{t_0^3}{3t^3}-\frac{t_0}{t^5}\\+\frac{\tan^{-1}t_0}{t^5}-\frac{t_0^4}{t^4}+\frac{t_0^4\tan^{-1}t}{t^5}+\frac{t_0^5}{t^5}-\frac{t_0^4\tan^{-1}t_0}{t^5}\end{array}\right)\right]=\infty.$$

it follows theorem 2-5 that the equation is oscillatory.

Remark 2-5 : the Theorem (3-4) extends the results of Philos [29]and [23].

Theorem (2-6) :

Suppose that the O_2,O_6 and O_9 hold and

O_{17} $0 < k_7 \le \frac{f(y)}{y} \le k_5 \quad \forall y \neq 0$

Moreover, assume that there exists a differentiable function

$$\rho : [t_0,\infty) \to (0,\infty)$$

such that

$$\rho(t) > 0,\ \dot{\rho}(t) \ge 0 \text{ and } \left(r(t)\dot{\rho}(t)\right)^{\cdot} \le 0, \forall t \ge t_0$$

O_{18} $\lim_{t\to\infty}\int_{t_0}^{t}\rho(s)q(s)ds = \infty$,

O_{19} $\lim_{t\to\infty}\int_{t_0}^{t}\left(\frac{1}{r(s)\rho(s)}\int_{t_0}^{s}\rho(u)q(u)du\right)ds = \infty$,

then every solution of super linear equation (2-1) is oscillatory.

Proof :

Let $x(t)$ for $t \geq T_1 \geq t_0$, it follows from Lemma 2-1 that $\dot{x}(t) > 0$ on $[T_2, \infty), T_2 \geq T_1$.

Multiplying (2-1) by $\frac{\rho(t)}{g(x(t))}$, we obtain

$$\frac{\rho(t)\left(r(t)\Psi(x(t))f(\dot{x}(t))\right)^{\cdot}}{g(x(t))} + \rho(t)q(t) = 0$$

Then, for every $t \geq T_2$ we get

$$\int_{T_2}^{t} \frac{\rho(s)\left(r(s)\Psi(x(s))f(\dot{x}(s))\right)^{\cdot}}{g(x(s))} ds + \int_{T_2}^{t} \rho(s)q(s)ds = 0$$

$$\left.\frac{\rho(s)\left(r(s)\Psi(x(s))f(\dot{x}(s))\right)}{g(x(s))}\right|_{T_2}^{t} - \int_{T_2}^{t} \frac{\dot{\rho}(s)\left(r(s)\Psi(x(s))f(\dot{x}(s))\right)}{g(x(s))} ds$$

$$+ \int_{T_2}^{t} \frac{\rho(s)\left(r(s)\Psi(x(s))f(\dot{x}(s))\right)}{g^2(x(s))} g'(x(s))\dot{x}(s)ds + \int_{T_2}^{t} \rho(s)q(s)ds = 0$$

Then

$$\frac{\rho(t)r(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))} - \frac{\rho(t)r(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))} \leq \int_{T_2}^{t} \frac{\dot{\rho}(t)\left(r(t)\Psi(x(t))f(\dot{x}(t))\right)}{g(x(t))} ds - \int_{T_2}^{t} \rho(s)q(s)ds \qquad (2\text{-}9)$$

By O_{17} in inequality (2-9), we have

$$k_7 \frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))} \le \frac{\rho(T_2)r(T_2)\Psi(x(T_2))f(\dot{x}(T_2))}{g(x(T_2))} + k_3 k_5 \int_{T_2}^{t} \frac{r(s)\dot{\rho}(t)\dot{x}(s)}{g(x(s))} ds$$

$$- \int_{T_2}^{t} \rho(s)q(s)ds$$

By the Bonnet theorem, for a fixed $\xi \in [T_2, t]$ we get

$$k_3 k_5 \int_{T_2}^{t} \frac{r(s)\dot{\rho}(s)\dot{x}(s)}{g(x(s))} = k_3 k_5 \left(r(T_2)\dot{\rho}(T_2) \right) \int_{T_2}^{\xi} \frac{\dot{x}(s)}{g(x(s))} ds = k_3 k_5 \left(r(T_2)\dot{\rho}(T_2) \right) \int_{x(T_2)}^{x(\xi)} \frac{du}{g(u)}$$

$$= k_3 k_5 N_1 = N < \infty$$

Let

$$c_1 = \frac{\rho(T_2)r(T_2)\Psi(x(T_2))f\left(\dot{x}(T_2)\right)}{g(x(T_2))} + N$$

Then, we have

$$k_7 \frac{\rho(T_2)r(T_2)\Psi(x(T_2))f\left(\dot{x}(T_2)\right)}{g(x(T_2))} \le c_1 - \int_{T_2}^{t} \rho(s)q(s)ds$$

But,

$\lim_{t \to \infty} \int_{T_2}^{t} \rho(s)q(s)ds = \infty$ then, there exists $t \ge T_3 \ge T_2$

Such that

$$\int_{T_1}^{t} \rho(s)q(s)ds \geq 2c_1 ,$$

Implies

Then, for all $t \geq T_2$, we have,

$$k_7 \frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))} \leq -\frac{1}{2}\int_{T_1}^{t} \rho(s)q(s)ds$$

$$\frac{\dot{x}(t)}{g(x(t))} \leq \frac{-1}{2k_3k_7}\frac{1}{r(s)\rho(s)}\int_{T_1}^{t} \rho(s)q(s)ds$$

Thus, for every $t \geq T_2$, we obtain

$$\int_{T_1}^{t} \frac{\dot{x}(s)ds}{g(x(s))} \leq \frac{-1}{2k_3k_7}\int_{T_1}^{t}\left(\frac{1}{r(s)\rho(s)}\int_{T_1}^{t} \rho(u)q(u)du\right)ds$$

Using O_{19} , we get

$$\int_{x(T_1)}^{x(t)} \frac{du}{g(u)} \to -\infty \quad as \ t \to \infty ,$$

which contradicts, Hence, the proof is completed .

Example (2-6) :

Consider the differential equation

$$\left[\left(\frac{t^2+2}{4t^2+1}\right)\left(11+\frac{x^{12}(t)}{x^{12}(t)+1}\right)\left(3\dot{x}(t)+\frac{\dot{x}^7(t)}{\dot{x}^6(t)+1}\right)\right]^{\cdot}+\left(t^2+2\right)\left(1+\cos t\right)x^7(t)=0 \ , for \ t \geq t_0 \geq 1$$

We note that

$$(1)\ 0<r(t)=\frac{t^2+2}{4t^2+1} \quad and \ \lim_{t\to\infty}\frac{1}{r(t)}=\lim_{t\to\infty}\frac{4t^2+1}{t^2+2}=4>0 \quad for\ t\geq t_0\geq 1,$$

$$(2)\ 0<11\leq\Psi(x(t))=11+\frac{x^{12}(t)}{x^{12}(t)+1}\leq 12 \quad \forall x\in R,$$

$$(3)\ 0<3<\frac{f(y)}{y}=3+\frac{y^6}{y^6+1}<4 \quad \forall y\neq 0,$$

$$(4)\ xg(x)=x.x^7=x^8>0 \ and \ g'(x)=7x^6>0 \ \forall x\neq 0 \quad ,$$

$$(5)\ \liminf_{t\to\infty}\int_T^t q(s)ds=\liminf_{t\to\infty}\int_T^t (s^2+2)(1+\cos s)ds=$$

$$=\liminf_{t\to\infty}\left[\frac{t^3}{3}+t^2\sin t+2t\cos t-2\sin t+2t+2\sin t-\frac{T^3}{3}\right.$$

$$\left.-T^3\sin T-2T\cos T+2\sin T-2T-2\sin T\right]$$

$$=\liminf_{t\to\infty} t^3\left[\frac{1}{3}+\frac{\sin t}{t}+\frac{2\cos t}{t^2}-\frac{2\sin t}{t^3}+\frac{2t}{t^2}+\frac{2\sin t}{t^3}-\frac{T^3}{3t^2}\right.$$

$$\left.-\frac{T^2\sin T}{t^3}-2T\frac{\cos T}{t^3}+\frac{2\sin T}{t^3}-\frac{2T}{t^3}-\frac{2\sin T}{t^3}\right]=\infty>0 \text{ , for all large T.}$$

Let $\rho(t)=\dfrac{4t^2+1}{t^2+2}$ we have $\dot{\rho}(t)=\dfrac{14t}{(t^2+2)^2}>0$ and

$$\left(r(t)\dot{\rho}(t)\right)^{\cdot}=\frac{14\left[-12t^4-8t^2+2\right]}{\left((t^2+2)(4t^2+1)\right)^2}<0 \ for\ t\geq t_0\geq 1,$$

$$(6)\lim_{t\to\infty}\int_{t_0}^{t}\rho(s)q(s)ds = \lim_{t\to\infty}\int_{t_0}^{t}\frac{4s^2+1}{s^2+2}(s^2+2)(1+\cos s)ds$$

$$= \lim_{t\to\infty}\int_{t_0}^{t}\left(4s^2+4s^2\cos s+1+\cos s\right)ds$$

$$= \lim_{t\to\infty}\left[\frac{4}{3}t^3+4t^2\sin t+8t\cos t-8\sin t+t+\sin t\right.$$

$$\left.-\frac{4}{3}t_0^3-4t_0^2\sin t_0-8t_0\cos t_0+8\sin t_0-t_0-\sin t_0\right]$$

$$= \lim_{t\to\infty}t^3\left[\frac{4}{3}+\frac{4\sin t}{t}+\frac{8\cos t}{t^2}-\frac{8\sin t}{t^3}+\frac{1}{t^2}+\frac{\sin t}{t^3}\right.$$

$$\left.-\frac{4t_0^3}{3t^3}-\frac{4t_0^2\sin t_0}{t^3}-\frac{8t_0\cos t_0}{t^3}+\frac{8\sin t_0}{t^3}-\frac{t_0}{t^3}-\frac{\sin t_0}{t^3}\right]=\infty ,$$

it follows from Theorem (2-6)) that the equation is oscillatory.

Remark 2-6

Theorem (2-6) extends the results of Grace and Lalli [11]and [23].

Theorem 2-7 :

Suppose that O_2, O_7, O_8, O_9 and O_{17} hold and moreover , assume that there exists a differentiable function

$$\rho(t):[t_0,\infty)\to(0,\infty)$$

such that

$$\rho(t) > 0, \dot{\rho}(t) \geq 0 \text{ and } \left(r(t)\dot{\rho}(t) \right)^{\bullet} \leq 0 \quad \text{for all } t \geq t_0$$

and the continuous functions

$$h, H : D \equiv \{(t,s) : t \geq s \geq t_0\} \rightarrow R.$$

where H has a continuous and non positive partial derivative on D with respect to the second variable such that

$$H(t,t) = 0 \quad \text{for } t \geq t_0, \; H(t,s) > 0 \; \text{for } t > s \geq t_0,$$
$$\frac{-\partial}{\partial s} H(t,s) = h(t,s)\sqrt{H(t,s)} \quad \text{for all } (t,s) \in D$$

O_{20} $\int^{\infty} \frac{\Psi(u)}{g(u)} du < \infty \quad \text{and} \quad \int^{-\infty} \frac{\Psi(u)}{g(u)} du < \infty.$

O_{21} $\limsup_{t \to \infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} r(s)\,\rho(s)\,h^2(t,s)\,ds < \infty$.

then equation (2-1)is oscillatory .

Proof :

Let x(t) be a nonoscillatory solution of equation (2-1) , say x(t) > 0 for $t \geq T_1 \geq t_0$ .it follows from Lemma 2-1 that $\dot{x}(t) > 0 \quad \text{for } t \geq T_2 \geq t_0$.

Define

$$\omega(t) = \frac{r(t)\rho(t)\Psi(x(t)) f(\dot{x}(t))}{g(x(t))}$$

Then, for every $t \geq T_2$,we get

$$\dot{\omega}(t) = -\rho(t)q(t) + \frac{r(t)\dot{\rho}(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))} - \frac{\rho(t)r(t)\Psi(x(t))f(\dot{x}(t))g'(x(t))\dot{x}(t)}{g^2(x(t))}$$

Hence, for all $t \geq T_2$ we have

$$\dot{\omega}(t) \leq -\rho(t)q(t) + \frac{r(t)\dot{\rho}(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))} - \frac{k}{r(t)\rho(t)\Psi(x(t))}\omega^2(t)\frac{\dot{x}(t)}{f(\dot{x}(t))}.$$

From the O_2 and the O_{17} we get

$$\dot{\omega}(t) \leq -\rho(t)q(t) + \frac{\dot{\rho}(t)}{\rho(t)}\omega(t) - \frac{k}{k_3 k_7}\frac{1}{r(t)\rho(t)}\omega^2(t) \qquad T_2 \geq t_0$$

Thus ,for every $t \geq T_2 \geq t_0$

$$\int_{T_2}^{t} H(t,s)\rho(s)q(s)\,ds \leq -\int_{T_2}^{t} H(t,s)\dot{\omega}(s)ds + \int_{T_2}^{t}\frac{\dot{\rho}(s)}{\rho(s)}H(t,s)\omega(s)\,ds - B_1\int_{T_2}^{t}\frac{H(t,s)}{r(s)\rho(s)}\omega^2(s)ds$$

where $B_1 = \dfrac{kk_7}{k_3}$ is a positive constant

$$= H(t,T_2)\omega(T_2) - \int_{T_2}^{t} -\frac{\partial}{\partial s}H(t,s)\omega(s)ds + \int_{T_2}^{t}\frac{\dot{\rho}(s)}{\rho(s)}H(t,s)\omega(s)\,ds$$

$$-B_1\int_{T_2}^{t}\frac{H(t,s)}{r(s)\rho(s)}\omega^2(s)ds \qquad (2\text{-}10)$$

Now we note that

$$\int_{T_2}^{t} \frac{\dot{\rho}(s)}{\rho(s)} H(t,s)\omega(s)\,ds = \int_{T_2}^{t} H(t,s)\dot{\rho}(s) \frac{r(s)\Psi(x(s)) f(\dot{x}(s))}{g(x(s))}\,ds$$

$$\leq k_5 \int_{T_2}^{t} \left[\frac{-\partial}{\partial s} H(t,s) \int_{T_2}^{s} \left(r(u)\,\dot{\rho}(u) \right) \frac{\Psi(x(s))\,\dot{x}(u)}{g(x(u))}\,du \right] ds$$

By the Bonnet theorem, for a fixed $s \geq T_2$ and for some $m_t \in [T_2, s]$ we get

We have

$$\int_{T_2}^{s} r(u)\,\dot{\rho}(u) \frac{\Psi(x(u))\,\dot{x}(u)}{g(x(u))}\,du = r(T_2)\,\dot{\rho}(T_2) \int_{T_2}^{m_t} \frac{\Psi(x(u))\,\dot{x}(u)}{g(x(u))}\,du$$

$$= r(T_2)\,\dot{\rho}(T_2) \int_{x(T_2)}^{x(m_t)} \frac{\Psi(y)}{g(y)}\,dy$$

$r(T_2)\,\dot{\rho}(T_2) > 0$ *and* And hence , since

$$\int_{x(T_2)}^{x(m_t)} \frac{\Psi(y)}{g(y)}\,dy < \int_{x(T_2)}^{\infty} \frac{\Psi(y)}{g(y)}\,dy$$

We have ,

$$\int_{T_2}^{t} r(u)\,\dot{\rho}(u) \frac{\Psi(x(u))\,\dot{x}(u)}{g(x(u))}\,du \leq k_8 \quad ; k_8 = r(T_2)\,\dot{\rho}(T_2) \int_{x(T_2)}^{\infty} \frac{\Psi(y)}{g(y)}\,dy$$

Thus . (2-10) be comes

$$\int_{T_2}^{t} H(t,s)\,\rho(s)\,q(s)\,ds \leq \omega(T_2)H(t,T_2) - \int_{T_2}^{t} \left[h(t,s)\sqrt{H(t,s)}\,\omega(s) \right] ds + k_5 k_8 H(t,T_2)$$

$$- B_1 \int_{T_2}^{t} \frac{H(t,s)}{r(s)\rho(s)} \omega^2(s)\,ds$$

$$\leq H(t,T_2)\omega(T_2)+k_5k_8H(t,T_2)-\int_{T_2}^{t}h(t,s)\sqrt{H(t,s)}\omega(s)\,ds$$

$$-\ B_1\int_{T_2}^{t}\frac{H(t,s)}{r(s)\rho(s)}\omega^2(s)\,ds$$

$$\leq -\left[\int_{T_1}^{t}\left(\sqrt{\frac{B_1H(t,s)}{r(s)\rho(s)}}\,\omega(s)+\frac{h(t,s)\sqrt{H(t,s)}}{2\sqrt{B_1}}\right)^2-\frac{h^2(t,s)H(t,s)}{\dfrac{4B_1H(t,s)}{r(s)\rho(s)}}\right]ds$$

$$+\ H(t,T_2)(k_5k_8+\omega(T_2))$$

$$\leq H(t,T_2)(k_5k_8+\omega(T_2))+\int_{T_1}^{t}\frac{r(s)\rho(s)}{4B_1}h^2(t,s)\,ds \qquad (2\text{-}11)$$

Now, dividing by $H(t,t_0)$ and take the upper limit as $t\to\infty$ and from (2-11) and case21 we get

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\rho(s)q(s)\,ds<\infty.$$

which contradicts O_7 .Hence , the proof is completed .

__Example 2-7__ :

Consider the following differential equation

$$\left[\left(\frac{t}{t+1}\right)\left(\frac{x^4(t)}{x^4(t)+1}\right)\left(9\dot{x}(t)+\frac{\dot{x}^9(t)}{\dot{x}^8(t)+1}\right)\right]^{\bullet}+\frac{1}{t}x^3(t)=0 \qquad for\ t\geq t_0>0$$

We note that

1- $r(t)=\dfrac{t}{t+1}>0$ *and* $\lim\limits_{t\to\infty}\dfrac{1}{r(t)}=\lim\limits_{t\to\infty}\dfrac{t+1}{t}=1>0$ *for* $t\geq t_0>0$,

2- $0<k_1\leq\Psi(x(t))=\dfrac{x^4(t)}{x^4(t)+1}<1$ *for all* $x\in\mathbb{R}$.

3- $\liminf\limits_{t\to\infty}\displaystyle\int_T^t q(s)\,ds=\liminf\limits_{t\to\infty}\int_T^t\frac{ds}{s}=\liminf\limits_{t\to\infty}\left[\ln t-\ln T\right]=\infty>0$.

4- $0<9<\dfrac{f(y)}{y}=9+\dfrac{y^8}{y^8+1}<10$ *for all* $y\neq 0$

Let $\rho(t)=1$ *we have* $\dot{\rho}(t)=0$ *and* $\left(r(t)\dot{\rho}(t)\right)^{\bullet}=0$ *for* $t\geq t_0>0$

Let $H(t,s)=(t-s)^2$ *for* $t\geq s\geq t_0>0$, *we get*

5- $$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\rho(s)q(s)\,ds=\limsup_{t\to\infty}\frac{1}{(t-t_0)^2}\int_{t_0}^{t}(t-s)^2\,(1)\,\frac{1}{s}\,ds$$

$$=\limsup_{t\to\infty}\frac{1}{(t-t_0)^2}\left[t^2\ln s-2st+\frac{s^2}{2}\right]_{t_0}^{t}$$

$$=\limsup_{t\to\infty}\frac{1}{(t-t_0)^2}\left[t^2\ln t-2t^2+\frac{t^2}{2}-t^2\ln t_0+2tt_0-\frac{t_0^2}{2}\right]$$

$$=\limsup_{t\to\infty}\frac{t^2}{(t-t_0)^2}\left[\ln t-2+\frac{1}{2}-\ln t_0+\frac{2t_0}{t}-\frac{t_0^2}{2t^2}\right]$$

$$=\infty\ .$$

6- $$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}r(s)\rho(s)h^2(t,s)\,ds=\limsup_{t\to\infty}\frac{1}{(t-t_0)^2}\int_{t_0}^{t}\frac{s}{s+1}(1)(4)\,ds$$

$$=\limsup_{t\to\infty}\frac{4}{(t-t_0)^2}\left[s-\ln(s+1)\right]_{t_0}^{t}$$

$$=\lim_{t\to\infty}\frac{4t-4\ln(t+1)-4t_0+4\ln(t_0+1)}{t^2-2tt_0+t_0^2}$$

$$=0<\infty,$$

7- $\int^{\infty}\frac{\Psi(u)}{g(u)}du=\int^{\infty}\frac{u^4}{u^4+1}\times\frac{1}{u^3}du=\int^{\infty}\frac{u}{1+u^4}du=\frac{1}{2}\tan^{-1}u^2\Big|^{\infty}<\infty$ *and*

$$\int^{-\infty}\frac{\Psi(u)}{g(u)}du=\int^{-\infty}\frac{u^4}{u^4+1}\,\frac{1}{u^3}du=\frac{1}{2}\,Tan^{-1}u^2\Big|^{-\infty}<\infty,$$

it follows from Theorem 2-7that the equation is oscillatory.

Remark2-7 :

Theorem 2-7 extends the results of Grace [12] and [23].

Theorem 2-8 :

Suppose that O_2 ,O_3,O_6 hold and O_7 moreover ,assume that there exists a differentiable function

$$\rho:[t_0,\infty)\to(0,\infty)$$

such that

$$\rho(t)>0\ ,\dot{\rho}(t)\geq 0\quad for\ \ all\ t\geq t_0$$

and continuous functions

$$h\ ,H\ :D\equiv\{(t,s):t\geq s\geq t_0\}\to \mathrm{R}\ ,$$

where H has a continuous and non positive partial derivative on D with respect to the second variable such that

$$H(t,t)=0\quad for\ t\geq t_0\qquad ,\qquad H(t,t_0)>0\qquad for\ t>s\geq t_0,$$
$$\frac{-\partial}{\partial s}H(t,s)=h(t,s)\sqrt{H(t,s)}\qquad for\ all\ (t,s)\in D,$$

$$O_{22}\ 0 < \inf_{s \ge t_0}\left(\liminf_{t \to \infty} \frac{H(t,s)}{H(t,t_0)} \right) \le \infty$$

If there exist a continuous function Ω *on* $[t_0,\infty)$ *such that*

$$O_{23}\ \limsup_{t\to\infty} \frac{1}{H(t,T_2)} \int_{T_2}^{t} H(t,s)\,\rho(s)q(s)\,ds$$

$$-\limsup_{t\to\infty} \frac{1}{H(t,T_2)} \int_{T_1}^{t} \frac{r(s)\rho(s)}{4B_2}\left(h(t,s) - \frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)} \right)^2 ds \ge \Omega(T_2)$$

for every $T_2 \ge t_0$.

$$O_{24}\ \int_{t_0}^{\infty} \frac{\Omega_+^2(s)}{r(s)\,\rho(s)}\,ds = \infty,\ \textit{where}\ \Omega_+(t) = \max\{\Omega(t),0\},$$

then equation(2-1) is Oscillatory.

<u>**Proof**</u>:

Let x(t) be a non oscillatory solution of equation (2-1), assume that $x(t) \ne 0$ *for* $t \ge T_1 \ge t_0$

Define

$$\omega(t) = \frac{\rho(t)r(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))}$$

Thus, for every $t \ge T_2$, we obtain

$$\dot{\omega}(t) = -\rho(t)q(t) + \frac{r(t)\dot{\rho}(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))} - \frac{\rho(t)r(t)\Psi(x(t))f(\dot{x}(t))g'(x(t))\dot{x}(t)}{g^2(x(t))}$$

Hence , for every $t \geq T_2$, we get

$$\dot{\omega}(t) \leq -\rho(t)q(t) + \frac{\dot{\rho}(t)}{\rho(t)}\omega(t) - \frac{kk_4}{k_1}\frac{1}{r(t)\rho(t)}\omega^2(t)$$

Thus , for every $t \geq T_2 \geq t_0$, *we have*

$$\int_{T_2}^{t} H(t,s)\,\rho(s)q(s)\,ds \leq -\int_{T_2}^{t} H(t,s)\,\dot{\omega}(s)\,ds + \int_{T_2}^{t}\frac{\dot{\rho}(s)}{\rho(s)}H(t,s)\,\omega(s)\,ds$$

$$-B_2\int_{T_2}^{t}\frac{H(t,s)}{r(s)\rho(s)}\omega^2(s)\,ds\ .$$

where $B_2 = \dfrac{kk_4}{k_1}$ is a positive constant

$$\leq -\left[H(t,s)\,\omega(s)\Big|_{T_2}^{t} - \int_{T_2}^{t}\frac{\partial}{\partial s}H(t,s)\,\omega(s)\,ds\right] + \int_{T_2}^{t}\frac{\dot{\rho}(s)}{\rho(s)}H(t,s)\,\omega(s)\,ds$$

$$-B_2\int_{T_2}^{t}\frac{H(t,s)}{r(s)\rho(s)}\omega^2(s)ds$$

Then ,

$$\int_{T_2}^{t}H(t,s)\rho(s)q(s)\,ds\le H(t,T_2)\,\omega(T_2)-\int_{T_2}^{t}h(t,s)\sqrt{H(t,s)}\;\omega(s)\,ds+\int_{T_2}^{t}\frac{\dot{\rho}(s)}{\rho(s)}\;H(t,s)\omega(s)\,ds$$

$$-B_2\int_{T_2}^{t}\frac{H(t,s)}{r(s)\rho(s)}\omega^2(s)ds$$

$$=H(t,T_2)\omega(T_2)+\int_{T_2}^{t}\frac{r(s)\rho(s)}{4B_2}\left[h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right]^2$$

$$-\int_{T_2}^{t}\left[\sqrt{\frac{B_2H(t,s)}{r(s)\rho(s)}}\,\omega(s)+\frac{1}{2}\sqrt{\frac{r(s)\rho(s)}{B_2}}\left[h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right]\right]^2ds$$

$$\limsup_{t\to\infty}\frac{1}{H(t,T_2)}\int_{T_2}^{t}\left[H(t,s)\rho(s)q(s)-\frac{r(s)\rho(s)}{4B_2}\left[h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right]^2\right]ds\le\omega(T_2)$$

$$-\liminf_{t\to\infty}\frac{1}{H(t,T_2)}\int_{T_2}^{t}\left[\sqrt{\frac{B_2H(t,s)}{r(s)\rho(s)}}\omega(s)+\frac{1}{2}\sqrt{\frac{r(s)\rho(s)}{B_2}}\left(h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right)\right]^2ds$$

From O_{24} we get

$$\omega(T_2)\ge\Omega(T_2)$$

$$+\liminf_{t\to\infty}\frac{1}{H(t,T_2)}\int_{T_2}^{t}\left[\sqrt{\frac{B_2H(t,s)}{r(s)\rho(s)}}\omega(s)+\frac{1}{2}\sqrt{\frac{r(s)\rho(s)}{B_2}}\left(h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right)\right]^2ds$$

This shows that

$\omega(T_2)\ge\Omega(T_2)$ *for every* $T_2\ge t_0$ *and*

$$\liminf_{t\to\infty}\frac{1}{H(t,T_2)}\int_{T_2}^{t}\left[\sqrt{\frac{B_2H(t,s)}{r(s)\rho(s)}}\omega(s)+\frac{1}{2}\sqrt{\frac{r(s)\rho(s)}{B_2}}\left(h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right)\right]^2ds<\infty$$

Hence ,

$$\liminf_{t\to\infty}\frac{1}{H(t,T_2)}\int_{T_2}^{t}\frac{B_2}{r(s)\rho(s)}H(t,s)\omega^2(s)\,ds$$

$$+\liminf_{t\to\infty}\frac{1}{H(t,T_2)}\int_{T_2}^{t}\left(h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right)\sqrt{H(t,s)}\omega(s)\,ds<\infty$$

i.e., we have

$$\liminf_{t\to\infty}\left[U(t)+V(t)\right]<\infty \qquad (2\text{-}12)$$

where

$$U(t)=\frac{1}{H(t,T_2)}\int_{T_2}^{t}\frac{B_2}{r(s)\rho(s)}H(t,s)\omega^2(s)ds \qquad ,t\geq T_2$$

And

$$V(t)=\frac{1}{H(t,T_2)}\int_{T_2}^{t}\left[h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right]\sqrt{H(t,s)}\,\omega(s)ds$$

Now, suppose that

$$\int_{T_2}^{\infty}\frac{1}{r(s)\rho(s)}\omega^2(s)ds=\infty \qquad (2\text{-}13)$$

From the O_{22} , we have

$$\liminf_{t\to\infty}B_2\int_{T_2}^{t}\frac{H(t,s)}{H(t,T_2)}\frac{\omega^2(s)}{r(s)\rho(s)}ds=\infty$$

Thus,

$$\lim_{t\to\infty}U(t)=\infty \qquad (2\text{-}14)$$

Now consider a sequence $\{T_n\}$; $n=1,2,3,\ldots$ in $[t_0,\infty)$ with $\lim_{n\to\infty}T_n=\infty$ and such that

$$\lim_{n\to\infty}\left[U(T_n)+V(T_n)\right]=\liminf_{t\to\infty}\left[U(t)+V(t)\right]$$

By (2-12) there exists a constant M such that

$$U(T_n)+V(T_n)\le M \qquad ; n=1,2,... \tag{2-15}$$

Furthermore , (2-14) guarantees that

$$\lim_{n\to\infty} U(T_n)=\infty \tag{2-16}$$

And hence (2-15) gives

$$\lim_{n\to\infty} V(T_n)=-\infty \tag{2-17}$$

By taking into account (2-16) ,from (2-15) ,we derive

$$1+\frac{V(T_n)}{U(T_n)}<\frac{M}{U(T_n)}<\frac{1}{2} \qquad \textit{for all } n \textit{ is sufficiently large ,thus}$$

$$\frac{V(T_n)}{U(T_n)}<-\frac{1}{2} \qquad \textit{for all large } n, \tag{2-18}$$

From (2-17) and (2-18) , we have

$$\lim_{n\to\infty}\frac{V^2(T_n)}{U(T_n)}>\lim_{n\to\infty}\frac{\frac{1}{4}U^2(T_n)}{U(T_n)}=\frac{1}{4}\lim_{n\to\infty}U(T_n)=\infty \tag{1-19}$$

On the other hand ,by Schwars inequality ,we have for any positive integer n .

$$V^2(T_n)=\frac{1}{H^2(T_n,T_2)}\left[\int_{T_2}^{T_n}\left(h(T_n,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(T_n,s)}\right)\sqrt{H(T_n,s)}\omega(s)ds\right]^2$$

$$\leq \frac{1}{H(T_n,T_2)}\int_{T_2}^{T_n}\frac{r(s)\rho(s)}{B_2}\left(h(T_n,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(T_n,s)}\right)^2 ds$$

Then we

$$\times\int_{T_2}^{T_n}\frac{B_2}{r(s)\rho(s)}\frac{1}{H(T_n,T_2)}H(T_n,s)\omega^2(s)ds$$

have

$$V^2(T_n)\leq \frac{1}{B_2 H(T_n,T_2)}\int_{T_2}^{T_n}r(s)\rho(s)\left(h(T_n,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(T_n,s)}\right)^2 dsU(T_n)$$

or

$$\frac{V^2(T_n)}{U(T_n)}\leq \frac{1}{H(T_n,T_2)}\int_{T_2}^{T_n}\frac{r(s)\rho(s)}{B_2}\left(h(T_n,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(T_n,s)}\right)^2 ds$$

So, becomes of (2-19), we have

$$\lim_{n\to\infty}\frac{1}{H(T_n,T_2)}\int_{T_2}^{T_n}\frac{r(s)\rho(s)}{B_2}\left(h(T_n,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(T_n,s)}\right)^2 ds=\infty$$

Thus ,for all $t\geq T_2$ we have ,

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}r(s)\rho(s)\left(h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right)^2 ds=\infty$$

which contradicts O_6

Thus ,(2-13) fails, and hence

$$\int_{T_2}^{\infty}\frac{1}{r(s)\rho(s)}\omega^2(s)ds<\infty \qquad \textit{for all } T_2\geq t_0$$

Hence ,and since $\omega(T_2) \geq \Omega(T_2)$, we have ,

$$\int_{t_0}^{\infty} \frac{\Omega_+^2(s)}{r(s)\rho(s)} ds \leq \int_{t_0}^{\infty} \frac{1}{r(s)\rho(s)} \omega^2(s) ds < \infty$$

which contradicts O_{24} , hence the proof is completed.

Example 2-8 :

Consider the following differential equation

$$\left[\left(\frac{1}{t^6}\right)\left(\frac{x^4(t)}{x^4(t)+1}\right)\left(\dot{x}(t)+\frac{\dot{x}(t)}{x^2(t)+1}\right)\right]^{\cdot} + \frac{1}{t^3}x^3(t) = 0 \qquad for\ t > t_0 > 0$$

We note that

$$1 - r(t) = \frac{t}{t+1} > 0 \qquad for\ t \geq t_0 > 0,$$

$$2 - 0 < k_2 \leq \Psi(x(t)) = \frac{x^2(t)}{x^2(t)+2} < 1 \qquad for\ all\ t \geq t_0 > 0$$

$$3 - 0 < f^2(y) = y^2 + \frac{2y^2}{y^2+1} + \frac{y^2}{(y^2+1)^2} < 4y^2$$

$$\frac{1}{4} f^2(y) < y^2 < y^2 + \frac{y^2}{y^2+1} = yf(y) \qquad for\ all\ y \neq 0$$

$$4 - xg(x) = x^4 > 0 \quad and \quad g'(x) = 3x^2 > 0\ \forall x \neq 0$$

$$5 - \liminf_{t\to\infty} \int_T^t q(s) ds = \liminf_{t\to\infty} \int_T^t \frac{1}{s^3} ds = \frac{1}{2T^2} > 0 \qquad for\ t \geq T > 0$$

$$6-\int^{\infty}\frac{\Psi(u)}{g(u)}du=\int^{\infty}\frac{u^4}{u^4+1}\ \frac{1}{u^3}\ du=\int^{\infty}\frac{u}{2+u^4}du=\frac{1}{2}Tan^{-1}u^2\Big|^{\infty}<\infty \quad and$$

$$\int^{-\infty}\frac{\Psi(u)}{g(u)}du=\int^{-\infty}\frac{u^4}{u^4+1}\ \frac{1}{u^3}du=\frac{1}{2}Tan^{-1}u^2\Big|^{-\infty}<\infty$$

Let $\rho(t)=2>0$, *we have* $\dot{\rho}(t)=0$

And let $H(t,s)=(t-s)^2$ *for* $t\ge s\ge t_0$

Thus ,

$$6-\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}r(s)\rho(s)\left(h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right)^2 ds$$

$$=\limsup_{t\to\infty}\frac{8}{(t-t_0)^2}\int_{t_0}^{t}\frac{1}{s^6}\ ds$$

$$=\limsup_{t\to\infty}\left[\frac{8}{t^2-2tt_0+t^2}\ \frac{-1}{5s^2}\right]_{t_0}^{t}=0<\infty\ ,$$

$$7-0<\inf_{s\ge t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right]=\inf_{s\ge t_0}\left[\liminf_{t\to\infty}\frac{(t-s)^2}{(t-t_0)^2}\right]=1<\infty$$

$$8-\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\rho(s)q(s)-\frac{r(s)\rho(s)}{4B_2}\left[h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right]^2 ds$$

$$=\limsup_{t\to\infty}\frac{1}{(t-t_0)^2}\int_{t_0}^{t}\left((t-s)^2(1)\frac{2}{s^3}-\frac{1}{4B_2}\ \frac{2}{s^6}\ \right)ds$$

$$= 2\limsup_{t\to\infty} \frac{1}{(t-t_0)^2}\left[\int_{t_0}^{t}\left(\frac{t^2}{s^3}-\frac{2t}{s^2}+\frac{1}{s}\right)ds - \frac{4}{4B_2}\int_{t_0}^{t}\frac{1}{s^6}\,ds\right]$$

$$= \frac{1}{t_0^2} > \frac{1}{2t_0^2}$$

Set $\Omega(t_0) = \frac{1}{2t_0^2}$ we get

$$9- \int_{t_0}^{\infty}\frac{\Omega_*^2(s)}{r(s)\,\rho(s)}\,ds = \frac{1}{8}\int_{t_0}^{\infty}\frac{s^{-4}}{s^{-6}}\,ds = \infty$$

it follows from theorem 2-8 that the equation is oscillatory .s

Remark 2-8 :

Theorem 2-8 extends the results of Grace [12], extends the results of [1] and [23].

Theorem 2-9 :

Suppose that O_2,O_3,O_{22},O_{23}, and O_{24} hold and moreover ,assume that there exists a differentiable function

$$\rho : (t_0,\infty) \to (0,\infty)$$

such that

$$\rho(t)>0\ ,\dot{\rho}(t)\ge 0 \quad for\ \ all\ t \ge t_0$$

and the continuous functions

$$h, H : D \equiv \{(t,s): t \ge s \ge t_0\} \to \mathrm{R},$$

where H has a continuous and non positive partial derivative on D with respect to the second variable such that

$$H(t,t)=0 \qquad for\ t\geq t_0,\ H(t,s)>0\ \ for\ \ t>s\geq t_0,$$
$$-\frac{\partial}{\partial s}H(t,s)=h(t,s)\sqrt{H(t,s)} \qquad for\ \ all\ (t,s)\in D$$

and

O_{25} $\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\rho(s)q(s)ds<\infty.$

If there exists a continuous function Ω on $[t_0,\infty)$ such that

O_{26} $\liminf_{t\to\infty}\frac{1}{H(t,T_2)}\int_{T_2}^{t}\left[H(t,s)\rho(s)q(s)-\frac{r(s)\rho(s)}{4B_1}\left(h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right)^2\right]ds\geq$

$$\Omega(T_2) \qquad\qquad for\, T_2\geq t_0$$

then equation (2-1) is oscillatory.

__Proof__ :

Let $x(t)$ be a non oscillatory solution of equation (2-1), say $x(t)\neq 0$ for $t\geq T_1\geq t_0$, Furthermore

Define

$$\omega(t)=\rho(t)\frac{r(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))}$$

Thus ,for $t\geq T_2$, we obtain

$$\dot{\omega}(t) = -\rho(t)q(t) + \frac{r(t)\dot{\rho}(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))} - \frac{\rho(t)r(t)\psi(x(t))f(\dot{x}(t))g'(x(t))\dot{x}(t)}{g^2(x(t))}$$

Hence for all $t \geq T_2$, we have

$$\dot{\omega}(t) \leq -\rho(t)q(t) + \frac{\dot{\rho}(t)}{\rho(t)}\omega(t) - \frac{k}{k_3 k_5} \times \frac{1}{r(t)\rho(t)}\omega^2(t)$$

Thus, for every $t \geq T_2$ we have

$$\int_{T_2}^{t} H(t,s)\rho(s)q(s)ds \leq -\int_{T_2}^{t} H(t,s)\dot{\omega}(s)ds + \int_{T_2}^{t}\frac{\dot{\rho}(s)}{\rho(s)}H(t,s)\omega(s)ds - B_3\int_{T_2}^{t}\frac{H(t,s)}{r(s)\rho(s)}\omega^2(s)ds$$

$$\leq -\left[H(t,s)\omega(s)\Big|_{T_2}^{t} - \int_{T_2}^{t}\frac{\partial}{\partial s}H(t,s)\omega(s)ds \right] + \int_{T_2}^{t}\frac{\dot{\rho}(s)}{\rho(s)}H(t,s)\omega(s)ds - B_3\int_{T_2}^{t}\frac{H(t,s)}{r(s)\rho(s)}\omega^2(s)ds$$

where $B_3 = \dfrac{kk_3}{k_5}$ is a positive constant.

we have

$$\int_{T_2}^{t} H(t,s)\rho(s)q(s)ds \leq H(t,T_2)\omega(T_2) - \int_{T_2}^{t} h(t,s)\sqrt{H(t,s)}\,\omega(s)ds + k_5\int_{T_2}^{t}\frac{\dot{\rho}(s)}{\rho(s)}H(t,s)ds$$

$$-B_3\int_{T_2}^{t}\frac{H(t,s)}{r(s)\rho(s)}\omega^2(s)ds$$

$$= H(t,T_2) + \omega(T_2) + \int_{T_2}^{t} \frac{r(s)\rho(s)}{4B_3}\left[h(t,s) - \frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right]^2$$

$$- \int_{T_2}^{t}\left[\sqrt{\frac{B_3 H(t,s)}{r(s)\rho(s)}}\,\omega(s) + \frac{1}{2}\sqrt{\frac{r(s)\rho(s)}{B_3}}\left[h(t,s) - \frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right]\right]^2 ds$$

And hence for $t \geq T_2 \geq t_0$ we get

$$\liminf_{t\to\infty} \frac{1}{H(t,T_2)} \int_{T_2}^{t} H(t,s)\rho(s)q(s) - \frac{r(s)\rho(s)}{4B_3}\left[h(t,s) - \frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right]^2 ds$$

$$\leq \omega(T_2)$$

$$- \limsup_{t\to\infty} \frac{1}{H(t,T_2)} \int_{T_2}^{t}\left[\sqrt{\frac{B_3 H(t,s)}{r(s)\rho(s)}}\,\omega(s) + \frac{1}{2}\sqrt{\frac{r(s)\rho(s)}{B_3}}\left(h(t,s) - \frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right)\right]^2 ds$$

And there for by O_{26} we obtain

$$\omega(T_2) \geq \Omega(T_2)$$

$$+ \limsup_{t\to\infty} \frac{1}{H(t,T_2)} \int_{T_2}^{t}\left[\sqrt{\frac{B_3 H(t,s)}{r(s)\rho(s)}}\,\omega(s) + \frac{1}{2}\sqrt{\frac{r(s)\rho(s)}{B_3}}\left(h(t,s) - \frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right)\right]^2 ds$$

This shows that

$\omega(T_2) \geq \Omega(T_2)$ *for every* $T_2 \geq t_0$ *and*

$$\limsup_{t\to\infty} \frac{1}{H(t,T_2} \int_{T_2}^{t}\left[\sqrt{\frac{B_3 H(t,s)}{r(s)\rho(s)}}\,\omega(s) + \frac{1}{2}\sqrt{\frac{r(s)\rho(s)}{B_3}}\left(h(t,s) - \frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right)\right]^2 ds < \infty$$

Hence ,

$$\limsup_{t\to\infty}\frac{1}{H(t,T_2)}\int_{T_2}^{t}\frac{B_3}{r(s)\rho(s)}H(t,s)\omega^2(s)\,ds$$

$$+\limsup_{t\to\infty}\frac{1}{H(t,T_2)}\int_{T_2}^{t}\left(h(t,s)-\frac{\dot\rho(s)}{\rho(s)}\sqrt{H(t,s)}\right)\sqrt{H(t,s)}\,\omega(s)\,ds<\infty$$

i.e.

$\limsup_{t\to\infty}[U(t)+V(t)]<\infty$, where

$$U(t)=\frac{1}{H(t,T_2)}\int_{T_2}^{t}\frac{B_3}{r(s)\rho(s)}H(t,s)\omega^2(s)\,ds \quad ,t\ge T_2$$

And

$$V(t)=\frac{1}{H(t,T_2)}\int_{T_2}^{t}\left[h(t,s)-\frac{\dot\rho(s)}{\rho(s)}\sqrt{H(t,s)}\right]\sqrt{H(t,s)}\,\omega(s)\,ds$$

Now consider the O_{26} , we have

$$\Omega(T_2)\le\liminf_{t\to\infty}\left[\frac{1}{H(t,T_2)}\int_{T_1}^{t}H(t,s)\rho(s)q(s)\,ds-\frac{1}{4B_3}\frac{1}{H(t,T_2)}\right.$$
$$\left.\times\int_{T_1}^{t}r(s)\rho(s)\left[h(t,s)-\frac{\dot\rho(s)}{\rho(s)}\sqrt{H(t,s)}\right]^2 ds\right]$$

$$\le\liminf_{t\to\infty}\frac{1}{H(t,T_2)}\int_{T_1}^{t}H(t,s)\rho(s)q(s)-\liminf_{t\to\infty}\frac{1}{H(t,T_2)}\int_{T_1}^{t}\frac{r(s)\rho(s)}{4B_3}\left[h(t,s)-\frac{\dot\rho(s)}{\rho(s)}\sqrt{H(t,s)}\right]^2 ds$$

And so , by O_{26} ,we obtain

$$\liminf_{t\to\infty}\frac{1}{H(t,T_2)}\int_{T_1}^{t}\frac{r(s)\rho(s)}{4B_3}\left[h(t,s)-\frac{\dot\rho(s)}{\rho(s)}\sqrt{H(t,s)}\right]^2 ds<\infty$$

Then, by O_{23}, we obtain

$$\lim_{t\to\infty}\frac{1}{H(t,T_2)}\int_{T_2}^{t}\frac{r(s)\rho(s)}{4B_3}\left[h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right]^2 ds <\infty$$

This , shows that there exists a sequence $\{T_n\}$, $n = 1,2,3,\ldots\ldots$ in $[t_0,\infty)$ with $\lim_{n\to\infty} T_n = \infty$ and such that

$$\lim_{n\to\infty}\frac{1}{H(T_n,T_2)}\int_{T_2}^{T_n} r(s)\rho(s)\left[h(T_n,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(T_n,s)}\right]^2 ds < \infty$$

Now suppose that

$$\liminf_{t\to+\infty}[U(t)+V(t)]< \infty \qquad (2\text{-}20)$$

Let

$$\int_{T_2}^{\infty}\frac{1}{r(s)\rho(s)}\omega^2(s)\,ds$$

From the O_{22} , we have

$$\liminf_{t\to\infty} B_3\int_{T_2}^{t}\frac{H(t,s)}{H(t,T_2)}\times\frac{\omega^2(s)}{r(s)\rho(s)}ds = \infty$$

Thus .

$$\lim_{t\to\infty} U(t) = \infty ,$$

Hence

$$\lim_{t\to\infty} U(T_n) = \infty \qquad (2\text{-}21)$$

By inequality (2-20) there exists a constant F such that

$$U(T_n)+V(T_n)\le F \qquad ,n=1,2,3,\ldots$$

We have

$$\lim_{n\to\infty} V(T_n) = -\infty \tag{2-22}$$

By taking into account (2-21) and from (2-22), we derive $1+\frac{V(T_n)}{U(T_n)} < \frac{F}{U(T_n)} < \frac{1}{2}$ for all n is sufficiently large, thus,

$$\frac{V(T_n)}{U(T_n)} < -\frac{1}{2} \text{ for all large n .}$$

And

$$\lim_{n\to\infty} \frac{V^2(T_n)}{U(T_n)} < \infty \tag{2-23}$$

On the other hand, by Schwarz inequality, we have for any positive integer n

$$V^2(T_n) \le \frac{1}{H(T_n,T_2)} \int_{T_2}^{T_n} \frac{r(s)\rho(s)}{B_3} \left(h(T_n,s) - \frac{\dot{\rho}(s)}{\rho(s)} \sqrt{H(T_n,s)} \right) ds$$

$$\times \int_{T_2}^{T_n} \frac{B_3}{r(s)\rho(s)} \frac{1}{H(T_n,T_2)} H(T_n,s)\omega^2(s)\, ds$$

$$V^2(T_n) \le \frac{1}{B_3 H(T_n,T_2)} \int_{T_2}^{T_n} r(s)\rho(s) \left[h(T_n,s) - \frac{\dot{\rho}(s)}{\rho(s)} \sqrt{H(T_n,s)} \right]^2 ds\, U(T_n)$$

Or

$$\frac{V^2(T_n)}{U(T_n)} \le \frac{1}{H(T_n,T_2)} \int_{T_2}^{T_n} \frac{r(s)\rho(s)}{B_3} \left[h(T_n,s) - \frac{\dot{\rho}(s)}{\rho(s)} \sqrt{H(T_n,s)} \right]^2 ds$$

So, because of (2-23), we have

$$\lim_{n\to\infty}\frac{1}{H(T_n,T_2)}\int_{T_2}^{T_n}\frac{r(s)\rho(s)}{B_3}\left[h(T_n,T_2)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(T_n,s)}\right]^2 ds=\infty$$

Thus , for all $t \geq t_0$, we get ,

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int r(s)\rho(s)\left[h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right]^2 ds=\infty$$

Which contradicts O_{23} , thus suppose that

$$\int_{T_2}^{\infty}\frac{1}{r(s)\rho(s)}\omega^2(s)ds<\infty \qquad for \quad all\ T_2\geq t_0$$

Hence , and since

$$\Omega(T_2)\leq\omega(T_2)\,,$$

we get

$$\int_{t_0}^{\infty}\frac{\Omega_+^2(s)}{r(s)\rho(s)}ds\leq\int_{t_0}^{\infty}\frac{\omega^2(s)}{r(s)\rho(s)}ds<\infty$$

Which contradicts O_{26s} , hence , the proof is completed .

__Example3-9 :__

Consider the following differential equation:

$$\left[\left(\frac{1}{t^6}\right)\left(5+\frac{x^8(t)}{x^8(t)+1}\right)\left(\dot{x}(t)+\frac{\dot{x}(t)}{x^2(t)+2}\right)\right]^{\cdot}+\frac{1}{t^3}x^3(t)=0 \quad for\ t>t_0>0$$

We note that

1) $0<r(t)=\frac{1}{t^6}$ $\quad for\ t\geq t_0>0,$

2) $0<5\leq\Psi(x(t))=5+\frac{x^8(t)}{x^8(t)+1}<6 \quad for\ x\in\mathbb{R}$

3) $0<f^2(y)=y^2+\frac{2y^2}{y^2+2}+\frac{y^2}{(y^2+2)^2}<4y^2$

$$\frac{1}{4}f^2(y)<y^2<y^2+\frac{y^2}{y^2+2}=yf(y) \qquad for\ all\ y\neq 0$$

4) $\liminf\limits_{t\to\infty}\int_T^t q(s)ds=\liminf\limits_{t\to\infty}\int\frac{ds}{s^3}=\liminf\limits_{t\to\infty}\left[\frac{-1}{2t^2}+\frac{1}{2T^2}\right]=\frac{1}{2T^2}>0 \quad for\ t\geq T>0$

Let

$\rho(t)=2>0$ then we have

$$\dot{\rho}(t)=0\ \ and\ \left(r(t)\dot{\rho}(t)\right)^{\cdot}=0\ \ for\ t\geq t_0>0$$

and let

$$H(t,s)=(t-s)^2 \quad for\ \ t>s\geq t_0$$

5) $0<\inf\limits_{s\geq t_0}\left[\liminf\limits_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right]=\inf\limits_{s\geq t_0}\left[\liminf\limits_{t\to\infty}\frac{(t-s)^2}{(t-t_0)^2}\right]=1<\infty$

6) $\limsup\limits_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t r(s)\rho(s)\left[h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right]^2 ds$

$$= \limsup_{t\to\infty} \frac{4}{(t-t_0)^2} \int_{t_0}^{t} \frac{1}{s^6} ds = \limsup_{t\to\infty} \frac{8}{(t-t_0)^2} \left[\frac{-1}{5s^2} \right]_{t_0}^{t} = 0 < \infty$$

$$7)\ \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\rho(s)q(s) - \frac{r(s)\rho(s)}{4B_3} \left[h(t,s) - \frac{\dot{\rho}(s)}{\rho(s)} \sqrt{H(t,s)} \right]^2 \right] ds$$

$$= \liminf_{t\to\infty} \frac{1}{(t-t_0)^2} \int_{t_0}^{t} (t-s)^2 \frac{2}{s^3} ds - \frac{1}{4B_3} \int_{t_0}^{t} \frac{8}{s^6} ds$$

$$= 2 \liminf_{t\to\infty} \frac{1}{(t-t_0)^2} \left[\int_{t_0}^{t} \frac{t^2}{s^3} - \frac{2t}{s^2} + \frac{1}{s} - \frac{1}{B_3 s^6} \right] ds$$

$$= \frac{1}{t_0^2} > \frac{1}{2t_0^2}$$

Set

$$\Omega(T) = \frac{1}{2T^2} \quad \text{we get}$$

$$9)\ \int_{t_0}^{\infty} \frac{\Omega_+^2(s)}{r(s)\rho(s)} = \int_{t_0}^{\infty} (s^{-4} \times s^{-6})\, ds = \infty$$

it follows from theorem 2-9 that the equation is oscillatory .

Remark2-9 : theorem 2-9 extends the results of Grace [12] and [23].

Theorem 2-10 :

Suppose that O_2, O_8, O_9 and O_{15} hold and

O_{27}: $0 < \int_0^{\varepsilon} \frac{\Psi(u)}{g(u)} du < \infty$ *and* $\int_{-\varepsilon}^{0} \frac{\Psi(u)}{g(u)} du < \infty$ *for all* $\varepsilon > 0$

Assume that there exists a differentiable function

$$\delta : [t_0, \infty) \to (0, \infty)$$

such that

$$(r(t)\delta(t))^{\cdot} \geq 0,$$

O_{28} $\limsup_{t \to \infty} \frac{r(t)\delta(t)}{\int_{t_0}^{t} \delta(s)\, ds} < \infty$,

O_{29} $\lim_{t \to \infty} \frac{1}{\int_{t_0}^{t} \delta(s)\, ds} \int_{t_0}^{t} \left(\delta(s) \int_{t_0}^{s} q(u)\, du \right) ds = \infty$,

Then every solution of equation (2-1) is oscillatory .

Proof :

Let x(t) be anon oscillatory solution of equation (2-1) ,say x(t) > 0 ,for $t \geq T_1 \geq t_0$,it follows from lemma 2-1 ,we obtain $\dot{x}(t) > 0$ *on* $[T_2, \infty)$ *for all* $T_2 \geq T_1$

Define

$$\omega(t) = \frac{r(t)\Psi(x(t)) f(\dot{x}(t))}{g(x(t))}$$

thus ,for every $t \geq T_2 \geq t_0$, *we obtain*

$$\dot{\omega}(t) \le -q(t) - k_7 \frac{r(t)\Psi'(x(t))\dot{x}(t) g'(x(t)) \dot{x}(t)}{g(x(t))}$$

thus, for every $t \ge T_2$, *we have*

$$\omega(t) \le \omega(T_2) - \int_{T_2}^{t} q(s)\, ds - k_7 \int_{T_2}^{t} \frac{r(s)\Psi'(x(s))\dot{x}^2(s) g'(x(s))}{g^2(x(s))}\, ds$$

$$k_7 \frac{r(t)\Psi'(x(t))\dot{x}(t)}{g(x(t))} + k_7 \int_{T_2}^{t} \frac{r(s)\Psi'(x(s))\dot{x}^2(s) g'(x(s))}{g^2(x(s))}\, ds + \int_{T_2}^{t} q(s)\, ds \le \omega(T_2) = c$$

$$where \quad c = \frac{r(T_2)\Psi(x(T_2)) f(\dot{x}(T_2))}{g(x(T_2))}$$

Multiplying the last inequality by $\delta(t)$ and integrating, and dropping the second term in the last inequality, which is nonnegative, we obtain

$$k_7 \int_{T_2}^{t} (r(s)\delta(s)) \frac{\Psi(x(s))\dot{x}(s)}{g(x(s))}\, ds + \int_{T_2}^{t} \left(\delta(s) \int_{T_2}^{s} q(\tau)\, d\tau \right) ds \le c \int_{T_2}^{t} \delta(s)\, ds$$

Now we use the O_{27} and integral the last inequality by parts to obtain ,

$$k_7 r(t)\delta(t) \int_{T_2}^{t} \frac{\Psi(x(s))\dot{x}(s)}{g(x(s))}\, ds - k_7 \int_{T_2}^{t} (r(s)\delta(s))' \int_{T_2}^{s} \frac{\Psi(x(\tau))\dot{x}(\tau)}{g(x(\tau))}\, d\tau\, ds + \int_{T_2}^{t} \delta(s) \int_{T_2}^{s} q(\tau)\, d\tau\, ds$$

$$\le c \int_{T_2}^{t} \delta(s)\, ds$$

$$k_7 r(t)\delta(t) \int_{x(T_2)}^{x(t)} \frac{\Psi(u)}{g(u)}\, du - k_7 \int_{T_2}^{t} (r(s)\delta(s))' \int_{x(T_2)}^{x(s)} \frac{\Psi(u)}{g(u)}\, du\, ds + \int_{T_2}^{t} \delta(s) \int_{T_2}^{s} q(\tau)\, d\tau\, ds \le c \int_{T_2}^{t} \delta(s)\, ds$$

$$k_1 r(t)\delta(t)\int_0^{x(t)}\frac{\Psi(u)}{g(u)}\,du-k_1 r(t)\delta(t)\int_0^{x(T_2)}\frac{\Psi(u)}{g(u)}\,du-k_1\int_{T_2}^{t}\left(r(s)\delta(s)\right)'\int_0^{x(s)}\frac{\Psi(u)}{g(u)}du\,ds$$

$$+k_1\int_{T_2}^{t}\left(r(s)\delta(s)\right)'\int_0^{x(T_2)}\frac{\Psi(u)}{g(u)}du\,ds+\int_{T_2}^{t}\delta(s)\int_{T_2}^{s}q(\tau)d\tau\,ds\le c\int_{T_2}^{t}\delta(s)ds$$

Set

$$G(t)=\int_0^{x(t)}\frac{\Psi(u)}{g(u)}\,du \qquad ,we\ \ have$$

$$r(t)\delta(t)G(t)-r(t)\delta(t)G(T_2)-\int_{T_2}^{t}\left(r(s)\delta(s)\right)'G(s)ds+r(t)\delta(t)G(T_2)-r(T_2)\delta(T_2)G(T_2)$$

$$+\frac{1}{k_1}\int_{T_2}^{t}\delta(s)\int_{T_2}^{s}q(\tau)d\tau\,ds\le\frac{c}{k_1}\int_{T_2}^{t}\delta(s)\,ds \qquad (2-24)$$

we consider two possibilities . Either exists a sequence $\{T_n\}_{n=1,2,3,\ldots}$ such that

$$r(T_n)\delta(T_n)G(T_n)-\int_{t}^{T_n}\left(r(s)\delta(s)\right)'G(s)\,ds\ge 0$$

or there exists $T_4\ge T_2$ such that

$$r(t)\delta(t)G(t)-\int_{T_2}^{t}\left(r(s)\delta(s)\right)'G(s)\,ds\le 0 \quad for\ \ t\ge T_4 \qquad (2-25)$$

In the former case we get that a contradiction by dividing (2-24) by $\int_{T_2}^{t}\delta(s)\,ds$,we have

$$\frac{r(t)\delta(t)}{\int_{T_2}^{t}\delta(s)\,ds}\int_0^{x(t)}\frac{\Psi(u)}{g(u)}du-\frac{1}{\int_{T_2}^{t}\delta(s)\,ds}\int_{T_2}^{t}\left(r(s)\delta(s)\right)'\int_0^{x(s)}\frac{\Psi(u)}{g(u)}\,du\,ds+\frac{1}{k_1\int_{T_2}^{t}\delta(s)\,ds}\int_{T_2}^{t}\delta(s)\int_{T_2}^{s}q(\tau)d\tau\,ds$$

$$\leq \frac{c}{k_7 \int_{T_2}^{t} \delta(s)\, ds} \int_{T_2}^{t} \delta(s)\, ds + \frac{r(T_2)\delta(T_2)}{\int_{T_2}^{t} \delta(s)\, ds} \int_{0}^{x(T_2)} \frac{\Psi(u)}{g(u)}\, du$$

$$\frac{r(t)\delta(t)}{\int_{T_2}^{t} \delta(s)\, ds} \int_{0}^{x(t)} \frac{\Psi(u)}{g(u)}\, du \leq \frac{1}{\int_{T_2}^{t} \delta(s)\, ds} \int_{T_2}^{t} (r(s)\delta(s))' \int_{0}^{x(s)} \frac{\Psi(u)}{g(u)} du\, ds - \frac{1}{k_7 \int_{T_2}^{t} \delta(s)\, ds} \int_{T_2}^{t} \delta(s) \int_{T_2}^{s} q(\tau)\, d\tau\, ds$$

$$+ \frac{c}{k_7} + \frac{r(T_2)\delta(T_2)}{\int_{T_2}^{t} \delta(s)\, ds} \int_{0}^{x(T_2)} \frac{\Psi(u)}{g(u)}\, du$$

By taking the limit superior on both sides ,we get

$$\limsup_{t \to \infty} \frac{r(t)\delta(t)}{\int_{T_2}^{t} \delta(s)\, ds} \int_{0}^{x(t)} \frac{\Psi(u)}{g(u)}\, du \leq -\infty$$

which contradicts to the O_{28} . In the second case we write

$$R(t) = \int_{T_3}^{t} (r(s)\delta(s))' G(s)\, ds \ .$$

which is nonnegative and nondecreasing .we can assume that T_4 is chosen so that $R(t)>0$, $t \geq T_4$, from (2-25) we have,

$$r(t)\delta(t)G(t) \leq R(t) \qquad (2-26)$$

$$(r(t)\delta(t))' G(t) \leq \dot{R}(t) - (r(t)\delta(t))\dot{G}(t)$$

$$\left(r(t)\delta(t)\right)^{\bullet} G(t) \leq \dot{R}(t) \qquad (2-27)$$

From (2-26) and (2-27) ,we get

$$\left(r(t)\delta(t)\right)G(t)\dot{R}(t) - \left(r(t)\delta(t)\right)^{\bullet} G(t)R(t) \leq 0$$

Then

$$\left(r(t)\delta(t)\right)\dot{R}(t) - \left(r(t)\delta(t)\right)^{\bullet} \leq 0$$

$$\left(r(t)\delta(t)\right)\dot{R}(t) \leq \left(r(t)\delta(t)\right)^{\bullet} R(t) \ , \qquad (2-28)$$

Now,

$$\left(\frac{r(t)\delta(t)}{R(t)}\right)^{\bullet} = \frac{R(t)\left(r(t)\delta(t)\right)^{\bullet} - \left(r(t)\delta(t)\right)\dot{R}(t)}{R^2(t)}$$

From (2-28) ,we have

$$\left(\frac{r(t)\delta(t)}{R(t)}\right)^{\bullet} \geq 0$$

Thus ,for every $t \geq T$ we get

$$\int_T^t \left(\frac{r(s)\delta(s)}{R(s)}\right)^{\bullet} ds \geq 0$$

$$\frac{r(t)\delta(t)}{R(t)} \geq \frac{r(T)\delta(T)}{R(T)} \geq \frac{r(T)\delta(T)}{R(t)} \ ,$$

Then ,for all $t \geq T$, we get

$$R(t) \le \frac{R(T)}{r(T)\delta(T)} r(t)\delta(t)$$

$$0 < r(t)\delta(t)G(t) \le R(t) \le R(T)\frac{r(t)\delta(t)}{r(T)\delta(T)}$$

Hence

$$0 < G(t) \le \frac{R(t)}{r(t)\delta(t)} \le \frac{R(T)}{r(T)\delta(T)} \qquad\qquad for \quad t \ge T$$

It follows that G(t) is bounded ,Say by A_1 and noting that $(r(t)\delta(t))' \ge 0$ $for\ t \ge T_2$.

Returning to (2-24) we get

$$\frac{1}{k_7}\int_{T_2}^{t}\delta(s)\int_{T_2}^{s} q(\tau)d\tau\ ds \le \frac{c}{k_7}\int_{T_2}^{t}\delta(s)\,ds + r(T_2)\delta(T_2)G(T_2) - A_1 r(t)\delta(t) \qquad (2-29)$$

Now ,dividing by $\int_{t_0}^{t}\delta(s)\ ds$ and apply the O_{28} and taking into account (2-29) ,we drive

$$\lim_{t\to\infty}\frac{1}{\int_{t}^{t}\delta(s)\ ds}\int_{t_0}^{t}\left(\delta(s)\int_{t_0}^{s} q(u)du\right)ds < -\infty .$$

which contradicts O_{29} ,Hence,the proof is completed .

<u>Example 2-10</u> :

Consider the following differential equation

$$\left[\left(\frac{t+1}{t+2}\right)\left(\frac{x^4(t)}{x^4(t)+1}\right)\left(7\dot{x}(t)+\frac{\dot{x}^{3}(t)}{\dot{x}^{2}(t)+3}\right)\right]^{\cdot} + \left(\frac{1}{t}+\sin t\right)x^3(t) = 0 \qquad for\ t > t_0 \ge 0$$

We note that

$$1-\ r(t)=\frac{t+1}{t+2}>0 \quad and \quad \lim_{t\to\infty}\frac{1}{r(t)}=\lim_{t\to\infty}\frac{t+2}{t+1}=1>0 \ . \qquad for \quad t>t_0\geq 0 \ .$$

$$2-\ 0<k_2\leq\Psi(x(t))=\frac{x^4(t)}{x^4(t)+1}\leq 1 \qquad for\ all\ \ x\in\mathbb{R}\ ,\ \ .$$

$$3-\ \liminf_{t\to\infty}\int_T^t q(s)\,ds=\liminf_{t\to\infty}\int_T^t\left(\frac{1}{s}+\sin s\right)ds=\liminf_{t\to\infty}\left[\ln t-\cos t-\ln T+\cos T\right]$$
$$=\infty>0 \ ,$$

$$4\text{-}\ xg(x)=x^4>0\ \ and\ \ g'(x)=3x^2>0 \qquad for\ \ all\ x\neq 0\ .$$

$$5-\ \ 0<\int_0^\varepsilon\frac{\Psi(u)}{g(u)}du=\int_0^\varepsilon\frac{u^4}{u^4+1}\ \ \frac{1}{u^3}\,du=\int_0^\varepsilon\frac{u}{u^4+1}du=\int_0^\varepsilon\frac{u}{(u^2)^2+1}du=\frac{1}{2}\ \tan^{-1}u^2\Big|_0^\varepsilon$$
$$=\frac{1}{2}\ \tan^{-1}\varepsilon^2<\infty$$

$$\int_{-\varepsilon}^0\frac{\Psi(u)}{g(u)}\,du=\int_{-\varepsilon}^0\frac{u}{u^4+1}du=\frac{1}{2}\ \tan^{-1}\varepsilon^2<\infty \ ,$$

Let $\delta(t)=1$,we have $\left(r(t)\delta(t)\right)'=\dfrac{1}{(t+2)^2}>0$ for all $t\geq t_0>0$

Hence ,

$$6-\ \limsup_{t\to\infty}\frac{r(t)\delta(t)}{\int_{t_0}^{t}\delta(s)\,ds}=\limsup_{t\to\infty}\frac{t+1}{(t+2)(t-t_0)}=0<\infty\ ,$$

$$7-\lim_{t\to\infty}\frac{1}{\int_{t_0}^{t}\delta(s)\,ds}\int_{t_0}^{t}\delta(s)\int_{t_0}^{s}q(u)\,du\,ds\ =\lim_{t\to\infty}\frac{1}{t-t_0}\left[s\ln s-s-\sin s-s\ln t_0+s\cos t_0\right]_{t_0}^{t}$$

$$=\lim_{t\to\infty}\frac{t\ln t}{t-t_0}-\lim_{t\to\infty}\frac{\sin t}{t-t_0}-\ln t_0\lim_{t\to\infty}\frac{t}{t-t_0}+\cos\lim_{t\to\infty}\frac{t}{t-t_0}$$

$$+(t_0+\sin t_0-t_0\cos t_0)\lim_{t\to\infty}\frac{1}{t-t_0}=\infty\ ,$$

it follows from theorem 2-10 that the equation is oscillatory .

<u>Remark 2-10</u> : Theorem 2-10 extends the results of Grace and Lalli [11]and [23].

CHAPTER (3)

THE BOUNDNESS AND THE OSCILLATION OF THE EQUATION

$$\left(r(t)\Psi(x(t))f(\dot{x}(t))\right)^{\cdot} + q(t)g(x(t)) = H(t,\dot{x}(t),x(t))$$

3.1 Introduction:

In this chapter we study the boundness and the oscillation of the regular solutions of the equation

$$\left(r(t)\Psi(x(t))f(\dot{x}(t))\right)^{\cdot} + q(t)g(x(t)) = H(t,\dot{x}(t),x(t)) \qquad (3\text{-}1)$$

We shall discuss the boundness of equation (3-1) in the case where $\Psi(x(t))=1$ and the oscillation in the case where $\Psi(x(t))$ is of more general form. Special cases of the function f will be discussed.

3.2 Boundness of the solutions :

Definition 3.1

A solution $x(t)$ of equation (3-1) is bounded if there exists a positive constant M_0 such that $|x(t)| \le M_0 \qquad \forall t \ge T \ge t_0$.

In the following we discuss the sufficient conditions to ensure the boundness of the solutions of the following equation

$$\left(r(t)f(\dot{x}(t))\right)^{\cdot}+q(t)g(x(t))=H(t,\dot{x}(t),x(t)) \qquad (3\text{-}2)$$

where q and r are continuous functions on the interval $[t_0,\infty)$, $t_0 \geq 0$, r is a positive function, f is a continuous function on the real line R with $y f(y)>0\ \forall y \neq 0$ and $y f(y)-\int_0^y f(u)\,du \geq k_0 y^2$ for some $k_0>0$ and $\forall\ y \in R$, g is a continuous function on the real line R with $xg(x)>0\ \forall\ x \neq 0$ and H is a continuous function on $[t_0,\infty)\times R^2$ with $\left|H(t,\dot{x}(t),x(t))\right| \leq |p(t)|\quad \forall t \in [t_0,\infty)$ and x, $y \in R$.

Throughout this section. the following notation is used

$$G(x)=\int_0^x g(v)\,dv \qquad \text{and} \qquad R(t)=\int_{t_0}^{t}\frac{|p(s)|}{r(s)}\,ds$$

We rewrite equation(3-2) in the following equivalent form

$$\begin{cases}\dot{x}(t)=y \\ (f(y))^{\cdot}=\dfrac{H(t,y,x)-\dot{r}(t)f(y)-q(t)g(x(t))}{r(t)}\end{cases} \qquad (I)$$

<u>Theorem 3-1 :</u>

Suppose that

(B_1) $G(x(t))$ is bounded below and $G(x)\to\infty$ as $|x|\to\infty$.,

(B_2) q and r: $[t_0,\infty)\to(0.\infty)$ are nondecreasing functions on $[t_0,\infty)$ and r(t) is bounded $\forall t \geq t_0$,

(B_3) $\lim_{t\to\infty} R(t)<\infty$, Then, all solutions of the equation (3-2) are bounded.

Proof:

Since G(x) is bounded below and $G(x) \to \infty$ as $|x| \to \infty$,say $G(x) \geq -\beta_1$ for some $\beta_1 > 0$ for all $x \in R$ .We define the function V as

$$V(t) = \frac{G(x)+\beta_1}{r(t)} + \frac{yf(y) - \int_0^y f(u)du}{q(t)} \qquad ,t \geq t_0$$

Then, for all $t \geq t_0$. We have

$$\dot{V}(t) = \frac{\dot{x}(t)g(x(t))}{r(t)} - \frac{\left(G(x(t))+\beta_1\right)\dot{r}(t)}{r^2(t)} + \frac{y(f(y))^{\bullet} + \dot{y}f(y) - \dot{y}f(y)}{q(t)}$$
$$- \frac{\left(yf(y) - \int_0^y f(u)du\right)\dot{q}(t)}{q^2(t)} \qquad ,t \geq t_0$$

By (I) we get

$$\dot{V}(t) = \frac{y(t)g(x(t))}{r(t)} - \frac{\left(G(x)+\beta_1\right)\dot{r}(t)}{r^2(t)} + \frac{y(t)H(t,y(t),x(t))}{r(t)q(t)} - \frac{y(t)f(y)\dot{r}(t)}{r(t)q(t)}$$
$$- \frac{y(t)g(x(t))}{r(t)} - \frac{\left(yf(y) - \int_0^y f(u)du\right)}{q^2(t)}\dot{q}(t) \qquad ,t \geq t_0$$

Then , taking into account above conditions ,

$$\dot{V}(t) \leq \frac{y(t)H(t,y(t),x(t))}{r(t)q(t)} \qquad for\ t \geq t_0 \ ,$$

Then , for every $t \geq t_0$ we have

$$V(t) \leq V(t_0) + \int_{t_0}^{t} \frac{y(s)H(s, y(s), x(s))}{r(s)q(s)} ds$$

Hence , $\forall t \geq t_0$ we have

$$k_0 \frac{y^2(t)}{q(t)} \leq \frac{yf(y) - \int_0^y f(u)du}{q(t)} \leq V(t) \leq V(t_0) + \int_{t_0}^{t} \frac{y(s)H(s, y(s), x(s))}{r(s)q(s)} ds$$

Since $y \leq \frac{1}{2}(y^2 + 1)$ we obtain

$$\frac{y(t)}{q(t)} \leq \frac{1}{2q(t)} + \frac{V(t_0)}{2k_0} + \frac{1}{2k_0} \int_{t_0}^{t} \frac{y(s)H(s, y(s), x(s))}{r(s)q(s)} ds$$

Since q is nondecreasing function, we have

$$\left|\frac{y(t)}{q(t)}\right| \leq \beta_2 + \frac{1}{2k_0} \int_{t_0}^{t} \left|\frac{p(s)}{r(s)}\right| \left|\frac{y(s)}{q(s)}\right| ds$$

where $\beta_2 = \frac{1}{2q(t_0)} + \frac{V(t_0)}{2k_0}$ is a positive constant .

By the Gronwall's inequality , we get

$$\left|\frac{y(t)}{q(t)}\right| \leq \beta_2 + \exp\left[\frac{1}{2k_0} \int_{t_0}^{t} \frac{|p(s)|}{r(s)} ds\right] \leq \beta_3 < \infty \qquad (3\text{-}3)$$

Therefore,

$$V(t) \leq V(t_0) + \beta_3 \int_{t_0}^{t} \frac{|p(s)|}{r(s)} ds \leq \beta_4 < \infty$$

Hence , V(t) is bounded ,since $V(t) \geq \frac{G(x)+\beta_1}{r(t)}$

and r(t) is bounded , thus G(x) is bounded and hence x(t) is bounded .

This completes the proof of the theorem .

Remark 3-1: If $q(t)$ is bounded,then $y(t) = \dot{x}(t)$ is also bounded. This can be seen easily from (3-3).

__Example 3-1__

Consider the following differential equation

$$\left(\left(\frac{t}{t+1}\right)(\dot{x}(t)+\dot{x}(t))\right)^{\cdot} + (t^2+3t) = \frac{e^{-2t}}{t+1}\sin\dot{x}(t)\cos x(t) \quad for\ t \geq t_0 > 0$$

We note that

1- $0 < r(t) = \frac{t}{t+1} \leq 1 \quad and \quad \dot{r}(t) = \frac{1}{(t+1)^2} > 0 \qquad \forall\, t \geq t_0 > 0$,

2- $q(t) = t^2 + 3t > 0 \quad and \quad \dot{q}(t) = 2t + 3 > 0 \qquad for\ all\ t \geq t_0 > 0$

3- $\left|H(t,\dot{x}(t),x(t))\right| = \left|\frac{te^{-2t}\sin\dot{x}(t)\cos x(t)}{t+1}\right| \leq \left|\frac{te^{-2t}}{t+1}\right| = p(t) \quad for\ all\ \dot{x} \in \mathbb{R}\ and\ t \in [t_0,\infty)$

$$4 - yf(y) - \int_0^y f(u)du = y^2 + y^4 - \int_0^y (u+u^3)du = y^2 + y^4 - \frac{y^2}{2} - \frac{y^4}{4}$$

$$= \frac{y^2}{2} + \frac{3}{4}y^4 \geq \frac{1}{2}y^2 .$$

5- $xg(x) = x^4 > 0$ *for all* $x \neq 0$ *and* $G(x) = \int_0^x g(u)\,du = \int_0^x u^3\,du = \frac{x^4}{4} \geq 0 > -\beta_1$

for all $x \in R$. $\beta_1 > 0$ *and* $G(x) \to \infty$ *as* $|x| \to \infty$,

6- $R(t) = \int_{t_0}^{t} \frac{|p(s)|}{r(s)}\,ds = \int_{t_0}^{t} \frac{s e^{-7s}}{s+1} \times \frac{s+1}{s}\,ds = \frac{1}{7}(e^{-7t_0} - e^{-7t})$

Implies that

$$\lim_{t\to\infty} R(t) = \lim_{t\to\infty} \frac{1}{7}(e^{-7t_0} - e^{-7t}) = \frac{1}{7} e^{-7t_0} < \infty ,$$

it follows from Theorem (3-1) that all solutions of the equation is bounded .easily from (3-1).

Theorem 3-2 :

Suppose that the conditions (B_1) and (B_5) hold and

(B_1) $r : [t_0, \infty) \to (0, \infty)$ is nondecreasing and bounded function on $[t_0, \infty)$ and q(t) is a positive on $[t_0, \infty)$,

(B_5) $\gamma(t) = \frac{q(t) - 1}{r(t)}$ is a positive nonincreasing function on $[t_0, \infty)$,

then all solutions of (3-2) are bounded .

Proof :

Since G(x) is bounded below and $G(x) \to \infty$ *as* $|x| \to \infty$,say $G(x) \geq -\beta_1$ for some $\beta_1 > 0$ *and* $x \in R$. Now we define the function V as

$$V(t) = \frac{G(x) + \beta_1}{r(t)} + yf(y) - \int_0^y f(u)\,du \qquad ,t \geq t_0$$

Then for all $t \geq t_0$, we have

$$\dot{V}(t) = \frac{\dot{x}(t)g(x(t))}{r(t)} - \frac{(G(x)+\beta_1)\dot{r}(t)}{r^2(t)} + y(f(y))^{\bullet} + \dot{y}f(y) - \dot{y}f(y)$$

By (I) we get

$$\dot{V}(t) = \frac{y(t)g(x(t))}{r(t)} - \frac{(G(x(t)+\beta_1)\dot{r}(t)}{r^2(t)} + \frac{y(t)H(t,y(t),x(t))}{r(t)} - \frac{y(t)f(y)\dot{r}(t)}{r(t)} - \frac{y(t)q(t)g(x(t))}{r(t)}$$

Then , taking into account the above conditions , we obtain

$$\dot{V}(t) \leq \frac{y(t)H(t,\dot{x}(t),x(t))}{r(t)} + \frac{y(t)g(x(t))}{r(t)} - \frac{y(t)q(t)g(x(t))}{r(t)}$$

$$\dot{V}(t) \leq \frac{y(t)H(t,\dot{x}(t),x(t))}{r(t)} - \left(\frac{q(t)-1}{r(t)}\right)(y(t)g(x(t)))$$

Then , for every $t \geq t_0$, we get

$$V(t) \leq V(t_0) + \int_{t_0}^{t} \frac{y(s)H(s,\dot{x}(s),x(s))}{r(s)}ds - \int_{t_0}^{t}\left(\frac{q(s)-1}{r(s)}\right)y(s)g(x(s))ds$$

$$= V(t_0) + \int_{t_0}^{t} \frac{y(s)H(s,\dot{x}(s),x(s))}{r(s)}ds - \int_{t_0}^{t} \gamma(s)y(s)g(x(s))ds$$

By the Bonnet theorem there exists $a_t \in [t_0, t]$ such that

$$\int_{t_0}^{t}\gamma(s)y(s)g(x(s))ds = \gamma(t_0)\int_{t_0}^{a_t}g(x(s))\dot{x}(s)ds = \gamma(t_0)\int_{x(t_0)}^{x(a_t)}g(u)du$$

$$= \gamma(t_0)\int_0^{x(a_t)}g(u)du - \gamma(t_0)\int_0^{x(t_0)}g(u)du$$

$$= \gamma(t_0)G(x(a_t)) - \gamma(t_0)G(x(t_0))$$

Hence , $\forall t \geq t_0$, we have

$$V(t) \leq V(t_0) + \int_{t_0}^{t}\frac{y(s)H(s,\dot{x}(s),x(s))}{r(s)}ds - \gamma(t_0)G(x(a_t)) + \gamma(t_0)G(x(t_0))$$

Then , from the condition (B_1) we obtain

$$V(t) \leq V(t_0) + \gamma(t_0)G(x(t_0)) + \beta_1\gamma(t_0) + \int_{t_0}^{t}\frac{y(s)H(s,\dot{x}(s),x(s))}{r(s)}ds$$

Hence , for all $t \geq t_0$, we have

$$k_0 y^2 \leq yf(y) - \int_0^{y}f(u)du \leq V(t) \leq V(t_0) + \gamma(t_0)G(x(t_0)) + \beta_1\gamma(t_0)$$

$$+ \int_{t_0}^{t}\frac{y(s)H(s,\dot{x}(s),x(s))}{r(s)}ds$$

and so , we can obtain that

$$k_0\left(y^2+1\right) \leq k_0 + V(t_0) + \gamma(t_0)(G(x(t_0)) + \beta_1) + \int_{t_0}^{t}\frac{y(s)H(s,\dot{x}(s),x(s))}{r(s)}ds \qquad (3\text{-}4)$$

But $y \leq \frac{1}{2}(y^2+1)$ and $k_0>0$

Then $k_0 y \le \frac{k_0}{2}(y^2+1)$

By substituting in inequality (3-4) we obtain

$$y(t) \le \beta_4 + \frac{1}{2k_0}\int_{t_0}^{t}\frac{y(s)H(s,\dot{x}(s),x(s))}{r(s)}ds$$

where $\beta_4 = \frac{1}{2} + \frac{V(t_0)}{2k_0} + \frac{\gamma(t_0)G(x(t_0))}{2k_0} + \frac{\beta_1\gamma(t_0)}{2k_0}$ is a positive constant

There for , we have

$$|y(t)| \le \beta_4 + \frac{1}{2k_0}\int_{t_0}^{t}\frac{|p(s)|}{r(s)}|y(s)|ds$$

By the Gronwll's inequality , we get

$$|y(t)| \le \beta_4 \exp\frac{1}{2k_0}\int_{t_0}^{t}\frac{|p(s)|}{r(s)}ds \le \beta_5 < \infty \qquad (3\text{-}5)$$

Therefor ,

$$V(t) \le V(t_0) + \beta_5\int_{t_0}^{t}\frac{|p(s)|}{r(s)}ds \le V(t_0) + \gamma(t_0)[\beta_1 + G(x(t_0))] + \beta_5\int_{t_0}^{t}\frac{|p(s)|}{r(s)}ds < \infty$$

Hence , V(t) is bounded . Since $V(t) \geq \frac{G(x)+\beta_1}{r(t)}$ and r(t) is bounded , thus G(x) is bounded and hence x(t) is bounded .

This completes the proof of the theorem .

Example 3-2 :

Consider the following differential equation

$$\left[\left(\frac{t^2}{t^2+1}\right)\left(\dot{x}(t)+\dot{x}^5(t)\right)\right]^{\cdot}+\left(\frac{t^7e^{-t}}{1+t^6}+1\right)\left(x^9(t)+\frac{8x^7(t)}{1+x^8(t)}\right)=\frac{1}{2(t^2+1)^2}\times\frac{4\sin 4t}{(1+\cos^2 2t)}\ ,$$

$t \geq t_0 \geq 5$

We note that

(1) $1 > r(t) = \frac{t^2}{t^2+1} > 0$ *and* $\dot{r}(t) = \frac{2t}{(t+1)^2} > 0$ *for all* $t \geq t_0 \geq 5$.

(2) $\gamma(t) = \frac{q(t)-1}{r(t)} = \left[\frac{t^7e^{-t}}{1+t^2}+1-1\right]\frac{t^2+1}{t^2} = t^5e^{-t} > 0$,

$\dot{\gamma}(t) = 5t^4e^{-t} - t^5e^{-t} = t^4e^{-t}(5-t) \leq 0 \qquad \forall t \geq t_0 \geq 5,$

(3) $\left|H(t,\dot{x}(t),x(t))\right| = \left|\frac{1}{2(1+t^2)^2}\times\frac{\sin 4t}{(1+\cos^2 2t)}\right| \leq \left|\frac{\sin 4t}{(t^2+1)(1+\cos^2 2t)}\right| = |p(t)|$.

(4) $yf(y) - \int_0^y f(u)\,du = y^2 + y^6 - \frac{y^2}{2} - \frac{y^6}{6} = \frac{y^2}{2} + \frac{5}{6}y^6 \geq \frac{1}{2}y^2$.

(5) $xg(x) = x(x^9 + \frac{8x^7}{x^8+1}) = x^{10} + \frac{8x^8}{x^8+1} > 0$ *for all* $x \neq 0$ and

$$G(x) = \int_0^x g(u)du$$

$$= \int_0^x \left(u^9 + \frac{8u^7}{u^8+1} \right) du = \frac{u^{10}}{10} + \ln(u^8+1) \Big|_0^x = \frac{1}{10}x^{10} + \ln(1+x^8) \geq 0 > -\beta_1 \ \forall x \in R$$

and $\beta_1 > 0$ *and* $G(x) \to \infty$ *as* $|x| \to \infty$,

(6) $$R(t) = \int_{t_0}^t \frac{|p(s)|}{r(s)} ds = \int_{t_0}^t \frac{|\sin 4s|}{(s^2+1)(1+\cos^2 2s)} \times \frac{s^2+1}{s^2} ds$$.

$$= \int_{t_0}^t \frac{|\sin 4s|}{s^2(1+\cos^2 2s)} ds \leq \int_{t_0}^t \frac{ds}{s^2} = \frac{-1}{t} + \frac{1}{t_0}$$

Hence ,

$$\lim_{t\to\infty} R(t) \leq \lim_{t\to\infty} \left(\frac{1}{t_0} - \frac{1}{t}\right) = \frac{1}{t_0} < \infty ,$$

it follows from theorem (3-2) that all solutions of the equation is bounded .

Remark (3-2) :

Assuming, in equation (3-2), that $f(\dot{x}) = \dot{x}$, then Theorem 3-1 and Theorem 3-2 extend the results of Burton and Townsend [5], Burton [6], Olkhiuk [24] and Greaf and Spikes [13] to more general solutions.

3.3 oscillation of the solutions :

The oscillatory behavior for the Differential equation of the form

$$\left(r(t)\Psi(x(t))f(\dot{x}(t)) \right)^{\bullet} + q(t)g(x(t)) = H(t,\dot{x}(t),x(t)) \quad ,t \geq t_0 \qquad (3\text{-}1)$$

Will be discussed in this section. where q and r are continuous functions on the interval $[t_0,\infty)$, $t_0 \geq 0$. r(t) is a positive function , f is continuous function on the real line R with y f(y) > 0 for all $y \neq 0$, Ψ is continuous function on the real line R with $\Psi(x) > 0$ $\forall x \in$ R, g is continuously differentiable function on the real line R except possibly at 0 with x g(x) > 0 and $g'(x) \geq k > 0$ *for all* $x \neq 0$ and H is a continuous function on $[t_0,\infty) \times R^2$ with $\frac{H(t,y,x)}{g(x)} \leq p(t)$ $\forall t \in [t_0,\infty)$, $y \in$ R and $x \neq 0$.

Throughout this study, we restrict our attention only to the solutions of the differential equation (3-1) which exists on some ray $[t_0,\infty)$, $t_0 \geq 0$ may depend on a particular Solution

<u>**Theorem 3-3**</u> :

Suppose that

(1) $0 < r(t)$ $\forall t \geq t_0$.

(2) $0 < k_2 \leq \Psi(x(t)) \leq k_3$ *for all* $x \in$ R ,

(3) $0 < k_4 \leq \frac{f(y)}{y} \leq k_5$ *for all* $y \neq 0$,

(4) $\frac{H(t,\dot{x}(t),x(t))}{g(x(t))} \leq p(t)$ *for all* $x \neq 0$, $\dot{x}(t) \in$ R $t \in [t_0,\infty)$,

(5) $\int_{t_0}^{\infty} \frac{ds}{r(s)} = \infty$,

(6) $\int_{t_0}^{\infty} [q(s) - p(s)]ds = \infty$,

then every solutions of equation (3-1) is oscillatory .

Proof :

Let x(t) be a non oscillatory solution of (3.1) , say x(t) > 0 for $t \ge t_1 \ge t_0$, then

$$\left(\frac{r(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))}\right)^{\bullet} = \frac{\left(r(t)\Psi(x(t))f(\dot{x}(t))\right)^{\bullet}}{g(x(t))} - \frac{r(t)\Psi(x(t))f(\dot{x}(t))g'(x(t))\dot{x}(t)}{g^2(x(t))}$$

$$= \frac{H(t,\dot{x}(t),x(t))}{g(x(t))} - q(t) - \frac{r(t)\Psi(\dot{x}(t))f(\dot{x}(t))g'(x(t))\dot{x}(t)}{g^2(x(t))}$$

$$\le p(t) - q(t) \qquad (3\text{-}6)$$

Then, for every $t \ge t_1$, we have

$$\frac{r(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))} \le \frac{r(t_1)\Psi(x(t_1))f(\dot{x}(t_1))}{g(x(t_1))} - \int_{t_1}^{t}[q(s) - p(s)]\,ds$$

By condition (6) there exists $T_0 \ge t_1$ such that

$$f(\dot{x}(t)) < 0 \qquad for\ t \ge T_0 \ge t_1$$

Then , $\forall t \ge T_0$, we have

$$\dot{x}(t) < 0 \qquad for\ t \ge T_0 .$$

The condition (6), also implies that, there exists $T \ge T_0$ such that

$$\int_{T_0}^{T}[q(s) - p(s)]\,ds = 0 \ and \ \int_{T}^{t}[q(s) - p(s)]ds \ge 0 \qquad for\ t \ge T$$

Integrating Equation (3-1) by parts , we have

$$r(t)\Psi(x(t))f(\dot{x}(t)) = r(T)\Psi(x(T))f(\dot{x}(T)) + \int_T^t H(s,\dot{x}(s),x(s))\,ds - \int_T^t g(x(s))q(s)\,ds$$

$$\leq r(T)\Psi(x(T))f(\dot{x}(T)) - \int_T^t g(x(s))[q(s)-p(s)]\,ds$$

$$= r(T)\Psi(x(T))f(\dot{x}(T)) - g(x(t))\int_T^t [q(s)-p(s)]\,ds + \int_T^t \dot{x}(s)g'(x(s)) \times$$

$$\int_T^s [q(u)-p(u)]\,du\,ds$$

Hence , for $t \geq T$, we obtain

$$r(t)\Psi(x(t))f(\dot{x}(t)) \leq r(T)\Psi(x(T))f(\dot{x}(T))$$

Then , for $t \geq T$, we have

$$k_3k_5 r(t)\dot{x}(t) \leq r(T)\Psi(x(T))f(\dot{x}(T)) \qquad for\ t \geq T$$

Thus

$$\dot{x}(t) \leq \frac{r(T)\Psi(x(T))f(\dot{x}(T))}{k_3k_5}\ \frac{1}{r(t)}$$

By integrating the last inequality from T to t we obtain

$$x(t) \leq \frac{r(T)\Psi(x(T))f(\dot{x}(T))}{k_3k_5}\int_T^t \frac{ds}{r(s)} \to -\infty \qquad as\ t \to \infty$$

which is a contradiction to the fact that x(t)>0, Hence , the proof is completed.

Example 3-3 :

Consider the following differential equation

$$\left[\left(\frac{t^2}{t+1}\right)\left(2+\frac{x^2(t)}{x^2(t)+1}\right)\left(3\dot{x}(t)+\frac{\dot{x}^3(t)}{\dot{x}^2(t)+1}\right)\right]' + \left(\frac{1}{2}+\sin t\right)x(t) = \frac{x^5\cos t\sin\dot{x}}{t^2(x^4+1)} \quad , t>0$$

We note that

1- $0 < r(t) = \dfrac{t^2}{t+1}$ *and* $\displaystyle\int_{t_0}^{\infty}\frac{ds}{r(s)} = \int_{t_0}^{\infty}\frac{s+1}{s^2}\,ds = \left[\ln s - \frac{1}{s}\right]_{t_0}^{\infty} = \infty$, $\quad , t>0$.

2- $0 < 2 \le \Psi(x(t)) = 2 + \dfrac{x^2(t)}{x^2(t)+1} < 3 \quad$ *for* $x \in \mathbb{R}$.

3- $0 < 3 < \dfrac{f(y)}{y} = 3 + \dfrac{y^2}{y^2+1} < 4 \qquad$ *for all* $y \neq 0$,

4- $\dfrac{H(t,\dot{x}(t),x(t))}{g(x(t))} = \dfrac{x^5\cos t\sin\dot{x}}{t^2(x^4+1)}\times\dfrac{1}{x} = \dfrac{x^4\cos t\sin\dot{x}}{t^2(x^4+1)} \le \dfrac{1}{t^2} = p(t)$

$\quad \forall\,\dot{x}\in\mathbb{R}, x\in\mathbb{R}$ *and* $t\in[t_0,\infty)$.

5- $\displaystyle\int_{t_0}^{\infty}[q(s)-p(s)]\,ds = \int_{t_0}^{\infty}\left(\frac{1}{2}+\sin s-\frac{1}{s^2}\right)ds = \frac{1}{2}s-\cos s+\frac{1}{s}\Big|_{t_0}^{\infty} = \infty$,

it follows from Theorem (3-1) that the equation is oscillatory .

Remark (3-1) : Theorem (3-1)extends the results of Grafe , Rankin and Spikes [14] and extends the results of [20]

Theorem 3-4 :

Suppose that the conditions (2), (3), (4), and (5) hold and

(7) $\int_{t_0}^{\infty}[q(s)-p(s)]ds<\infty$,

(8) $\liminf_{t\to\infty}\int_{T}^{t}[q(s)-p(s)]ds\geq 0$ *for all* t large T.

(9) $\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{r(s)}\int_{s}^{\infty}[q(u)-p(u)]du\,ds=\infty$,

then the super linear differential equation (3-1) is oscillatory.

<u>**Proof**</u> :

Let $x(t)$ be a non oscillatory solution of equation (3-1), say $x(t)>0$ for $t\geq t_1\geq t_0$, then from equation (3-1) we have

$$\left(\frac{r(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))}\right)^{\cdot}=\frac{H(t,\dot{x}(t),x(t))}{g(x(t))}-q(t)-\frac{r(t)\Psi(x(t))f(\dot{x}(t))g'(x(t))\dot{x}(t)}{g^2(x(t))}$$

$$\leq p(t)-q(t)$$

Then, for every $b\geq t_1$, we have

$$\frac{r(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))}\leq\frac{r(b)\Psi(x(b))f(\dot{x}(b))}{g(x(b))}-\int_{b}^{t}[q(s)-p(s)]ds$$

Now, if $\dot{x}(t)>0$ for all $t\geq b$, then by conditions (2) and (3) we get

$$k_2k_4\frac{r(t)\dot{x}(t)}{g(x(t))}\leq\frac{r(b)\Psi(x(b))f(\dot{x}(b))}{g(x(b))}-\int_{b}^{t}[q(s)-p(s)]ds$$

By condition (7), we obtain

$$0 \le \frac{r(b)\Psi(x(b))f(\dot{x}(b))}{g(x(b))} - \int_b^{\infty}[q(s)-p(s)]ds$$

Hence, for all $t \ge b$ we obtain

$$\int_t^{\infty}[q(s)-p(s)]ds \le \frac{r(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))} \le k_3 k_5 \frac{r(t)\dot{x}(t)}{g(x(t))}$$

Now, by integrating the last inequality we obtain

$$\int_b^t \frac{1}{r(s)} \int_s^{\infty}[q(u)-p(u)]du\,ds \le k_3 k_5 \int_b^t \frac{\dot{x}(s)\,ds}{g(x(s))} = k_3 k_5 \int_{x(b)}^{x(t)} \frac{du}{g(u)}$$

Hence, for all $t \ge b$ we get

$$\int_b^t \frac{1}{r(s)} \int_s^{\infty}[q(u)-p(u)]du\,ds \le k_3 k_5 \int_{x(b)}^{x(t)} \frac{du}{g(u)} < \infty \qquad as \quad t \to \infty$$

This is a contradiction to (9).

If $\dot{x}(t)$ changes signs, then there exists a sequence $\{c_n\} \to \infty$ such that $\dot{x}(c_n) < 0$. Choose N large enough so that (8) holds, we have

$$\frac{r(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))} \le \frac{r(c_N)\Psi(x(c_N))f(\dot{x}(c_N))}{g(x(c_N))} - \int_{c_N}^{t}[q(s)-p(s)]ds$$

So,

$$\lim_{t\to\infty} k_3 \frac{r(t)f(\dot{x}(t))}{g(x(t))} \le \frac{r(c_N)\Psi(x(c_N))f(\dot{x}(c_N))}{g(x(c_N))} + \lim_{t\to\infty}\left\{-\int_{c_N}^{t}[q(s)-p(s)]ds\right\} < 0$$

Hence,

$$\lim_{t\to\infty} f(\dot{x}(t)) < 0.$$

It follows that

$$\lim_{t\to\infty} \dot{x}(t) < 0 .$$

This is contradiction to the fact that $\dot{x}(t)$ oscillatory.

Then, there exists $t_2 \geq t_1$ such that $\dot{x}(t) < 0$ for $t \geq t_2$, and by condition (8) there exists $T_2 \geq T_0$ such that

$$\int_{T_1}^{t} [q(s) - p(s)] ds \geq 0 \text{ for all } t \geq T_1 .$$

Choosing $T_1 \geq t_2$ as indicated, and then integration (3-1), we have

$$r(t)\Psi(x(t)) f(\dot{x}(t)) \leq r(T_1)\Psi(x(T_1)) f(\dot{x}(T_1)) - g(x(t)) \int_{T_1}^{t} [q(s) - p(s)] ds$$
$$+ \int_{T_1}^{t} \dot{x}(s) g'(x(s)) \int_{T_1}^{u} [q(u) - p(u)] du\, ds$$

Hence, for every $t \geq T_1$, we have

$$k_3 k_5 \dot{x}(t) \leq r(T_1)\Psi(x(T_1)) f(\dot{x}(T_1))$$

$$x(t) \leq x(T_1) + \frac{r(T_1)\Psi(x(T_1)) f(\dot{x}(T_1))}{k_3 k_5} \int_{T_1}^{t} \frac{ds}{r(s)} \to -\infty \quad as\, t \to \infty$$

which is a contradiction to the fact that x(t)>0 for $t \geq t_1$, hence the proof is completed.

Example (3-4) :

Consider the differential equation

$$\left(t\left(8+\frac{x^4(t)}{x^4(t)+1}\right)\left(7\dot{x}(t)+\frac{\dot{x}^5(t)}{\dot{x}^4+1}\right)\right)^{\bullet}+\frac{1}{t^2}x^3(t)=\frac{x^3\cos x\sin 2\dot{x}}{t^3} \quad ,t>1$$

We note that

$$1)\ 0<r(t)=t \ \ and \ \ \int_{t_0}^{\infty}\frac{ds}{r(s)}=\int_{t_0}^{\infty}\frac{1}{s}ds=\ln s\Big|_{t_0}^{\infty}=\infty \qquad for\ \ t>1\ ,$$

$$2)\ 0<8\le\Psi(x(t))=8+\frac{x^4(t)}{x^4(t)+1}<9 \qquad for\ \ all\ \ x\in\mathbb{R}\ \ ,$$

$$3)\ 0<7<\frac{f(y)}{y}=7+\frac{y^4}{y^4+1}<8 \quad for\ all\ \ y\neq 0\ ,$$

$$4)\ \frac{H(t,\dot{x}(t),x(t))}{g(x(t))}=\frac{x^3\cos x\sin 2\dot{x}}{x^3\,t^3}\le\frac{1}{t^3}=p(t)\ \ ,\forall\,\dot{x}\in\mathbb{R}\ ,x\in\mathbb{R}\ \ and\ t\in[t_0,\infty).$$

$$5)\ \int_{t_0}^{\infty}[q(s)-p(s)]\ ds=\int_{t_0}^{\infty}\left(\frac{1}{s^2}-\frac{1}{s^3}\right)ds=\frac{-1}{s}+\frac{1}{2s^2}\Bigg|_{t_0}^{\infty}<\infty\ \ ,$$

$$6)\ \liminf_{t\to\infty}\int_{T}^{t}[q(s)-p(s)]ds=\liminf_{t\to\infty}\int_{T}^{t}\left[\frac{1}{s^2}-\frac{1}{s^3}\right]ds$$

$$=\liminf_{t\to\infty}\left(\frac{-1}{s}+\frac{1}{2s^2}\Bigg|_{T}^{t}\right)=\frac{1}{T}-\frac{1}{2T^2}>0.$$

7) $\lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{r(s)} \int_{s}^{\infty} [q(u)-p(u)]\,du\,ds = \lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{s} \int_{t_0}^{t} \left(\frac{1}{u^2}-\frac{1}{u^3}\right) du\,ds$

$$= \lim_{t\to\infty}\left(\frac{1}{t}-\frac{1}{4t^2}+\left(\frac{1}{t_0}-\frac{1}{2t_0^2}\right)\ln t-\frac{1}{t_0}+\frac{1}{4t_0^2}-\left(\frac{1}{t_0}-\frac{1}{2t_0^2}\right)\ln t_0\right)$$

$$= \infty .$$

8) $\int_{\varepsilon}^{\infty} \frac{du}{g(u)} = \int_{\varepsilon}^{\infty} \frac{du}{u^3} = \frac{1}{2\varepsilon^2} < \infty$ *and* $\int_{-\varepsilon}^{-\infty} \frac{du}{g(u)} = \int_{-\varepsilon}^{-\infty} \frac{du}{u^3} = \frac{1}{2\varepsilon^2} < \infty$ *for all* $\varepsilon > 0$,

it follows from Theorem (3-4) that the equation is oscillatory .

Remark3-4 : Theorem (3-4) extends the results of Grafe,Rankin and Spikes [14] and extends the results of [20].

Theorem (3-5) :

If the conditions (1), (2) and (4) hold and

(10) $\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} [q(u)-p(u)]\,du\,ds = \infty$.

(11) $yf(y) \geq k^*(f(y))^2$ *for all* $y \in \mathbb{R}, k^* > 0$.

then equation (3-1) is oscillatory.

Proof:

Without loss of generality, we may assume that there exists a solution $x(t) > 0$ then on $[t_0,\infty)$ for some $T \geq t_0 \geq 0$.

Now. From equation (3-1) we get,

$$\left(\frac{r(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))}\right)^{\cdot}=\frac{\left(r(t)\Psi(x(t))f(\dot{x}(t))\right)^{\cdot}}{g(x(t))}-\frac{r(t)\Psi(x(t))f(\dot{x}(t))g'(x(t))\dot{x}(t)}{g^{2}(x(t))}$$

$$=\frac{H(t,x(t),\dot{x}(t))}{g(x(t))}-q(t)-\frac{r(t)\Psi(x(t))f(\dot{x}(t))g'(x(t))\dot{x}(t)}{g^{2}(x(t))}$$

$$\leq p(t)-q(t)-\frac{r(t)\Psi(x(t))g'(x(t))\dot{x}(t)f(\dot{x}(t))}{g^{2}(x(t))}$$

Thus,for every $T\geq t_0$ we have

$$\frac{r(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))}\leq\frac{r(T)\Psi(x(T))f(\dot{x}(T))}{g(x(T))}-\int_T^t(q(s)-p(s))\,ds$$
$$-\int_T^t\frac{r(s)\Psi(x(s))g'(x(s))f(\dot{x}(s))\dot{x}(s)}{g^{2}(x(s))}\,ds$$

From last inequality we have

$$\frac{r(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))}+\int_T^t\frac{r(s)\Psi(x(s))g'(x(s))\dot{x}(s)f(\dot{x}(s))}{g^{2}(x(s))}\,ds+\int_T^t(q(s)-p(s))\,ds\leq c$$

where c is a constant.

From the conditions, we get

$$\frac{r(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))}+kk^{*}\int_T^t\frac{r(s)\Psi(x(s))f^{2}(\dot{x}(s))}{g^{2}(x(s))}\,ds+\int_T^t(q(s)-p(s))\,ds\leq c$$

Asecond integrating yields

$$\frac{1}{t}\int_T^t \frac{r(s)\Psi(x(s))f(\dot{x}(s))}{g(x(s))}ds + \frac{B}{t}\int_T^t\int_T^s \left(\frac{r(u)\Psi(x(u))f(\dot{x}(u))}{g(x(u))}\right)^2 du\,ds + \frac{1}{t}\int_T^t\int_T^s [q(u)-p(u)]\,du\,ds$$

$$\leq c\left(1-\frac{T}{t}\right)$$

where $B = \dfrac{kk_3k^*}{k_1}$ is a positive constant.

For $t \geq T$, by condition (10), we get

$$\lim_{t\to\infty}\frac{1}{t}\int_T^t \frac{r(s)\Psi(x(s))f(\dot{x}(s))}{g(x(s))}ds = -\infty \tag{3-7}$$

Defining:

$$R(t) = \left|\int_T^t \frac{r(s)\Psi(x(s))f(\dot{x}(s))}{g(x(s))}ds\right|$$

and applying Schwarz's inequality, we obtain

$$R^2(t) = \left|\int_T^t \frac{r(s)\Psi(x(s))f(\dot{x}(s))}{g(x(s))}ds\right|^2 \leq \int_T^t \left|\frac{r(s)\Psi(x(s))f(\dot{x}(s))}{g(x(s))}\right|^2 ds \times \int_T^t |1|^2 ds$$

$$= (t-T)\int_T^t \left|\frac{r(s)\Psi(x(s))f(\dot{x}(s))}{g(x(s))}\right|^2 ds \qquad for\ t \geq T$$

$$R^2(t) \leq t\int_T^t \left|\frac{r(s)\Psi(x(s))f(\dot{x}(s))}{g(x(s))}\right|^2 ds \tag{3-8}$$

By condition (10) implies that for sufficiently large t, say $t \geq T_1 \geq T$, we get

$$\frac{-1}{t}R(t)+\frac{B}{t}\int_T^t\int_T^s\left(\frac{r(u)\Psi(x(u))f(\dot{x}(u))}{g(x(u))}\right)^2 du\,ds\le 0$$

Then, $\forall t\ge T_1$, we get

$$\frac{-R(t)}{t}+\frac{B}{t}\int_T^t\frac{R^2(s)}{s}ds\le 0$$

It follows that

$$\frac{B}{t}\int_T^t\frac{R^2(s)}{s}ds\le\frac{R(t)}{t}$$

Thus, for all $t\ge T_1$ last inequality becomes

$$\frac{B^2}{t^2}\left\{\int_T^t\frac{R^2(s)}{s}ds\right\}^2\le\frac{R^2(t)}{t^2}$$

Now, we define

$$\Phi(t)=\int_T^t\frac{R^2(s)}{s}ds$$

Then,

$$\frac{B^2}{t}\le\frac{\Phi'(t)}{\Phi^2(t)}\qquad\forall t\ge T_1$$

Integrating, from T_1 *to* t we have

$$B^2\ln\left(\frac{t}{T_1}\right)\le\frac{1}{\Phi(T_1)}-\frac{1}{\Phi(t)}\le\frac{1}{\Phi(T_1)}.$$

This contradiction, hence, completes the proof of theorem.

Example (3-5):

Consider the following differential equation

$$\left[\frac{t^2}{t^2+1}\left(4+\frac{x^6(t)}{2x^6(t)+1}\right)\left(\dot{x}(t)+\frac{\dot{x}(t)}{\dot{x}^2(t)+1}\right)\right]^{\cdot}+t^3x^3(t)=\frac{x^{11}(t)}{x^8(t)+1}\sin t\,\frac{\dot{x}^2(t)}{\dot{x}^2(t)+1}\quad for\ t>0$$

We note that

1) $0<r(t)=\dfrac{t^2}{t^2+1}<1 \quad for\ t>0,$

2) $0<4\le\Psi(x(t))=4+\dfrac{x^6(t)}{x^6(t)+1}<5 \quad for\ x\in R,$

3) $xg(x)=x^4>0\ and\ g'(x)=3x^2>0\ \forall x\neq 0,$

4) $\dfrac{H(t,\dot{x}(t),x(t))}{g(x(t))}=\dfrac{x^8(t)}{x^8(t)+1}\sin t\,\dfrac{\dot{x}^2(t)}{\dot{x}^2(t)+1}\le\sin t,$

5) $yf(y)=y^2+\dfrac{y^2}{y^2+1}>0\ for\ all\ y\neq 0\ and\ f^2(y)=y^2+\dfrac{2y^2}{y^2+1}+\dfrac{y^2}{(y^2+1)^2}\le 4y^2$

$$\frac{1}{4}f^2(y)\le y^2\le y^2+\frac{y^2}{y^2+1}=yf(y)\quad\forall y\in R$$

6) $\displaystyle\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}[q(u)-p(u)]\,du\,ds=\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}[u^3-\sin u]\,du\,ds$

$$=\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\left[\frac{s^4}{4}+\cos s-\frac{t_0^4}{4}-\cos t_0\right]ds$$

$$=\lim_{t\to\infty}\frac{1}{t}\left[\frac{t^4}{4}+\sin t-\frac{t_0^4 t}{4}-t\cos t_0-\frac{t_0^5}{20}-\sin t_0\right]=\infty.$$

it follows from Theorem (3-5) that the equation is oscillatory.

Remark (3-5) Theorem (3-5) includes Theorem (4) of Greaf, Rankin and Spike[14] and extends the results of [20].

Theorem (3-6)

Suppose that the conditions (2)-(4) and (8) hold, and

(13) $\int_{t_0}^{\infty} \frac{ds}{r(s)} = A$,

(14) $\int_{t_0}^{\infty} \frac{1}{r(s)} \int_{t_0}^{s} [q(u) - p(u)] du\, ds = \infty$,

then all solutions of sub linear equation (3-1) are oscillatory.

Proof :

Suppose that $x(t)$ is a solution of (3-1) with $x(t) > 0$ for $t \geq T \geq t_0$, then

$$\left(\frac{r(t)\Psi(x(t)) f(\dot{x}(t))}{g(x(t))} \right)^{\bullet} = \frac{\left(r(t)\Psi(x(t)) f(\dot{x}(t)) \right)^{\bullet}}{g(x(t))} - \frac{r(t)\Psi(x(t)) f(\dot{x}(t)) g'(x(t)) \dot{x}(t)}{g^2(x(t))}$$

$$= \frac{H(t, \dot{x}(t), x(t))}{g(x(t))} - q(t) - \frac{r(t)\Psi(x(t)) f(\dot{x}(t)) g'(x(t)) \dot{x}(t)}{g^2(x(t))}$$

$$\leq p(t) - q(t)$$

Then, for every $t \geq T$, we have

$$\frac{r(t)\Psi(x(t)) f(\dot{x}(t))}{g(x(t))} \leq \frac{r(T)\Psi(x(T)) f(\dot{x}(T))}{g(x(T))} - \int_{T}^{t} [q(s) - p(s)] ds$$

If $\dot{x}(t) > 0$ for $t \geq T$ we have

$$k_2 k_4 \frac{r(t)\dot{x}(t)}{g(x(t))} \leq \frac{r(T)\Psi(x(T))f(\dot{x}(T))}{g(x(T))} - \int_T^t [q(s) - p(s)]ds$$

Then

$$\frac{r(t)\dot{x}(t)}{g(x(t))} \leq M' - \frac{1}{k_2 k_4}\int_T^t [q(s) - p(s)]ds$$

where $$M' = \frac{r(T)\Psi(x(T))f(\dot{x}(T))}{k_2 k_4 g(x(T))}$$

Now, multiplying the last inequality by $\frac{1}{r(t)}$ we obtain.

$$\frac{\dot{x}(t)}{g(x(t))} \leq \frac{M'}{r(t)} - \frac{1}{k_2 k_4}\frac{1}{r(t)}\int_T^t [q(s) - p(s)]ds$$

then, for every $t \geq T$, we get

$$\int_T^t \frac{\dot{x}(s)}{g(x(s))} ds \leq \int_T^t \frac{M'}{r(s)} ds - \frac{1}{k_2 k_4}\int_T^t \frac{1}{r(s)}\int_T^s [q(u) - p(u)]du\, ds$$

$$\int_{x(T)}^{x(t)} \frac{du}{g(u)} \leq \frac{1}{k_2 k_4}\left[\int_T^t \frac{M}{r(s)} ds - \int_T^t \frac{1}{r(s)}\int_T^s [q(u) - p(u)]du\, ds\right] ,$$

where $M = M' k_2 k_4$

From conditions (13)and (14) we obtain $\forall t \geq T$ that

$$I(t) = \int_{x(T)}^{x(t)} \frac{du}{g(u)}$$

Then, $\forall t \geq T$ we obtain

$$\int_{x(T)}^{x(t)} \frac{du}{g(u)} \to -\infty \quad as \quad t \to \infty,$$

This is a contradiction.

Now, if $\dot{x}(t) < 0$, we obtain

$$k_3 k_5 \frac{r(t)\dot{x}(t)}{g(x(t))} \leq \frac{r(T)\Psi(x(T))f(\dot{x}(T))}{g(x(T))} - \int_T^t [q(s) - p(s)] ds$$

Then,

$$\frac{\dot{x}(t)}{g(x(t))} \leq \frac{M''}{r(t)} - \frac{1}{k_3 k_5} \frac{1}{r(t)} \int_T^t [q(s) - p(s)] ds,$$

where $$M'' = \frac{r(T)\Psi(x(T))f(\dot{x}(T))}{k_3 k_5 g(x(T))}$$

For all $t \geq T$ we obtain

$$\int_T^t \frac{\dot{x}(s)}{g(x(s))} ds \leq \int_T^t \frac{M''}{r(s)} ds - \frac{1}{k_3 k_5} \int_T^t \frac{1}{r(s)} \int_T^s [q(u) - p(u)] du\, ds \ ,$$

$$= \frac{1}{k_3 k_5} \left[\int_T^t \frac{M}{r(s)} ds - \int_T^t \frac{1}{r(s)} \int_T^s [q(u) - p(u)] du\, ds \right]$$

where

$$M = M'' k_3 k_4$$

From conditions (13) and (14) we have that

$$I(t) = \int_T^t \frac{\dot{x}(s)}{g(x(s))} ds \to -\infty \quad as\ t \to \infty$$

Hence, since $x(t) < x(T)$, then for all $t \geq T$, we have

$$I(t) = \int_{x(T)}^{x(t)} \frac{du}{g(u)} = -\int_{x(t)}^{x(T)} \frac{du}{g(u)} = \left[\int_0^{x(t)} \frac{du}{g(u)} - \int_0^{x(T)} \frac{du}{g(u)} \right] \geq -\int_0^{x(T)} \frac{du}{g(u)} > -\infty ,$$

which is again a contradiction.

Now if $\dot{x}(s)$ changes signs, then there exists a sequence $\{c_N\}$ such that $\dot{x}(c_N) < 0$. Choose N large enough so that (8) hold, we get

$$\frac{r(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))} \leq \frac{r(c_N)\Psi(x(c_N))f(\dot{x}(c_N))}{g(x(c_N))} - \int_{c_N}^t [q(s) - p(s)] ds$$

$$\lim_{t\to\infty} \frac{r(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))} \leq \frac{r(c_N)\Psi(x(c_N))f(\dot{x}(c_N))}{g(x(c_N))} + \lim_{t\to\infty}\left\{ -\int_{c_N}^t [q(s) - p(s)] ds \right\} < 0$$

Hence,

$$\lim_{t\to\infty} f(\dot{x}(t)) < 0,$$

It follows that

$$\lim_{t\to\infty} \dot{x}(t) < 0,$$

which contradicts the fact that $\dot{x}(t)$ oscillates .this complete the proof .

Example(3-6):

Consider the differential equation

$$\left[\left(t^3\left(\frac{1}{2}+\frac{x^4(t)}{x^4(t)+1}\right)\left(17\dot{x}(t)+\frac{\dot{x}^{10}(t)}{\dot{x}^9(t)+1}\right)\right]^{\bullet} + (t+2)x^{\frac{1}{3}}(t) = \frac{x^{\frac{4}{3}}(t)}{4\left[|x(t)|+1\right]}, t>0$$

We note that

1) $0<r(t)=t^3$ *and* $\int_{t_0}^{\infty}\frac{ds}{r(s)}=\int_{t_0}^{\infty}\frac{ds}{s^3}=\frac{1}{2t_0^2}$ $\quad t>0$,

2) $0<\frac{1}{2}\le\Psi(x(t))=\left(\frac{1}{2}+\frac{x^4(t)}{x^4(t)+1}\right)\le 1$ for $x\in\mathrm{R}$,

3) $0<17\le\frac{f(y)}{y}=17+\frac{y^9}{y^9+1}$ *for all* $y\neq 0$,

4) $\frac{H(t,\dot{x}(t),x(t))}{g(x(t))}=\frac{x^{\frac{4}{3}}(t)}{4\left[|x(t)|+1\right]}\times\frac{1}{x^{\frac{1}{3}}(t)}=\frac{x(t)}{4\left[|x(t)|+1\right]}\le\frac{1}{4}$

5) $\int_0^{\varepsilon}\frac{du}{g(u)}=\int_0^{\varepsilon}\frac{du}{u^{\frac{1}{3}}}=\frac{3}{2}\varepsilon^{\frac{2}{3}}<\infty$ and $\int_0^{-\varepsilon}\frac{du}{g(u)}=\int_0^{-\varepsilon}\frac{du}{u^{\frac{1}{3}}}=\frac{3}{2}(-\varepsilon)^{\frac{2}{3}}<\infty$ for every $\varepsilon>0$,

6) $\liminf_{t\to\infty}\int_T^t [q(s)-p(s)]ds = \liminf_{t\to\infty}\int_T^t \left[2+s-\frac{1}{4}\right]ds = \liminf_{t\to\infty}\left[\frac{7}{4}s+\frac{s^2}{2}\right]_T^t = \infty > 0,$

7) $$\int_{t_0}^{\infty}\frac{1}{r(s)}\int_{t_0}^{s}[q(u)-p(u)]du\,ds$$

$$= \int_{t_0}^{\infty}\frac{1}{s^3}\int_{t_0}^{s}\left[2+u-\frac{1}{4}\right]du\,ds$$

$$= \int_{t_0}^{\infty}\frac{1}{s^3}\left[\frac{7}{4}s+\frac{s^2}{2}-\frac{7}{4}t_0-\frac{t_0^{\,2}}{2}\right]ds$$

$$= \frac{-7}{4s}+\frac{1}{2}\ln s-\frac{7t_0}{8s^2}+\frac{t_0^2}{4s^2}\Bigg|_{t_0}^{\infty} = \infty,$$

it follows from Theorem (3-6) that equation is oscillatory .

Remark (3-6) : Theorem (3-6)is an extension of Theorem(8) of Graf, Rankin and Spikes **[14]**] and extends the results of [20].

Theorem 3-7 :

suppose that conditions (1) –(4) hold, and moreover assume that there exists a differentiable function

$$\rho:[t_0,\infty)\to(0,\infty)$$

such that

$$\rho(t)>0,\ \dot{\rho}(t)\geq 0 \text{ and } \left(r(t)\dot{\rho}(t)\right)^{\bullet} \leq 0$$

If for every large T there exists $T_1 \geq T$ such that

16) $\liminf_{t\to\infty} \int_{T_1}^{t} \rho(s) \int_{T_1}^{s} [q(u) - p(u)] du\, ds > -\infty$

17) $\int_{t_0}^{\infty} \rho(s)[q(s) - p(s)] ds = \infty$,

18) $\int_{t_0}^{\infty} \frac{ds}{r(s)\rho(s)} = \infty$,

then all solutions super-linear of equation (3-1)are oscillatory.

Proof :

Let $x(t)$ be a non oscillatory solution of equation (3-1), say $x(t) > 0$ for $t \geq t_1 \geq t_0$.

$$\left(\frac{r(t)\Psi(x(t)) f(\dot{x}(t))}{g(x(t))} \right)^{\bullet} = \frac{\left(r(t)\Psi(x(t)) f(\dot{x}(t)) \right)^{\bullet}}{g(x(t))} - \frac{r(t)\Psi(x(t)) f(\dot{x}(t)) g'(x(t)) \dot{x}(t)}{g^2(x(t))}$$

$$\leq \frac{H(t, \dot{x}(t), x(t))}{g(x(t))} - q(t)$$

$$\leq -\left[q(t) - p(t) \right]$$

Now, we consider the following three cases for the behavior of $\dot{x}(t)$.

Case (1)

If $\dot{x}(t)$ oscillates, choose $T \geq t_1$ so that $\dot{x}(T) = 0$ and condition (16) hold, then integrating (3-6) we obtain

$$\frac{r(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))} \leq -\int_{T}^{t}[q(s)-p(s)]ds$$

Multiplying by $\dot{\rho}(t)$ and integrating from T_1 to t we obtained

$$\int_{T_1}^{t}\dot{\rho}(s)\frac{r(s)\Psi(x(s))f(\dot{x}(s))}{g(x(s))}ds \leq -\int_{T_1}^{t}\dot{\rho}(s)\int_{T_1}^{s}[q(u)-p(u)]\,du\,ds < A \qquad (3-9)$$

where A aconstant

Now

$$\rho(t)\left(\frac{r(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))}\right)^{\bullet} \leq -\int_{T}^{t}[q(s)-p(s)]ds \qquad (3\text{-}10)$$

Thus, for every $t \geq T_1$ we get

$$\int_{T_1}^{t}\rho(s)\left(\frac{r(s)\Psi(x(s))f(\dot{x}(s))}{g(x(s))}\right)^{\bullet}ds \leq -\int_{T_1}^{t}\rho(s)[q(s)-p(s)]ds$$

$$\frac{\rho(t)r(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))} \leq \frac{\rho(T_1)r(T_1)\Psi(x(T_1))f(\dot{x}(T_1))}{g(x(T_1))} + \int_{T_1}^{t}\frac{\dot{\rho}(s)r(s)\Psi(x(s))f(\dot{x}(s))}{g(x(s))}ds - \int_{T_1}^{t}\rho(s)[q(s)-p(s)]ds$$

From inequality (3-7) we obtain

$$\frac{\rho(t)r(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))} \leq \frac{\rho(T_1)r(T_1)\Psi(x(T_1))f(\dot{x}(T_1))}{g(x(T_1))} - \int_{T_1}^{t}\dot{\rho}(s)\int_{T}^{s}[q(u)-p(u)]du\,ds - \int_{T_1}^{t}\rho(s)[q(s)-p(s)]ds$$

$$\leq \frac{\rho(T_1)r(T_1)\Psi(x(T_1))f(\dot{x}(T_1))}{g(x(T_1))} + A - \int_{T_1}^{t} \rho(s)[q(s) - p(s)]ds$$

$$\lim_{t\to\infty} \frac{\rho(t)r(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))} \leq A' - \lim_{t\to\infty} \int_{T_1}^{t} \rho(s)[q(s) - p(s)]ds\,,$$

where $$A' = A + \frac{\rho(T_1)r(T_1)\Psi(x(T_1))f(\dot{x}(T_1))}{g(x(T_1))}$$

Hence ,

$$\lim_{t\to\infty} \frac{\rho(t)r(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))} \to -\infty$$

Then, it follows

$$\lim_{t\to\infty} f(\dot{x}(t)) < 0$$

Hence ,

$$\lim_{t\to\infty} \dot{x}(t) < 0$$

which is a contradiction to the fact that $\dot{x}(t)$ is oscillatory.

Case (2)

Now, if $\dot{x}(t) > 0$ and by integrating (3-9) for every $t \geq T_2$ we have

$$\frac{\rho(t)r(t)\Psi(x(t))f(\dot{x}(t))}{g(x(t))} \leq A'' - \int_{T_2}^{t} \rho(s)[q(s) - p(s)]\,ds + \int_{T_2}^{t} \frac{\dot{\rho}(s)r(s)\Psi(x(s))f(\dot{x}(s))}{g(x(s))}ds$$

where $A'' = \dfrac{\rho(T_2)r(T_2)\Psi(x(T_2))f(\dot{x}(T_2))}{g(x(T_2))}$

From the conditions (2)and (3) we obtain

$$\frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))} \le \frac{A''}{k_2k_4} - \frac{1}{k_2k_4}\int_{T_1}^{t}\rho(s)[q(s)-p(s)]\,ds + \frac{k_3k_5}{k_2k_4}\int_{T_2}^{t}\frac{\dot{\rho}(s)r(s)\dot{x}(s)}{g(x(s))}\,ds$$

By the Bonnet theorem, for a fixed $c_t \in [T_2, t]$ such that

$$\int_{T_2}^{t}\left(r(u)\dot{\rho}(u)\right)\frac{\dot{x}(u)}{g(x(u))}\,du = r(T_2)\dot{\rho}(T_2)\int_{T_2}^{c_t}\frac{\dot{x}(u)}{g(x(u))}\,du$$

$$= r(T_2)\dot{\rho}(T_2)\int_{x(T_2)}^{x(c_t)}\frac{dy}{g(y)}$$

Hence the last inequality be comes

$$\frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))} - \frac{A''}{k_2k_4} + \frac{1}{k_2k_4}\int_{T_2}^{t}\rho(s)[q(s)-p(s)]\,ds \le \frac{k_3k_5}{k_2k_4}r(T_2)\dot{\rho}(T_2)\int_{x(T_2)}^{x(t)}\frac{dy}{g(y)}$$

$$\frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))} \le \frac{A''}{k_2k_4} - \frac{1}{k_2k_4}\int_{T_1}^{t}\rho(s)[q(s)-p(s)]\,ds + \frac{k_3k_5}{k_2k_4}r(T_2)\dot{\rho}(T_2)\int_{x(T_2)}^{x(t)}\frac{dy}{g(y)}$$

By condition(17) we obtain

$$\lim_{t\to\infty}\frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))} \to -\infty$$

It following

$$\lim_{t\to\infty} \dot{x}(t) < 0$$

which is contradiction to the fact $\dot{x}(t) > 0$.

Case (3)

If $\dot{x}(t) < 0$ for $t \ge t_3$ for some $t_3 \ge t_2$

Multiplying equation (3-1) by $\rho(t)$ and integrating, we have

$$\rho(t)\left(r(t)\Psi(x(t))f(\dot{x}(t))\right)^{\bullet} + \rho(t)q(t)g(x(t)) = \rho(t)H(t,\dot{x}(t),x(t)) \quad ,t \ge t_0$$

$$\rho(t)\left(r(t)\Psi(x(t))f(\dot{x}(t))\right)^{\bullet} \le -\rho(t)g(x(t))\left[q(t) - p(t)\right]$$

Now,

$$\left(\rho(t)r(t)\Psi(x(t))f(\dot{x}(t))\right)^{\bullet} \le -\rho(t)g(x(t))\left[q(t) - p(t)\right] + \dot{\rho}(t)r(t)\Psi(x(t))f(\dot{x}(t))$$

Then, for every $t \ge t_3$, we have

$$\left(\rho(t)r(t)\Psi(x(t))f(\dot{x}(t))\right)^{\bullet} = \rho(t)\left(r(t)\Psi(x(t))f(\dot{x}(t))\right)^{\bullet} + \dot{\rho}(t)r(t)\Psi(x(t))f(\dot{x}(t))$$
$$\le -\rho(t)g(x(t))\left[q(t) - p(t)\right].$$

Then there exists $t_3 \ge t_0$ such that

$$\int_{t_3}^{t_3} \rho(s)\left[q(s) - p(s)\right]ds = 0 \text{ and } \int_{t_3}^{t} \rho(s)\left[q(s) - p(s)\right]ds \ge 0$$

And integrating equation(3-1) we have

$$\rho(t)r(t)\Psi(x(t))f(\dot{x}(t)) \le \rho(t_3)r(t_3)\Psi(x(t_3))f(\dot{x}(t_3)) \;\; - \int_{t_3}^{t}\rho(s)g(x(s))\left[q(s)-p(s)\right]ds$$

$$k_3k_5\rho(t)r(t)\dot{x}(t) \le \rho(t_3)r(t_3)\Psi(x(t_3))f(\dot{x}(t_3)) - g(x(t))\int_{t_3}^{t}\rho(s)\left[q(s)-p(s)\right]ds$$

$$+\int_{t_3}^{t}\dot{x}(s)\,g(x(s))\int_{t_3}^{s}\rho(u)\left[q(u)-p(u)\right]du\,ds$$

$$\le \rho(t_3)r(t_3)\Psi(x(t_3))f(\dot{x}(t_3))$$

$$\dot{x}(t) \le \frac{\rho(t_3)r(t_3)\Psi(x(t_3))f(\dot{x}(t_3))}{k_3k_5}\,\frac{1}{\rho(t)r(t)}$$

Thus, for every $t_3 \ge t$ we get

$$x(t) \le x(t_3) + \frac{\rho(t_3)r(t_3)\Psi(x(t_3))f(\dot{x}(t_3))}{k_3k_5}\int_{t_3}^{t}\frac{ds}{\rho(s)r(s)} \to -\infty \quad as \quad t\to\infty$$

which is agene contradiction. Hence the proof is completed .

<u>**Example (3-7)**</u> :

Consider the differential equation

$$\left(\frac{1}{t}\left(5+\frac{x^6(t)}{x^6(t)+1}\right)\left(13\dot{x}(t)+\frac{\dot{x}(t)}{\dot{x}^2(t)+1}\right)\right)^{\bullet} + \left(s+\frac{\sin s}{s}\right)x^5(t) = \frac{2x^8\sin t\cos(\dot{x}(t)+1)}{(x^7+1)t^3} \;\;, t>0$$

We note that

$1) 0 < r(t) = \frac{1}{t} \qquad for > 0$

$2) 0 < 5 \le \Psi(x(t)) = 5 + \frac{x^6(t)}{x^6(t)+1} < 6 \qquad for\ all\ x \in R$.

$3) 0 < 13 \le \frac{f(y)}{y} = 13 + \frac{1}{y^2+1} < 14 \qquad for\ all\ y \ne 0$

$$4) \frac{H(t,\dot{x}(t),x(t))}{g(x(t))} = \frac{2x^{12}\sin t\cos(\dot{x}+1)}{(x^7+1)t^3} \times \frac{1}{x^5} = \frac{2x^7\sin t\cos(\dot{x}+1)}{(x^7+1)t^3} \le \frac{2}{t^3}, t > 0$$

Let

$\rho(t) = t$ then $\dot{\rho}(t) = 1$, $\left(r(t)\dot{\rho}(t)\right)^{\cdot} = \frac{-1}{t^2} < 0$ and

$$5) \int_{t_0}^{\infty} \rho(s)[q(s) - p(s)]ds = \int_{t_0}^{\infty} s\left[s - \frac{2}{s^3}\right]ds = \frac{s^3}{3} + \frac{2}{s}\Big|_{t_0}^{\infty} = \infty .$$

$$6) \int_{t_0}^{\infty} \frac{ds}{r(s)\rho(s)} = \int_{t_0}^{\infty} s\,ds = \frac{s^2}{2}\Big|_{t_0}^{\infty} = \infty ,$$

$$7) \liminf_{t\to\infty} \int_{T_1}^{t} \dot{\rho}(s)\int_{T_1}^{s}[q(u) - p(u)]du\,ds = \liminf_{t\to\infty} \int_{T_1}^{t}\left(\frac{u^2}{2} + \frac{1}{u^2}\right)_{T_1}^{s} ds =$$

$$\liminf_{t\to\infty}\left(\frac{t^3}{6} - \frac{1}{t} - (\frac{T_1}{2} - \frac{1}{T_1^2})t - \frac{T_1^3}{6} + \frac{1}{T_1} + (\frac{T_1^2}{2} - \frac{1}{T_1^2})T_1\right)$$

$$= \infty > -\infty .$$

It follows from Theorem (3-7) that the equation is oscillatory.

Remark (3-7): Theorem (3-7) extends the results of Graf, Rankin and Spikes [14].

CHAPTER (4)

SUMMARY

In this study we consider the problem of determining the oscillation and boundedness criteria for second order nonlinear differential equations with variable coefficients.

We apply the averaging technique to discuss the oscillation of those equations. In many instances our results will include, as special cases, known oscillation theorems for les general equations.

REFERENCES

[1] Aydin, Tiryaki and Agacikzafer, Oscillation criteria for second order non linear differential equations with damping, Tark J. Math. 24 (2000) 185-196.

[2]F.V.Atkinson, On second order non linear oscillations . pacific. J. Math. 5(1955). 643-647.

[3] R.G.Bartle, The elements of real analysis, Seven Edtion ,John Wiley and Sons (1976). pp 233.

[4] N.P.Bhatia, Some oscillation theorems for second order differential equations . J. Math. Anal. Appl. 15 (1966) ,442-446.

[5] T.A.Burton and C.G.Townsend, On the generalized lineard equation with forcing fn.,J. diff. eq. 4 (1968), 620-633.

[6] T.A.Burton, On the eq. $\ddot{x}+f(x)h(\dot{x})\dot{x}+g(x)=e(t)$, Ann. Math. Pura. Appl. 85 (1970), 277-285.

[7] E.M.El-Abbasy , Oscillation behavior of solutions of forced second order differential equations with alternating coefficients.Apl.Comp.6(1994)1-13 .

[8] M.El-Abbasy , On the oscillation of nonlinear second order differential equations. Appl.Math.Comp.8(2000)76-83 .

[9] L.Erbe , Oscillation theorems for second order nonlinear differential equations.Proc.Amer.Math.Soc.24(1970).811-814.

[10] .B.Fite , Concerning the zeros of the solutions of certain differential equations.Trans.Amer.Math.Soc.19(1918) , 341-352 .

[11] .R.Grace and B.S.Lalli , Oscillation theorems for certain second perturbed nonlinear differential equations , J.Math.Anal.Appl.77(1980).205-214 .

[12] S.RGrace , Oscillation theorems for nonlinear differential equations of second order, J.Math.Anal.Appl.171(1992),220-241.

[13] J.R.Graef and P.W.Spikes , Asymptotic behavior of solutions of a second order nonlinear differential equation . J. Differential equations 17(1975).461-476 .

[14] J.R.Greaf , S.M.Rankin and P.W.Spikes, Oscillation theorems for perturbed nonlinear differential equations , J.Math.Anal.Appl.65(1978),375-390 .

[15] P.Hartman,Non-oscillatory linear differential equations of second order, Amer.J.Math.74(1952),389-400.

[16] I.V.Kamenev . Integral criterion for oscillation of linear differential equations of second order,Math. Zametki 23 (1978), 249-251.

[17] L.T.kiguradze , Anote on the oscillation of solutions of the equation , casopis pest. Math. 92(1967),343-350 . $\ddot{u}+a(t)|u|^{n}\operatorname{sgn} u=0$

[18] H.J.Li . Oscillation criteria for half-linear second order differential equations, Hiroshima Math. J.25(1995), 571-583 .

[19] W.T.Li and R.P.Agarwal , Interval oscillation criteria for second order nonlinear differential equations , Ukrainian Math. J. 53(9)(2001) , 1391-1604 .

[20] W.Li and S.S.Cheng , An oscillation criteria for non homogeneous half-linear differential equations, Appl. Math. Lett.15(2002),256-263 .

[21] R.A.Moore , The behavior of solutions of a linear differential equation of second order, pacific J.Math. 5(1955),125-145.

[22] J.Ohriska and A. Zulova, oscillation criteria for second order non linear diff. eq. Im Preprint series A, 10 (2004),1-11.
[23] J.Ohriska and Antonia Zulova, Oscillation criteria for second order nonlinear differential equation, IM Preprint series A, Vol. 10(2004), 1-11.

[24] S.N.Olekhnik,the boundednes of solutions of a second order diff. eq. , (1973),1530-1534.

[25] H.Onose ,oscillations criteria for second order non linear differential equations ,Proc.Amer. Math. Soc. 51(1975),67-73.

[26] Ch. G. Philos, oscillation of second order linear ordinary differential equations with alternating Coefficients ,Bull Astral. Math. Soc. 27(1983),307-313.

[27] Ch. G. Philos , An oscillation criterion for second order sub-linear ordinary differential equations , Bull. Pol. Aca. Sci. Math. 32(1984), 567-572.
[28]________,Integral averages and second order super-linear oscillation , Math. Nachr.120(1985), 127-138.

[29]_________ , oscillation criteria for second order super-linear differential equations, Canada, J. Math. 41(1989), 321-340.

[30]_________ , oscillation theorems for linear differential equations of second order Sond. Arch. Math, 53(1989), 482-492.

[31]Ch.G. Philos and I. K. Purnaras, on the oscillation of second order nonlinear differential equations , Arch. Math. 159(1992),260-271.

[32]E.Popa, Sur quelques criteria's d'oscillations pour les equation differentials linears du second order , Rev. Roumaine. Math. Pures apple. 26(1981), 135-138.

[33]S.L.Ross, differential equations 3[rd]ed John Wiley and Sons (1984),pp580.

[34]V. N.Sevelo , Problems, methods and fundamental results in the theory of oscillation of solutions of nonlinear non autonomous ordinary differential equations, Proc. 2[nd]All – Union Conf. Theo. Appl. Mech, Moscow,(1965),pp. 142-157.

[35] C.Sturm, "sur les equations differentielles lineaires du second order" J. math. Pures Appl. 1(1836), 106-186.

[36] P.Waltman, An oscillation criterion for a non linear second order equation , J. Math. Anal. Appl. 10(1965), 439-441.

[37] A. Wintner , A criterion of oscillatory stability , Quart. Appl. Math. 7(1949), 115-117.

[38] J.S.W.Wong, A second order non linear oscillation theorems, Proc. Amer. Math. Soc., 40(1973),487-491.

[39] F. H. Wong and C. C . Yeh , Oscillation criteria for second order super-linear differential equations ,Math. Japonica. 37(1992),573-584.

[40] J. S. W. Wong and C. C. Yeh, An oscillation criterion for second order sub-linear differential equations J. Math. And. Appl. ,171(1992),346-351.

[41]J. Yan, Oscillation theorems for second order linear differential equations with damping , Proc. Amer. Math. Soc. 98(1986), 276-282.

[42]C.C. Yeh, An Oscillation criteria for second order nonlinear differential equations with functional arguments, Math. Anal. Appl., 76(1980),72-76.

ملخص الرسالة

نعتبر في هذه الدراسة مسألة تعيين شروط التذبذب و المحدودية للمعادلات التفاضلية الغير خطية ذات الرتبة الثانية و المعاملات المتغيرة.

نطبق طريقة متوسطات التكامل لبحث تذبذب هذه المعادلات- و في كثير من مقترحاتنا نتائجنا تشمل كحالات خاصة ـ نظريات التذبذب المعروفة لمعادلات أقل عمومية.

والأستاذ الفاضل [illegible] .

- كافة الطلبة وأخص بالذكر أ. [illegible] لما أبدى لي

من رعاية واهتمام والأستاذ [illegible] .

كذلك أذكر زميلي في الحاسوب لما قدمه لي من إمكانيات كان لها

بالغ الأثر في إنجاح هذا العمل.

- أسرتي الصغيرة المعهد العالي لإعداد المعلمين.

الشكر والتقدير

لا يسعني في البداية إلا أن أتقدم بأسمى آيات الشكر والعرفان وخالص التقدير والامتنان إلى كل من ساهم بشكل أو بآخر في إتمام هذا العمل وإظهاره إلى حيز الوجود.

وأخص بالشكر أستاذي الكريم الدكتور نجيبي [illegible] الذي أشرف على رسالتي، فكانت أفكاره نبراسا أهتدي بنوره... فله مني كل الشكر...

كما أشكر كلا من:

- لجنة التحكيم.
- قسم الرياضيات لما أبدوه لي من رعاية وتشجيع.
- الدكتور [illegible].

الإهــــــــــــــــداء

إلى رمز الوفاء ،،، من كانوا وراء نجاحي وتفوقـــــي

أعطوا بسخاء لكون المقابل .... النجـــــــــــاح

أسرتـــي الكريمـــة

إلى النبراس الوضاء، من أناروا لنا دروب المجد والتفوق ،

ومهدوا لنا الطريق لنمضي نحو الأفضل نحو الآمال ونحو

النجاح.

إلى كل من كان له باع أو ذراع في علم الرياضيات.

إلى روح القائل :

((من يخشى السؤال من أجل المعرفة فإنه ارتضى الجهل وهذا ما لا يقبله العقل)).

الدكتور مصباح العروي رحمه الله ..... أهدي هذا العمل.

كليــــة العلــــوم

قســـم الرياضيات

عنـــوان البحـــث

((متـوسطات التكامـل و تذبذب المعـادلات التفاضلية Super linear

مـن الرتبـة الثانيـة))

مقدمـــة مـــن الطـــالبة

أمباركـة عبد الله صـالحين

* * لجنـــة المناقشـــة :

الدكتور / نبيل زكـــي فـريد ....

(مشـــرف الرســـالة)

الدكتور / فتح الله صالح المستيري ....

(ممتحـن داخلـــي)

الدكتور / مصطفى الشريف بدر الدين ....

(ممتحـن خارجـي)

يعتمـد :-

د. محمد علي سالم الفرجاني

أمين اللجنة الشعبية لكلية العلوم

جامعة التحدى

كلية العلوم

قسم الرياضيات

متوسطات التكامل وتذبذب المعادلة التفاضلية SUPERLINEAR ذات الرتبة الثانية

هذا البحث مقدم من ضمن متطلبات التخرج لنيل درجة الماجستير

في علوم الرياضيات مقدم من:

امباركة عبدالله صالحين عجروش

تحت اشراف

د. نبيل زكى فريد

سرت-ليبيا

2006-2005