University of AL-Tahdi

Faculty of Science

Department of Mathematics

Laplace Transform and Z-transform Theory and Applications

A dissertation submitted to the department of mathematics in partial fulfillment of the requirements for the degree of master of science in mathematics

By

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Dedication

To My Family

Salah

Acknowledgment

الحمد لله

I would like to express my gratitude and appreciation for all who helped me to complete this thesis.

The guidance, encouragement and inspiration of Dr. Fathalla S. Mestiri, my major a supervisor, are kindly acknowledged and deeply appreciated special thanks to the department of mathematics.

Contents

Chapter One

Laplace Transform

1-1 Bilateral Laplace Transform [9, 10, 14]:

In this part we discuss how a function $F(s)$ can be expressed as an integral of a function $f(t)$ multiplied by e^{-st} and integrated with respect to t from $t = -\infty$ to $t = \infty$ where s may be a complex number.

To this end; let us consider the function $f(t)\, e^{-\sigma t}$ for a real constant σ.

If $f(t)$ has a finite number of maximal, minima, and has a finite number of points of discontinuities within any finite interval on the t-axis and if,

$$\int_{-\infty}^{\infty} \left| f(t)\, e^{-\sigma t} \right| dt < \infty \; .$$

In other words $f(t)$ satisfies Dirichlet conditions, and hence the Fourier transform of the function $f(t)\, e^{-\sigma t}$ exists. Let us denote the Fourier transform of $f(t)\, e^{-\sigma t}$ by $F(\alpha)$, this transform is given by

$$F(\alpha) = \int_{-\infty}^{\infty} f(t)\, e^{-\sigma t}\, e^{-i\alpha t}\, dt = \int_{-\infty}^{\infty} f(t)\, e^{-(\sigma + i\alpha)t}\, dt . \qquad (1\text{-}1)$$

Also, the inverse Fourier transformation formula yields

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha)\, e^{(\sigma + i\alpha)t}\, d\alpha . \qquad (1\text{-}2)$$

Letting $s = \sigma + i\alpha$, we rewrite (1-1) and (1-2) as

$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt \qquad (1\text{-}3)$$

$$f(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} F(s)\, e^{st}\, ds\ . \qquad (1\text{-}4)$$

As defined in (1-3), $F(s)$ is known as the bilateral Laplace transform of the function $f(t)$, and denoted by $L_b\{f(t)\}$ and (1-4) is the corresponding inverse transform formula denoted by $L_b^{-1}\{F(s)\}$.

The derivation of formulas in (1-3) and (1-4) is not formal. Based on the above we can prove that if $f(t)$ is a function of bounded variations and if

$$\int_{-\infty}^{\infty} \left| f(t)\, e^{-st} \right| dt < \infty\ ,$$

for some s in the complex plane, then the bilateral Laplace transform of $f(t)$ as shown in (1-3) exists.

Also, $f(t)$ can be derived from $F(s)$ by integrating (1-4) over a vertical line form $s = \sigma - i\infty$ to $s = \sigma + i\infty$ in the complex plane.

1-2 Region of Convergence [9, 10, 14]:

Since the bilateral Laplace transform is an improper integral, it exits only for those value of the variable s for witch the integral converges, and we are going to show the regions of convergence in the following examples.

Example (1-1):

Consider the function $g_1(t) = A\, e^{\beta t} u(t),\ \beta > 0$ (Figure 1-1).

where $u(t) = \begin{cases} 1\ , & t > 0 \\ 0\ , & t < 0 \end{cases}$ is the usual step function.

The bilateral Laplace transform is

$$G_1(s) = \int_{-\infty}^{\infty} A e^{\beta t} u(t)\, e^{-st}\, dt = A\int_{0}^{\infty} e^{-(s-\beta)t}\, dt \quad = A\int_{0}^{\infty} e^{(\beta-\sigma)t}\, e^{-i\alpha t}\, dt \quad (1\text{-}5)$$

with $s = \sigma + i\alpha$.

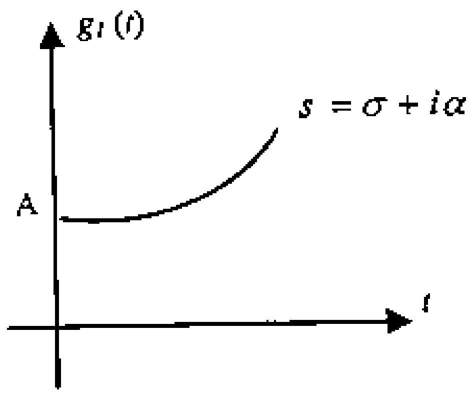

Figure (1-1)

It can be found that the intergral (1-5) converges if $\beta - \sigma$ is negative.

When $\sigma > \beta$ the function $e^{(\beta-\sigma)t}$ approaches zero as t approaches positive infinity.

The region of convergence (R. O. C) is defined by the fact that $\sigma > \beta$.

For any value of s in the complex plane where $\sigma > \beta$ we find that bilateral Laplace transform exists.

Evaluating the integral (1-5).

$$G_1(s) = \frac{A}{s-\beta} \quad ; \quad \sigma = \mathrm{Re}(s) > \beta$$

where $\mathrm{Re}(s)$ denote the real part of s.

The transform result $G_1(s)$ goes to ∞ at a finite value as s, which means that at $s=\beta$, this point in the complex plane is known as a pole of $G_1(s)$. The points in the complex plane where the transform goes to zero known as zero of $G_1(s)$.

It is often help full to plot the locations of the finite poles and zeros of bilateral Laplace- domain function in the complex plane.

The pole-zero plot for $G_1(s)$ is shown in figure (1.2) to gether with the region of convergence (R. O. C) in the complex plane.

The (R. O. c) is the part of the complex plane for which the real part of s is greater than β.

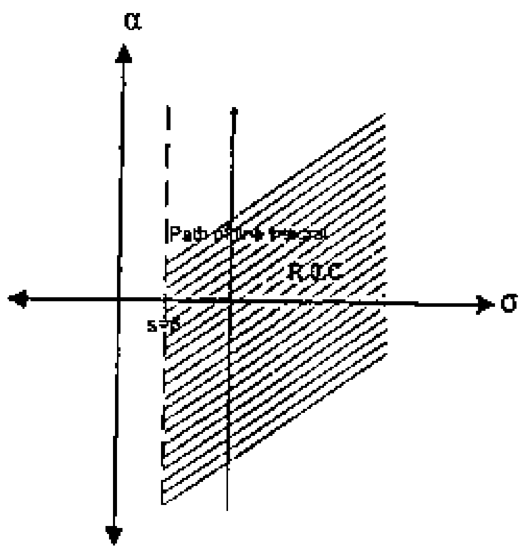

Figure (1-2)

Example (1-2):

Consider the function $g_2(t)=Ae^{-\beta t}\,u(-t)=g_1(-t) \quad \sigma>0$ (Figure 1-3).

The bilateral Laplace transform is

$$G_2(s)=\int_{-\infty}^{\infty} Ae^{-\beta t}\,u(-t)\,e^{-st}\,dt = A\int_{-\infty}^{0^-} Ae^{-(s+\beta)t}\,dt \qquad (1\text{-}6)$$

The integral converges if $\sigma < -\beta$ and the transform is

$$G_2(s) = \frac{-A}{s+\beta} = G_1(-s)\ ,\quad \sigma < -\beta\ .$$

The pole-zero plot and region of convergence for this function are shown in figure (1-4).

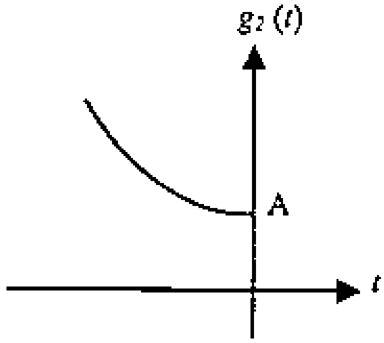

Figure (1-3)

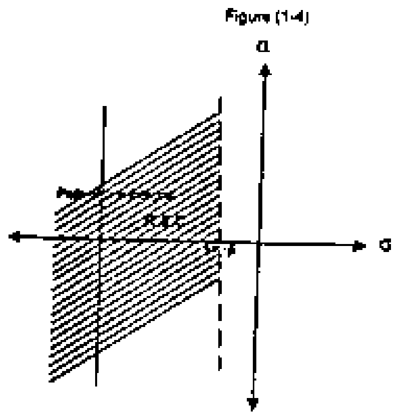

Figure (1-4)

If the function of the be transformed is $g(t) = Ae^{\beta t}$;

Then the bilateral Laplace transform becomes

$$G(s) = \int_{-\infty}^{\infty} Ae^{\beta t}\, e^{-st} = A\int_{-\infty}^{\infty} e^{\beta t}\, e^{-\sigma t}\, e^{-i\alpha t}\, dt = \int_{-\infty}^{\infty} e^{(\beta-\sigma)t}\, e^{-i\alpha t}\, dt\,.$$

The above integral does not converge.

It is not possible to evaluate the integral at one of its limits either the lower limit or the upper limit.

Example (1-3):

Find the bilateral Laplace transform of

$$f(t) = \begin{cases} \sin \pi t\,, & 0 \le t \le 2 \\ 0\,, & otherwise \end{cases}$$

Solution: using the definition

$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt = \int_{0}^{2} e^{-st} \sin \pi t \, dt$$

$$= \frac{e^{-st}\left[(-s) \sin \pi t - \pi \cos \pi t\right]}{s^2 + \pi^2}\Bigg|_0^2$$

$$= \frac{1}{s^2 + \pi^2}\left[e^{-2s}\left[(-s) \sin 2\pi - \pi \cos 2\pi\right] - \left[(-s) \sin(0) - \pi \cos(0)\right]\right]$$

$$= \frac{1}{s^2 + \pi^2}\left[-\pi\, e^{-2s} - (-\pi)\right]$$

$$= \frac{\pi - \pi e^{-2s}}{s^2 + \pi^2} = \frac{\pi(1 - e^{-2s})}{s^2 + \pi^2} .$$

The region of convergence of $F(s)$ is the entire complex plane.

Example (1-4):

Find the bilateral Laplace transform of $f(t) = \dfrac{e^{-at}}{1 + t^2}$.

Solution:

$$F(s) = \int_{-\infty}^{\infty} \frac{e^{-st}\, e^{-at}}{1 + t^2}\, dt .$$

We note that the region of convergence is the single vertical line at $\mathrm{Re}(s) = -\alpha$, because

$$F(s) = \int_{-\infty}^{\infty} \frac{e^{\alpha t}\, e^{-\alpha t}}{1 + t^2}\, dt = \int_{-\infty}^{\infty} \frac{1}{1 + t^2}\, dt$$

$$= \tan^{-1} t \Big|_{-\infty}^{\infty} = \tan^{-1} \infty - \tan^{-1}(-\infty) = \frac{\pi}{2} - \left(\frac{-\pi}{2}\right) = \pi .$$

If $\mathrm{Re}(s) \neq -\alpha$ then the integral $\int_{-\infty}^{\infty} \frac{e^{-st}\, e^{-\alpha t}}{1+t^2}\, dt$ does not converge.

Example (1-5):

For the function $f(t) = \begin{cases} t, & t > 0 \\ e^{-t}, & t < 0 \end{cases}$

the bilateral Laplace transform does not exists.

Since

$$\int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt = \int_{-\infty}^{0} e^{-(1+s)t}\, dt + \int_{0}^{\infty} t\, e^{-st}\, dt \ .$$

The first integral converges if $\mathrm{Re}(s) < -1$ and the second integral converges if $\mathrm{Re}(s) > 0$ since these two regions do not intersect then $f(t)$ has no region of convergence.

1-3 Unilateral Laplace Transform [6, 13, 16, 17]:

Definition (1-1):

The unilateral Laplace transform of a function $f(t)$ is defined by

$$F(s) = \int_{0}^{\infty} f(t)\, e^{-st}\, dt , \tag{1-7}$$

and denoted by $L\{f(t)\}$

provided that the integral (1-7) converges.

The unilateral and bilateral Laplace transform are naturally equivalent for function that are zero for $t < 0$.

then $\left|L\{f(t)\}\right| \leq \frac{K}{s-\alpha} = M < \infty$,

or $\quad -M \leq L\{f(t)\} \leq M$.

Hence the existence of $L\{f(t)\}$ □.

Remark (1-2):

The condition in the theorem (1-1) is sufficient but not necessary.

Example (1-6):

$$f(t) = \frac{1}{\sqrt{t}}, \qquad |f(t)| > K e^{\alpha t} \text{ as } t \to 0^+ .$$

But $L\{f(t)\}$ exists, since

$\int_{0^+}^{\infty} t^{-\frac{1}{2}} e^{-st}\, dt$ is convergent.

To see this we let $st = x$ then $s\,dt = dx$

$$\int_{0^+}^{\infty} \left(\frac{x}{s}\right)^{-\frac{1}{2}} e^{-x} \frac{dx}{s} = \frac{1}{\sqrt{s}} \int_{0^+}^{\infty} x^{-\frac{1}{2}} e^{-x}\, dx$$

$$= \frac{\Gamma 1/2}{\sqrt{s}} = \frac{\sqrt{\pi}}{\sqrt{s}} .$$

Theorem (1-2) [11, 16]:

If the integral $\int_0^{\infty} f(t)\, e^{-st}\, dt$ converges for $s = s_0$

then it also converges for all values of Re $(s) > s_0$.

Proof:

Let $L\{f(\tau)\} = \int_0^\infty f(\tau)\, e^{-s\tau}\, d\tau = F(s)$,

and $\psi(t) = \int_0^t e^{-s_0 \tau} f(\tau)\, d\tau;\ t > 0.$ (1-8)

We observe that $\lim_{t \to \infty} \psi(t) = F(s_0)$, by hypothesis; and we see that $\psi(0) = 0$, also from (1-8) we have $\dfrac{d\psi(t)}{dt} = e^{-s_0 t}\, f(t)$.

If we take $R > 0$ and choose $\epsilon > 0$ so that $0 < \epsilon < R$.

Then $\int_0^R e^{-st} f(t)\, dt = \int_0^R e^{-(s-s_0)t}\, e^{-s_0 t}\, f(t)\, dt$

$$= \int_0^R e^{-(s_0-s)t}\, \psi'(t)\, dt = e^{-(s-s_0)t}\, \psi(t) \Big|_0^R + (s-s_0) \int_0^R e^{-(s-s_0)t}\, \psi(t)\, dt. \qquad (1\text{-}9)$$

Now if we restrict s such that $\text{Re}(s) > s_0$ and $R \to \infty$,

then (1-9) becomes $(s - s_0) \int_0^\infty e^{-(s_0-s)t}\, \psi(t)\, dt$,

provided the integral $\int_0^\infty e^{-(s-s_0)t}\, \psi(t)\, dt$ converges.

But it does so, and even converges absolutely, as it can be seen from the facts that

(i) $\psi(t)$ is bounded over $t > 0$.

(ii) $\int_0^\infty e^{-(s-s_0)t}\, dt$ converges absolutely for $\text{Re}(s) > s_0$.

Thus the required integral converges,

and in fact $\int_0^\infty e^{-st} f(t)\, dt = (s - s_0) \int_0^\infty e^{-(s-s_0)t}\, \psi(t)\, dt$ □.

Remark (1-3):

If the Laplace integral diverge for any particular value of s, say $s = s_0$ then it diverge for all $\operatorname{Re}(s) < s_0$.

For example:

The unilateral Laplace transform of e^{3t} is $\frac{1}{s-3}$ it diverges at $s = 3$, and also diverges for all $\operatorname{Re}(s) < 3$.

Let $s = -3$ then $\int_0^\infty e^{3t} e^{-(-3)t} \, dt = \int_0^\infty e^{6t} \, dt$ it diverges.

1-4 Properties of the bilateral Laplace transform [9, 10, 14]:

In this section we are going to take up properties of the bilateral Laplace transform.

We shall develop those properties of bilateral Laplace transforms which will permit us to find transforms of many functions.

The unilateral and bilateral Laplace transforms have many properties in common; although there are important differences.

1- Linearity:

Theorem (1-3):

If $L_b\{f(t)\}$ and $L_b\{g(t)\}$ exist for some value of s.

Then so also does $L_b\{\alpha f(t) + \beta g(t)\}$ for that s;

and $L_b\{\alpha f(t) + \beta g(t)\}$

$$= \alpha L_b\{f(t)\} + \beta L_b\{g(t)\} \text{ for all constants } \alpha \text{ and } \beta.$$

Proof:

$$L_b\{\alpha f(t)+\beta g(t)\}=\int_{-\infty}^{\infty} e^{-st}\{\alpha f(t)+\beta g(t)\}\,dt$$

$$=\alpha\int_{-\infty}^{\infty} e^{-st} f(t)\,dt+\beta\int_{-\infty}^{\infty} e^{-st} g(t)\,dt=\alpha\, L_b\{f(t)\}+\beta\, L_b\{g(t)\}\square$$

2- Time scaling:

Theorem (1-4):

If $L_b\{f(t)\}=F(s)$, then $L_b\{f(at)\}=\dfrac{1}{|a|}F(\dfrac{s}{a})$ where are $a\neq 0$

Proof:

(i) If $a>0$ $(a=|a|)$

$$L_b\{f(at)\}=\int_{-\infty}^{\infty} f(at)\,e^{-st}\,dt.$$

Let $x=at$ then $dx=a\,dt$, therefore

$$L_b\{f(at)\}=\int_{-\infty}^{\infty} f(x)\,e^{-x(\frac{s}{a})}\frac{dx}{a}=\frac{1}{a}\int_{-\infty}^{\infty} f(x)\,e^{-\frac{s}{a}x}\,dx$$

$$=\frac{1}{|a|}F(\frac{s}{a}).$$

(ii) If $a<0$ $(a=-|a|)$

$$L_b\{f(at)\}=\int_{-\infty}^{\infty} e^{-st} f(at)\,dt.$$

Let $x=-at$ then $dx=-a\,dt$, therefore

$$L_b\{f(t)\}=\int_{-\infty}^{\infty} e^{-x(\frac{s}{a})} f(-x)\,\frac{dx}{-a}$$

$$=\frac{-1}{a}\int_{-\infty}^{\infty} f(-x)\,e^{-\frac{s}{a}(-x)}\,dx=\frac{-1}{a}F(\frac{s}{a})$$

$$=\frac{1}{|a|}F(\frac{s}{a}).$$

Form (i) and (ii) we get

$$L_b\{f(at)\} = \frac{1}{|a|} F(\frac{s}{a}) \ \square .$$

3- Frequency Scaling:

Theorem (1-5):

If $L_b\{f(t)\} = F(s)$, then $F(as) = \dfrac{1}{|a|} L_b\left\{f(\dfrac{t}{a})\right\}$ where $a \neq 0$.

Proof:

Using the scaling property of bilateral Laplace transform

$$L_b\{f(at)\} = \frac{1}{|a|} F(\frac{s}{a}).$$

Let $\beta = \dfrac{1}{a}$ then $L_b\left\{f(\dfrac{t}{\beta})\right\} = |\beta|\ F(\beta s)$, or $\dfrac{1}{|\beta|}\ L_b\left\{f(\dfrac{t}{\beta})\right\} = F(\beta s)$,

therefore the frequency scaling property is

$$\frac{1}{|a|} L_b\left\{f(\frac{t}{a})\right\} = F(as);\ a \neq 0 \quad \square .$$

4- Shifting theorem on s:

Theorem (1-6):

If $L_b\{f(t)\} = F(s)$, then $L_b\{e^{at} f(t)\} = F(s-a)$ *for* $Re(s) > a$.

Proof:

$$F(s-a)=\int_{-\infty}^{\infty} e^{-(s-a)t} f(t)\,dt$$

$$=\int_{-\infty}^{\infty} e^{-st}\left[e^{at} f(t)\right]dt \;=\; L_b\left\{e^{at} f(t)\right\} \;\square .$$

5- Shifting Theorem on t:

Theorem (1-7):

If $\quad L_b\{f(t)\}=F(s)$, then $L_b\{f(t-a)\}=e^{-as}\,F(s)$.

Proof:

$$L_b\,\{f(t-a)\}=\int_{-\infty}^{\infty} e^{-st} f(t-a)\,dt .$$

Let $\quad x=t-a\;$ then $dx=dt$,

therefore $\displaystyle\int_{-\infty}^{\infty} e^{-s(x+a)}\, f(x)\,dx=\int_{-\infty}^{\infty} e^{-sx}\, e^{-sa}\, f(x)\,dx$

$$=e^{-sa}\int_{-\infty}^{\infty} e^{-sx}\, f(x)\,dx=e^{-sa}\,F(s)\quad \square .$$

6- Convolution:

Definition (1-2):

The convolution of $g(t)$ with $f(t)$ is denoted by $f*g$ and defined by $\quad g(t)*f(t)=\int_{-\infty}^{\infty} g(\tau)\,f(t-\tau)\,d\tau .$

Note that

$$g(t) * f(t) = f(t) * g(t)$$

Since

$$g(t) * f(t) = \int_{-\infty}^{\infty} g(\tau)\, f(t-\tau)\, d\tau$$

Let $t - \tau = \mu$ then $d\tau = -\, d\mu$,

therefore $g(t) * f(t) = -\int_{\infty}^{-\infty} g(t-\mu)\, f(\mu)\, d\mu$

$$= \int_{-\infty}^{\infty} g(t-\mu)\, f(\mu)\, d\mu = f(t) * g(t).$$

Theorem (1-8):

Let $G(s)$ and $F(s)$ be bilateral Laplace transforms of $g(t)$ and $f(t)$ respectively then

$$L_b\{g(t) * f(t)\} = G(s) \,.\, F(s)$$

Proof:

From the definition of the bilateral Laplace transform

$$G(s)\ F(s) = \int_{-\infty}^{\infty} e^{-st} g(t)\, dt \quad \int_{-\infty}^{\infty} e^{-st}\ f(t)\, dt$$

$$= \int_{-\infty}^{\infty} e^{-s\sigma}\ g(\sigma)\, d\sigma \int_{-\infty}^{\infty} e^{-s\tau}\ f(\tau)\, d\tau$$

$$= \int_{\tau=-\infty}^{\infty} \int_{\sigma=-\infty}^{\infty} e^{-s(\sigma+\tau)}\, g(\sigma)\, f(\tau)\, d\sigma\, d\tau \ .$$

Let $t = \tau + \sigma$ τ fixed $\sigma = t - \tau$ $d\sigma = dt$

$$= \int_{\tau=-\infty}^{\infty} \int_{t=-\infty}^{\infty} e^{-st}\, g(t-\tau)\, f(\tau)\, dt\, d\tau$$

$$= \int_{t=-\infty}^{\infty} \int_{\tau=-\infty}^{\infty} e^{-st}\, g(t-\tau)\, f(\tau)\, d\tau\, dt$$

$$= \int_{-\infty}^{\infty} e^{-st} \left[\int_{-\infty}^{\infty} g(t-\tau)\, f(\tau)\, d\tau \right] dt = L_b\{g(t) * f(t)\} \;\square .$$

7- Frequency Convolution:

Theorem (1-9):

Let $G(s)$ and $F(s)$ be bilateral Laplace transform of $g(t)$ and $f(t)$ respectively then

$$L_b\{g(t)\; f(t)\} = \frac{-i}{2\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} G(w)\, F(s-w)\, dw .$$

Proof:

By definition

$$L_b\{g(t)\; f(t)\} = \int_{-\infty}^{\infty} g(t)\, f(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} \left[\frac{-i}{2\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} G(w)\, e^{wt}\, dw \right] f(t)\, e^{-st}\, dt$$

$$= \frac{-i}{2\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} G(w) \left[\int_{-\infty}^{\infty} f(t)\, e^{-(s-w)t}\, dt \right] dw$$

$$= \frac{-i}{2\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} G(w)\, F(s-w)\, dw \;\square .$$

8- Differentiation:

Theorem (1-10):

If $L_b\{f(t)\} = F(s)$, then $L_b\left\{\dfrac{d^n f(t)}{dt^n}\right\} = s^n F(s)$

provided $\dfrac{d^k}{dt^k} f(t) = f^{(k)}(t) \to 0$ as $t \to \pm\infty$, $K = 0, 1, 2, \ldots$

Proof:

We will use the proof by induction

1) For $n=1$, integration by parts gives.

$$L_b\left\{\frac{d f(t)}{dt}\right\} = e^{-st} f(t) \Big|_{-\infty}^{\infty} + s\int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt ,$$

because $F(s)$ exists, since $f(t)$ evaluated at $t = -\infty$ and $t = \infty$ is zero,

thus $L_b\left\{\dfrac{d f(t)}{dt}\right\} = s F(s)$

2) We assume it is true for the integer $n = k$.

3) We need to prove that statement must be true for $n = k + 1$.

$$L_b\left\{\frac{d^{k+1} f(t)}{dt^{k+1}}\right\} = e^{-st} f^{(k)}(t) \Big|_{-\infty}^{\infty} + s\int_{-\infty}^{\infty} f^{(k)}(t)\, e^{-st}\, dt$$

$$= s\left[s^k F(s)\right] = s^{k+1} F(s).$$

Therefore

$$L_b\left\{\frac{d^n (t)}{dt^n}\right\} = s^n F(s) \quad \square .$$

9- Frequency Differentiation:

Theorem (1-11):

If $L_b\{f(t)\} = F(s)$, then $L_b\{(-t)^n f(t)\} = \dfrac{d^n}{ds^n} F(s)$.

Proof:

We shall prove the theorem by induction,

Step (i) If $n = 1$

$$\frac{dF(s)}{ds} = \frac{d}{ds}\int_{-\infty}^{\infty} e^{-st} f(t)\, dt$$

$$= \int_{-\infty}^{\infty} \frac{\partial}{\partial s}\left[e^{-st} f(t)\right] dt$$

$$= (-1)\int_{-\infty}^{\infty} e^{-st}\left[t f(t)\right] dt = -L_b\{t f(t)\}.$$

Step (ii) assume that $L_b\{(-t)^k f(t)\} = \dfrac{d^k}{ds^k} F(s)$ is for $n = k$.

Step (iii) we need to prove that the statement is true for $n = k + 1$

$$\frac{d^{k+1}}{ds^{k+1}} F(s) = \frac{d}{ds}\left[\frac{d^k}{ds^k} F(s)\right]$$

$$= \frac{d}{ds}\left[(-1)^k L_b\{t^k f(t)\}\right]$$

$$= \frac{d}{ds}\left[(-1)^k \int_{-\infty}^{\infty} e^{-st}\left[t^k f(t)\right] dt\right.$$

$$= (-1)^k \int_{-\infty}^{\infty} \frac{\partial}{\partial s} e^{-st}\left[t^k f(t)\right] dt$$

$$= (-1)^k \int_{-\infty}^{\infty} (-t)\, e^{-st} \left[t^k f(t)\right] dt$$

$$= (-1)^{k+1} \int_{-\infty}^{\infty} e^{-st} \left[t^{k+1} f(t)\right] dt .$$

From steps (i), (ii) and (iii) we have

$$L_b \left\{(-t)^n f(t)\right\} = \frac{d^n F(s)}{d s^n} \quad \square .$$

10- Integration:

Theorem (1-12):

Let $f(t)$ be piece- wise continuous and satisfy the exponential condition $|f(t)| < K e^{\alpha t}$ for some α and K, and if $L_b \left\{f(t)\right\} = F(s)$.

Then $L_b \left\{ \int_{-\infty}^{t} f(u)\, du \right\} = \dfrac{F(s)}{s}$,

and $L_b \left\{ \int_{t}^{\infty} f(u)\, du \right\} = \dfrac{F(s)}{s}$.

Proof:

Let $g(t) = \int_{-\infty}^{t} f(u)\, du$

$$|g(t)| = \left| \int_{-\infty}^{t} f(u)\, du \right| \le \int_{-\infty}^{t} |f(u)|\, du .$$

Since $f(t)$ is piece- wise continuous and satisfies $|f(t)| < K e^{\alpha t}$ for some α and K.

Then $|g(t)| \le \int_{-\infty}^{t} K e^{\alpha u}\, du = K \left.\frac{e^{\alpha u}}{\alpha}\right|_{-\infty}^{t} = \frac{K}{\alpha}(e^{\alpha t} - 0) = \frac{K}{\alpha} e^{\alpha t}.$

Furthermore $g(t)$ is continuous and $g'(t) = f(t)$.

Now; $L_b\{f(t)\} = L_b\{g'(t)\} = s L_b\{g(t)\}$

$$L_b\{f(t)\} = s\, L_b\{g(t)\}, \text{ then } L_b\{g(t)\} = \frac{1}{s} L\{f(t)\},$$

or
$$L_b\left\{\int_{-\infty}^{t} f(u)\, du\right\} = \frac{F(s)}{s} .$$

Similarly
$$L_b\left\{\int_{t}^{\infty} f(u)\, du\right\} = \frac{F(s)}{s} \quad \square .$$

1-5 Properties of the unilateral Laplace transform [9, 10, 14, 15, 16]:

Properties of bilateral Laplace transform hold for the unilateral Laplace transform with the exception of properties (8) and (10). There are also initial and final values theorems for unilateral Laplace transform.

11. Differentiation (Unilateral):

Theorem (1-13):

If $L\{f(t)\} = F(s)$.

Then $L\left\{\frac{d f(t)}{dt}\right\} = s F(s) - f(0)$ and in general, we have

$$L\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1} f(0) \ldots\ldots - f^{(n-1)}(0)$$

with $f^{(k)}(t) \to 0$ as $t \to \infty$, $K = 0, 1, 2, \ldots$

Proof:

We shall prove theorem by induction.

(i) $n = 1$ $\quad L\left\{\frac{d f(t)}{dt}\right\} = \int_0^\infty \frac{d f(t)}{dt} e^{-st}\, dt = f(t)\, e^{-st}\Big|_0^\infty$

$$+ s\int_0^\infty f(t)\, e^{-st}\, dt.$$

Now, $F(s)$ exists, since $f(t)$ evaluated at $t = \infty$ is zero.

Thus $L\left\{\frac{d f(t)}{dt}\right\} = sF(s) - f(0)$.

(ii) Assume the statement true for $n = k$ where k is any positive integer.

$$L\left\{\frac{d^k f(t)}{dt^k}\right\} = s^k F(s) - s^{k-1} f(0) \ldots - f^{(k-1)}(0).$$

(iii) Prove that the statement must be true for $n = k + 1$

$$L\left\{\frac{d^{k+1} f(t)}{dt^{k+1}}\right\} = \int_0^\infty \frac{d^{k+1} f(t)}{dt^{k+1}} e^{-st}\, dt = f^{(k)}(t)\, e^{-st}\Big|_0^\infty$$

$$+ s\int_0^\infty \frac{d^k f(t)}{dt^k} e^{-st}\, dt$$

$$= -f^{(k)}(0) + s\left[s^k F(s) - s^{k-1} f(0) - \ldots\ldots f^{(k-1)}(0)\right]$$

$$= s^{k+1} F(s) - s^k f(0) \ldots - f^{(k)}(0) \quad \square.$$

From (i), (ii), and (iii) the statement is true.

Remark (1-4):

The theorem (1-10) is similar to theorem (1.13) but when theorem (1-10) was used in solving initial value problem it was found that the solution function satisfies the differential equation but some times it does not satisfy the initial conditions.

This can be attributed to the integral limits which are $-\infty$ and ∞ that make resulting solution function from the theorem (1-10) independent of initial conditions.

The important question here is when the theorem will be effective in solving initial value problem, the answer to this question is to have the initial conditions equal to zero i.e.

$$y^{(n)}(0) = y^{(n-1)}(0) = \dots\dots\dots = y(0) = 0$$

Theorem (1-14):

If $f(t)$ satisfies conditions of the existence of $L\{f(t)\}$ and if $\lim_{t\to 0} \frac{f(t)}{t}$ exists, then $L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(\bar{s})\, d\bar{s}$.

Proof:

$$\int_s^{\infty} F(\bar{s})\, d\bar{s} = \int_s^{\infty}\left[\int_0^{\infty} e^{-\bar{s}t} f(t)\, dt\right] d\bar{s} .$$

Since that is possible to change the order of integration, we obtain (see figure (1-5)).

$$= \int_0^{\infty}\left[\int_s^{\infty} e^{-\bar{s}t} f(t)\, d\bar{s}\right] dt = \int_0^{\infty} f(t)\left[\int_s^{\infty} e^{-\bar{s}t}\, d\bar{s}\right] dt$$

$$= \int_0^{\infty}\left[f(t)\, \frac{e^{-\bar{s}t}}{-t}\Big|_s^{\infty}\right] dt$$

$$= \int_0^{\infty} \frac{f(t)}{t}\, e^{-st}\, dt = L\left\{\frac{f(t)}{t}\right\} \quad \square .$$

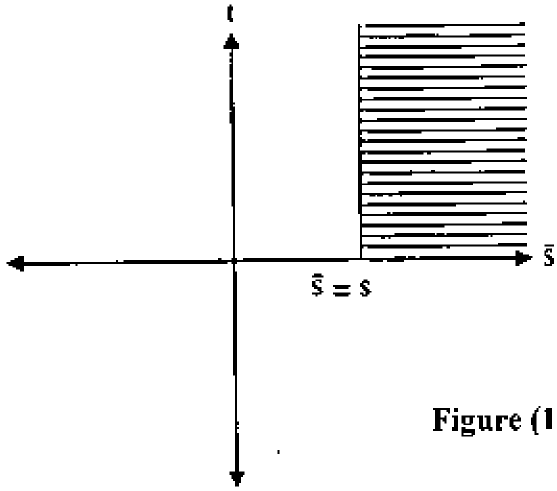

Figure (1-5)

Theorem (1-15):

Let $G(s)$ and $F(s)$ be unilateral Laplace transform of $g(t)$ and $f(t)$ respectively then

$$L\left\{\int_0^t g(t-\tau)\, f(\tau)\, d\tau\right\} = G(s)\,.\,F(s)$$

Proof:

$$G(s)\,.\,F(s) = \int_0^\infty e^{-st} g(t)\, dt \,.\, \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty e^{-s\sigma} g(\sigma)\, d\sigma \int_0^\infty e^{-s\tau} f(\tau)\, d\tau = \int_{\tau=0}^{\infty} \int_{\sigma=0}^{\infty} e^{-s(\sigma+\tau)} g(\sigma)\, f(\tau)\, d\sigma\, d\tau\ .$$

Let $\sigma + \tau = t$ $\qquad$ τ fixed

$\sigma = t - \tau$ $\qquad$ $d\sigma = dt$

$$= \int_0^\infty \int_\tau^\infty e^{-st} g(t-\tau)\, f(\tau)\, dt\, d\tau\ .$$

Since that it is possible to change the order of integration, we obtain (see figure (1.6)).

$$\int_0^\infty \int_0^t e^{-st} g(t-\tau)\, f(\tau)\, d\tau\, dt = L\left\{\int_0^t g(t-\tau)\, f(\tau)\, d\tau\right\}\ \square\,.$$

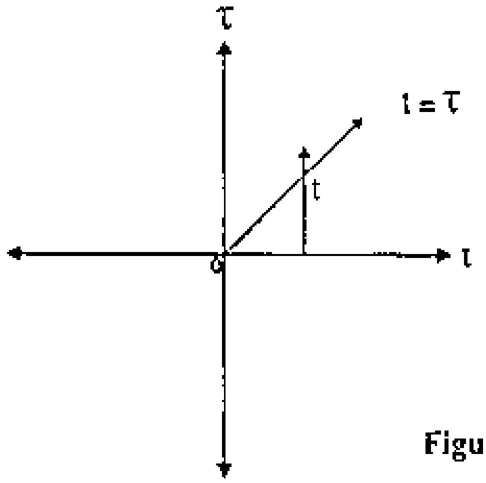

Figure (1-6)

Theorem (1-16):

If $f(t)$ is continuous except at points $t = a_i \;\; a_i > 0 \;\; (i = 1, 2, ..., n)$ than

$$L\{f'(t)\} = s\,L\{f(t)\} - f(0) - \sum_{i=1}^{n} \left[f(a_i^+) - f(a_i^-)\right]e^{-as}.$$

Proof:

$$L\{f'(t)\} = \sum_{i=1}^{n} \left[\int_0^{\bar{a}_i} f'(t)\,e^{-st}\,dt + \int_{a_i^+}^{\infty} f'(t)\,e^{-st}\,dt\right]$$

$$= \sum_{i=1}^{n} \left[f(t)\,e^{-st}\Big|_0^{\bar{a}_i} + s\int_0^{\bar{a}_i} f(t)\,e^{-st}\,dt + 0 - f(a_i^+)\,e^{-sa_i^+}\right.$$

$$\left. + s\int_{a_i^+}^{\infty} f(t)\,e^{-st}\,dt\,\right].$$

Since e^{-st} is continuous for all t therefore $\lim\limits_{t \to \bar{a}_i} e^{-st} = \lim\limits_{t \to a_i^+} e^{-st} = e^{-sa}$.

Now;

$$L\{f'(t)\} = \sum_{i=1}^{n} [\,f(a_i^-)\,e^{-sa} - f(a_i^+)\,e^{-sa} - f(0) +$$

$$s\int_0^{a_i^-} f(t)\,e^{-st}\,dt + s\int_{a_i^+}^{\infty} f(t)\,e^{-st}\,dt\,]$$

$$= s\,L\{f(t)\} - f(0) - \sum_{i=1}^{n} \left[f(a_i^+) - f(\bar{a}_i)\right]e^{-as} \quad \square.$$

12- Integration (Unilateral):

Theorem (1-17):

If $L\{f(t)\} = F(s)$ and $|f(t)| \le K e^{\alpha t}$ for some constants K and α.

Then $L\left\{\int_0^t f(\tau)\, d\tau\right\} = \dfrac{F(s)}{s}$, and $L\left\{\int_{-\infty}^t f(\tau)\, d\tau\right\} = \dfrac{F(s)}{s} + \dfrac{f^{(-1)}(0)}{s}$.

Where $f^{(-1)}(0) \equiv \lim\limits_{t\to 0} \int_{-\infty}^t f(\tau)\, d\tau$

Proof:

Let $g(t) = \int_0^t f(\tau)\, d\tau$, then

$$|g(t)| \le \int_0^t |f(\tau)|\, d\tau \le \int_0^t K e^{\alpha \tau}\, d\tau$$

$$= K \left.\frac{e^{\alpha \tau}}{\alpha}\right|_0^t = \frac{K}{\alpha}\left(e^{\alpha t} - 1\right)$$

Furthermore $g'(t) = f(t)$ excerpt for points at which $f(t)$ is not continuous.

Hence $L\{f(t)\} = L\{g'(t)\} = s\, L\{g(t)\} - g(0)$

$$L\{f(t)\} = s L\{g(t)\}$$

$$L\{g(t)\} = \frac{1}{s} L\{f(t)\}, \text{ then } \quad L\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{F(s)}{s}.$$

If the lower limit is $-\infty$, then we obtain

$$\int_{-\infty}^t f(\tau)\, d\tau = \int_{-\infty}^0 f(\tau)\, d\tau + \int_0^t f(\tau)\, d\tau$$

$$= f^{(-1)}(0) + \int_0^t f(\tau)\, d\tau.$$

Now $L\left\{\int_{-\infty}^{t} f(\tau)\,d\tau\right\} = L\left\{f^{-1}(0) + \int_{0}^{t} f(\tau)\,d\tau\right\}$

$$= \frac{f^{(-1)}(0)}{s} + \frac{F(s)}{s} \qquad \square .$$

13- Initial Value

Theorem (1-18):

Let $L\{g(t)\} = G(s)$ then $\lim_{s\to\infty} s\,G(s) = g(0)$.

Proof:

Using theorem (1-13).

$$L\left\{\frac{d}{dt}g(t)\right\} = \int_0^\infty \frac{d}{dt}\{g(t)\}\,e^{-st}\,dt = \{s\,G(s) - g(0)\}\,.$$

Let $s\to\infty$ then

$$\lim_{s\to\infty}\int_0^\infty \frac{d}{dt}\{g(t)\}\,e^{-st}\,dt = \lim_{s\to\infty}\{s\,G(s) - g(0)\}$$

$$\int_0^\infty \lim_{s\to\infty}\left\{\frac{d}{dt}\{g(t)\}\,e^{-st}\right\}dt = \lim_{s\to\infty}\{s\,G(s) - g(0)\}$$

Now,

$$\frac{d}{dt}\{g(t)\}\,e^{-st} \to 0 \quad \text{as} \quad s\to\infty.$$

Therefore $0 = \lim_{s\to\infty}\{s\,G(s) - g(0)\}$, that is $g(0) = \lim_{s\to\infty} s\,G(s)$ $\square$.

14- Final Value:

Theorem (1-19):

If $\lim_{t\to\infty} g(t)$ exists, then $\lim_{t\to\infty} g(t) = \lim_{s\to 0} s\, G(s)$.

Proof:

From theorem (1-13).

$$\lim_{s\to 0} \int_0^\infty \frac{d}{dt}\{g(t)\}\, e^{-st}\, dt = \lim_{s\to 0}\{sG(s) - g(0)\}$$

$$= \int_0^\infty \lim_{s\to 0}\left\{\frac{d}{dt}\{g(t)\}\, e^{-st}\right\} dt = \lim_{s\to 0}\left[sG(s) - g(0)\right]$$

$$\int_0^\infty \frac{d}{dt}\{g(t)\}\, dt = \lim_{s\to 0}\{sG(s) - g(0)\}$$

$$\lim_{t\to\infty}\left[g(t) - g(0)\right] = \lim_{s\to 0}\left[sG(s) - g(0)\right].$$

Then $\lim_{t\to\infty} g(t) = \lim_{s\to 0} s\, G(s)$ □.

Theorem (1-20):

The Laplace transform of a piece-wise continuous periodic function $f(t)$ with perioed P is

$$L\{f(t)\} = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t)\, dt$$

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st}\, f(t)\, dt = \int_0^p e^{-st}\, f(t)\, dt + \int_p^{2p} e^{-st}\, f(t)\, dt +$$

$$+ \int_{2p}^{3p} e^{-st}\, f(t)\, dt + \dots\dots .$$

Let $t = \tau + P$ then $dt = d\tau$

$$\int_{p}^{2p} e^{-st} f(t)\, dt = \int_{0}^{p} e^{-s(\tau+p)} f(\tau + p)\, d\tau = e^{-sp} \int_{0}^{p} e^{-s\tau} f(\tau)\, d\tau .$$

Let $t = \tau + 2p$, then $\int_{2p}^{3p} e^{-st} f(t)\, dt = \int_{0}^{p} e^{-2sp}\, e^{-s\tau} f(\tau)\, d\tau .$

Therefore $L\{f(t)\} = \int_{0}^{p} e^{-st} f(t)\, dt + e^{-sp} \int_{0}^{p} e^{-st} f(t)\, dt +$

$$+ e^{-2sp} \int_{0}^{p} e^{-st} f(t)\, dt + \ldots\ldots$$

$$= \left[1 + e^{-sp} + e^{-2sp} + \ldots\right] \int_{0}^{p} e^{-st} f(t)\, dt$$

$$= \frac{1}{1 - e^{-sp}} \int_{0}^{p} e^{-st} f(t)\, dt \qquad \square .$$

Table of Laplace transforms.

Table (A-1) Basic Laplace transforms and R. O. C.

Function	Transform	R. O. C
$f(t) = 1$	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$
$f(t) = t$	$\frac{1}{s^2}$	$\mathrm{Re}(s) > 0$
$f(t) = t^n$	$n!/s^{n+1}$	$\mathrm{Re}(s) > 0$
$f(t) = e^{-at}$	$1/s+a$	$\mathrm{Re}(s) > -a$
$f(t) = \cos wt$	s/s^2+w^2	$\mathrm{Re}(s) > 0$
$f(t) = \sin wt$	w/s^2+w^2	$\mathrm{Re}(s) > 0$
$f(t) = \sin hat$	a/a^2-s^2	$\mathrm{Re}(s) > 0$

Table (A-2) Laplace transform properties.

A function	Unilateral	Bilateral
$a f(t) + b g(t)$	$aF(s) + bG(s)$	$aF(s) + bG(s)$
$f(t-a)$	$e^{-sa} F(s)$	$e^{-as} F(s)$
$f(at)$	$\frac{1}{\lvert a \rvert} F(\frac{s}{a})$	$\frac{1}{\lvert a \rvert} F(\frac{s}{a})$
$e^{at} f(t)$	$F(s-a)$	$F(s-a)$
$-t f(t)$	$\frac{d}{ds} F(s)$	$\frac{d}{ds} F(s)$
$f(t) * g(t)$	$F(s) . G(s)$	$F(s) . G(s)$
$\frac{d}{dt} f(t)$	$sF(s) - f(0)$	$sF(s)$
$\int_{-\infty}^{t} f(t)\, dt$	$\frac{1}{s} \int_{-\infty}^{0} f(\tau)\, d\tau + \frac{F(s)}{s}$	$\frac{F(s)}{s}$

Table (A-3).

$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2 F(s) - s f(0) - f'(0)$
$t f(t)$	$-F'(s)$
$t f'(t)$	$-sF'(s) - F(s)$
$t f''(t)$	$-s^2 F'(s) - 2sF(s) + f(0)$
$t^2 f(t)$	$F''(s)$
$t^2 f'(t)$	$sF''(s) + 2F'(s)$
$t^2 f''(t)$	$s^2 F''(s) + 4sF'(s) + 2F(s)$

Table (A-4).

$f(t)$	$F(s)$
$-t\,f(t)$	$F'(s)$
$t^2 f(t)$	$F''(s)$
$f'(t)$	$sF(s) - f(0)$
$-t\,f'(t) - f(t)$	$sF'(s)$
$t^2 f'(t) + 2t\,f(t)$	$sF''(s)$
$f''(t)$	$s^2 F(s) - s\,f(0) - f'(0)$
$-t\,f''(t) - 2f'(t)$	$s^2 F'(s) + f(0)$
$t^2 f''(t) + 4t\,f'(t) + 2f(t)$	$s^2 F''(s)$

1-6 Elementary Evaluation of Laplace Transforms [12]:

In the present section we shall develop some elementary transforms by using theorems 1-3, 1-6, 1-11 and 1-13.

Let $L\left\{\frac{t^n}{n!}\right\}$ is denoted by $F_n(s)$ we shall first find the Laplace transform of $f(t) = 1$.

If $F_0(s) = L\{1\}$ then by theorem 1-13 we have

$$L\{0\} = s\,F_0(s) - 1 \quad \text{or} \quad F_0(s) = \frac{1}{s}\,.$$

Example (1.7):

Show that the Laplace transform of $f(t)=\frac{t^n}{n!}$ is $\frac{1}{s^{n+1}}$ for n any positive integer.

Solution:

By induction.

1. Let n=1, by theorem 1-13, $L\{1\}=sF_1(s)-f(0)$, or $\frac{1}{s}=sF_1(s)$ or

   $F_1(s)=\frac{1}{s^2}$.

2. We assume it is true for integer $n=k$, that is $F_k=\frac{1}{s^{k+1}}$.

3. We need to prove that statement must be true for $n=k+1$.

   By theorem 1-13.

   $L\left\{\frac{t^k}{k!}\right\}=F_k=sF_{k+1}(s)-f(0)$, or $\frac{1}{s^{k+1}}=sF_{k+1}(s)$,

   or $F_{k+1}(s)=\frac{1}{s^{k+2}}$.

   From (1), (2), and (3) we have $L\left\{\frac{t^n}{n!}\right\}=\frac{1}{s^{n+1}}$.

1-7 Some Difficult Transforms:

To develop less elementary transforms it is useful to prepare the table (A-3) we shall also need the following asymptotic theorems.

I) If $f(t)\sim At^{\alpha}$ $(\alpha>-1)$ as $t\to 0$.

then $F(s)\sim\frac{A\Gamma(\alpha+1)}{s^{\alpha+1}}$ as $s\to\infty$.

II) If $f(t) \sim Bt^{\beta}\ (\beta > -1)$ as $t \to \infty$, then $F(s) \sim \dfrac{B\Gamma(\beta+1)}{s^{\beta+1}}$ as $s \to 0$.

Example (1-8):

Find the transforms of Bessel's functions of the first kind $J_0(t)$ and $J_1(t)$.

Solution:

$J_0(t)$ is a solution of $ty'' + y' + ty = 0$.

Using the table A-3 and $J_0(0) = J_0'(0) = 0$ we get

$$(s^2+1)Y'(s) + sY(s) = 0,$$

by integration we get

$$Y(s) = \frac{c}{\sqrt{s^2+1}}.$$

Now;

$$J_0(t) \sim 1 \quad \text{as} \quad t \to 0, \text{ hence } Y(s) \sim \frac{1}{s} \quad \text{as} \quad s \to \infty,$$

but by inspection $Y(s) \sim \dfrac{c}{s}$, hence $c = 1$.

Therefore $\quad L\{J_0(t)\} = \dfrac{1}{\sqrt{s^2+1}}$.

Using the fact that $J_0'(t) = -J_1(t)$ and theorem (1-13) we get

$$L\{J_1(t)\} = sL[-J_0(t)] + J_0(0) = -\frac{s}{\sqrt{s^2+1}} + 1 = \frac{\sqrt{s^2+1}-s}{\sqrt{s^2+1}}.$$

Example (1-9):

Find Laplace transform of

a) $f(t)=\sin a\sqrt{t}$ and

b) $f(t)=\dfrac{\cos a\sqrt{t}}{\sqrt{t}}$

Solution:

a) By differentiating twice $f(t)$, one readily obtains.

$4t\, f''(t)+2f'(t)+a^2 f(t)=0$.

Using $f(0)=0$ and the table A-3.

$4s^2 F'(s)+(6s-a^2)F(s)=0$, thus $F(s)=\dfrac{c}{s^{\frac{3}{2}}}e^{-a^2/4s}$.

Now $f(t)\sim a\sqrt{t}$ as $t\to 0$, hence $F(s)\sim\dfrac{a\sqrt{\pi}}{2s^{\frac{3}{2}}}$ as $s\to\infty$,

by inspection $F(s)\sim\dfrac{c}{s^{\frac{3}{2}}}$, hence $c=\dfrac{a\sqrt{\pi}}{2}$.

Therefore $L\left\{\sin a\sqrt{t}\right\}=\dfrac{a\sqrt{\pi}}{2s^{\frac{3}{2}}}e^{-a^2/4s}$.

b) Now by theorem 1-13 we have

$L\left\{\dfrac{a\cos a\sqrt{t}}{2\sqrt{t}}\right\}=S\,L\left\{\sin a\sqrt{t}\right\}-\sin(0)$, thus $L\left\{\dfrac{\cos a\sqrt{t}}{\sqrt{t}}\right\}=\dfrac{\sqrt{\pi}}{\sqrt{s}}e^{-a^2/4s}$.

Example (1-10):

Find Laplace transform of $f(t)=J_0(a\sqrt{t})$

Solution:

By differentiating $f(t)$ and using the relations

i) $J_n'(t) = \frac{1}{2}\left[J_{n-1}(t) - J_{n+1}(t)\right].$

ii) $J_{n+1}(t) = \frac{2n}{t} J_n(t) - J_{n-1}(t).$

We have $4t\, f''(t) + 4f'(t) + a^2 f(t) = 0$, the transformed equation is

$$4s^2 F'(s) + (4s - a^2) F(s) = 0,$$ which gives at once

$$F(s) = \frac{c e^{-a^2/4s}}{s}.$$

Now, $f(t) \sim 1$ as $t \to 0$ and $F(s) \sim \frac{1}{s}$ as $s \to \infty$,

⋮

thus $c = 1$ and $L\left\{J_0(a\sqrt{t})\right\} = \frac{e^{-a^2/4s}}{s}$.

1-8 Inversion of The Laplace Transform [4, 10, 15, 16]:

The operation of finding the inverse of Laplace transform matches the following question, if we have been given the function $F(s)$ how could we find the function $f(t)$ if we have $L^{-1}\{F(s)\} = f(t)$.

The answer of the previous question is that we can find the function $f(t)$ by using the definition.

$$f(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} F(s)\, e^{st}\, ds. \qquad (1\text{-}10)$$

This method depends on evaluating (1-10) and requires an understanding of complex variables.

We recall that the integral in (1-10) is a line integral along a vertical line in the region of absolute convergence according to a result in the theory of function of a complex variable;

If $\quad F(s) \to 0 \qquad$ as $s \to \infty$.

then for $t > 0$, the value the line integral is equal to the sum of the residues of the function $F(s)\, e^{st}$ at the poles that are to the left of the vertical line $\mathrm{Re}(s) = \sigma$.

For $t < 0$ the value of the line integral is equal to the negative of the sum of the residues of the function $F(s)\, e^{st}$ at the poles that are to the right of the vertical line $\mathrm{Re}(s) = \sigma$, where the residues of function $F(s)\, e^{st}$ at a simple pole s_0 is

$$\mathrm{Res}\, F(s)\, e^{st}\Big|_{s=s_0} = \lim_{s \to s_0} \left\{ (s - s_0)\, F(s)\, e^{st} \right\}$$

and the residue of the function $F(s)\, e^{st}$ at a kth-order pole s_0 is

$$\mathrm{Res}\, F(s)\, e^{st}\Big|_{s=s_0} = \lim_{s \to s_0} \left\{ \frac{1}{(k-1)!} \frac{d^{k-1}}{d s^{k-1}} \left[(s - s_0)^k\, F(s)\, e^{st} \right] \right\}$$

Example (1-11):

If $F(s) = \dfrac{1}{s+3}$, $\quad \mathrm{Re}(s) > -3$ then find $f(t)$.

Solution:

$$f(t) = \frac{-i}{2\pi} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{st}}{s+3}\, ds .$$

Along a vertical line in the region $\text{Re}(s) > -3$ we obtain, (see figure (1-7)).

$$f(t) = \text{Res} \left. \frac{e^{st}}{s+3} \right|_{s=-3}$$

$$= \lim_{s \to -3} (s+3) \frac{e^{st}}{s+3} = e^{-3t} \ , \ t \geq 0 \ .$$

$$f(t) = 0 \ , \ t < 0 .$$

hence $f(t) = \begin{cases} e^{-3t}, & t \geq 0 \\ 0 \ , & t < 0 \end{cases}$

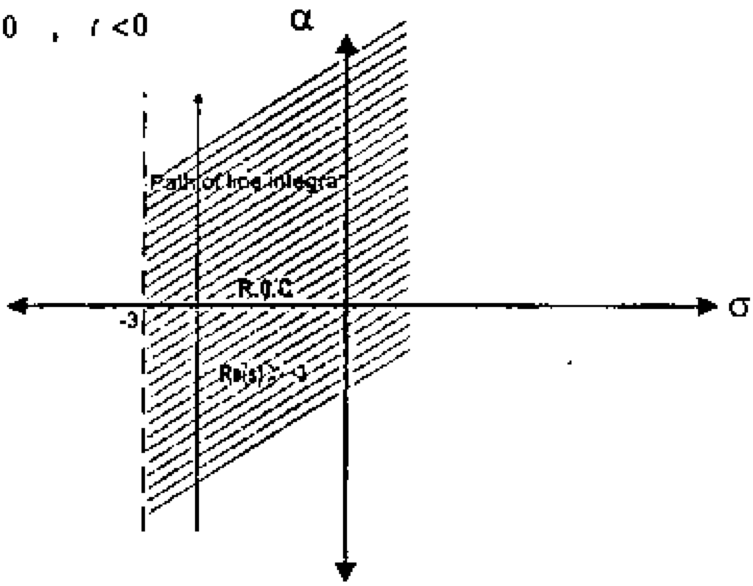

Figure (1-7)

Example (1-12):

If $F(s) = \dfrac{1}{(s+2)(s-3)^2}$, $-2 < \text{Re}(s) < 3$, then find $f(t)$.

Solution:

$$f(t) = \frac{-i}{2\pi} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{st}}{(s+2)(s-3)} ds ,$$

along a vertical line in the region $-2 < \text{Re}(s) < 3$ we obtain, (see figure 1-8).

$$f(t) = \text{Res}\, \frac{e^{st}}{(s+2)(s-3)^2}\bigg|_{s=-2}$$

$$= \lim_{s \to -2} (s+2) \frac{e^{st}}{(s+2)(s-3)^2} = \frac{1}{25} e^{-2t} \ , \ t \geq 0 \ .$$

$$f(t) = \text{Res}\, \frac{e^{st}}{(s+2)(s-3)^2}\bigg|_{s=3} = \lim_{s \to 3} \frac{1}{1!} \frac{d}{ds}\left[(s-3)^2 \left\{\frac{e^{st}}{(s+2)(s-3)^2}\right\}\right]$$

$$= \lim_{s \to 3} \frac{d}{ds}\left[\frac{e^{st}}{s+2}\right] = \lim_{s \to 3} \frac{(s+2)\, t e^{st} - e^{st}}{(s+2)^2}$$

$$= \frac{1}{4} t\, e^{3t} - \frac{1}{16} e^{3t} \ , \quad t < 0 \ .$$

Hence

$$f(t) = \begin{cases} \dfrac{1}{25} e^{-2t} & , \quad t \geq 0 \\ \dfrac{1}{4} t e^{3t} - \dfrac{1}{16} e^{3t} & , \quad t < 0 \end{cases}$$

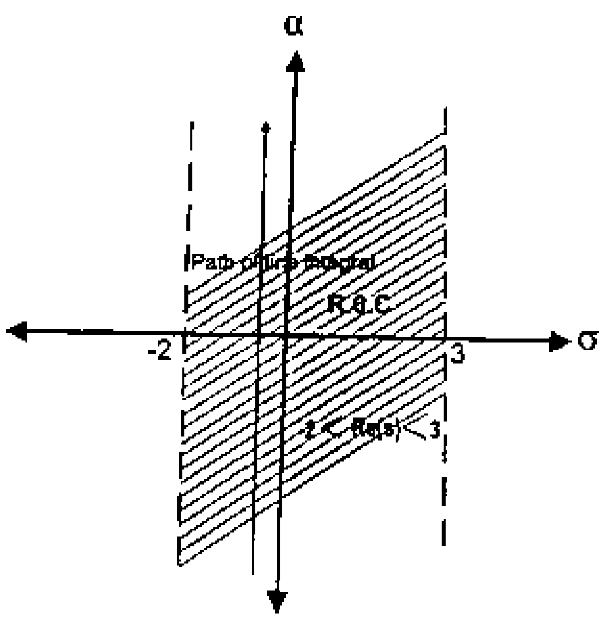

Figure (1-8)

1-9 Evaluation of inverse Laplace transform without contour integration [7]:

Method I:

Since $F(s) = L\{f(t)\}$ then we can evaluate $f(t)$ in two steps.

Step (1):

Try to find an ordinary differential equation satisfied by $F(s)$, of the form

$$\alpha_n(s)\frac{d^nF}{ds^n}+\alpha_{n-1}(s)\frac{d^{n-1}F}{ds^{n-1}}+....+\alpha_1(s)\frac{dF}{ds}+\alpha_0(s)F=G(s), \quad (1\text{-}11)$$

where $\alpha_i(s)\ (i=0,1,2,....,n)$ are at most polynomials in s and the inverse Laplace transform of $G(s)$ is known.

Step (2):

Taking the inverse Laplace transforms of each sides of equation (1-11) (use table A-4).

Then we have a linear differential equation with polynomial coefficients satisfied by the function $f(t)$ this equation usually easy to solve.

Example (1-13):

Find the inverse Laplace transform of $F(s)=\frac{1}{s}\tan^{-1}(s+a)$, where $a>0$.

Solution:

Let $L^{-1}\{F(s)\}=f(t)$.

Since $\lim\limits_{s\to\infty} sF(s)=f(0)$ then we have $f(0)=\frac{\pi}{2}$.

Now; by simple differentiation we find

$$s\frac{dF}{ds}+F=\frac{1}{1+(s+a)^2}.$$

Use table $(A-4)$ we have $-t\frac{df}{dt}=e^{-at}\sin t$,

then the solution is $f(t)=\frac{\pi}{2}-\int_0^t\frac{e^{-a\tau}\sin\tau}{\tau}\,d\tau$

Since $\int_0^1\frac{e^{-at}\sin bt}{t}\,dt=\tan^{-1}\frac{b}{a}+I\,E_1(a+ib)$,

where $E_1(z)=\int_z^\infty\frac{e^{-u}}{u}\,du=\mathrm{Re}(E_1(z))+i\,\mathrm{Im}(E_1(z))$

$z=x+iy$, $\mathrm{Re}\,E_1(z)$ and $\mathrm{Im}\,E_1(z)$ denote the real and imaginary parts of $E_1(z)$ respectively.

Therefore

$$L^{-1}\left\{\frac{1}{s}\tan^{-1}(s+a)\right\}=\tan^{-1}a-I\,E_1(at+it).$$

Example (1-14):

If $F(s)=\frac{e^{-a\sqrt{s}}}{s}$, then find $L^{-1}\{F(s)\}$, where $a\ge 0$.

Solution:

Let $L^{-1}\{F(s)\}=f(t)$.

Note that $F(s)=\frac{1}{s}$ when $a=0$ or $f(t)=1$ when $a=0$.

Now; differentiation of $F(s)$ shows that $F(s)$ satisfies the equation.

$$s^2 \frac{d^2F}{ds^2} + \frac{5}{2}s\frac{dF}{ds} - \frac{a^2}{4}sF + \frac{1}{2}F = 0.$$

Using table (A-4) we have

$$t^2 \frac{d^2f}{dt^2} + \left(\frac{3t}{2} - \frac{a^2}{4}\right)\frac{df}{dt} = 0. \qquad (1\text{-}12)$$

Now; $\lim\limits_{s \to \infty} sF(s) = f(0) = 0$, and $f(t) = 1$ when $a = 0$,

then the solution of (1-12) can be expressed in the form

$f(t) = c_1 + c_2 \int\limits_{\frac{a}{2t^{1/2}}}^{\infty} e^{-u^2}\,du$, where c_1 and c_2 are constants of integration.

The application of the boundary conditions $f(0)=0$ and $f(t)=1$ when $a=0$ gives $c_1 = 0$ and $c_2 = \frac{2}{\sqrt{\pi}}$ and hence

$$f(t) = \frac{2}{\sqrt{\pi}} \int\limits_{\frac{a}{2t^{1/2}}}^{\infty} e^{-u^2}\,du = erfc\left(\frac{a}{2\sqrt{t}}\right)$$

Example (1-15):

If $F(s) = \frac{\ln s}{s}$, then find $f(t) = L^{-1}\{F(s)\}$.

Solution:

$F(s)$ satisfies the differential equation

$$s\frac{dF}{ds} + F = \frac{1}{s} .$$

Using table (A-4) we have

$$-t\,\frac{df}{dt} = 1 \text{ , therefore } f(t) = -c - \ln t$$

Now;

$$\int_0^\infty e^{-st}(-c-\ln t)\,dt = \frac{\ln s}{s} .$$

In particular, when $s = 1$

$$\int_0^\infty e^{-t}\,(-c-\ln t)\,dt = 0 \text{ , giving } c = -\int_0^\infty e^{-t}\,\ln t\,dt .$$

Reference to a standard table of integrals identifies c as Eluler's number, 0.5772 ...

With the notation $\gamma = e^c$ we then have

$$L^{-1}\left\{\frac{\ln s}{s}\right\} = -\ln(\gamma t) .$$

Example (1-16):

If $F(s) = \frac{e^{-a\sqrt{s}}}{\sqrt{s}}$, then find $f(t) = L^{-1}\{F(s)\}$.

Solution:

Note that $F(s) \sim \frac{1}{\sqrt{s}}$ as $s \to 0$

Differentiation of $F(s)$ shows that $F(s)$ satisfies the equation

$$4s\, F''(s) + 6F' - a^2 F = 0$$

From the table (A-4) we have:

$$4t^2\, f'(t) + 2t\, f(t) - a^2 f(t) = 0\ , \text{ therefore } f(t) = \frac{c}{\sqrt{t}} e^{-\frac{a^2}{4t}}.$$

Now; $f(t) \sim \frac{c}{\sqrt{t}}$ as $t \to \infty$.

Hence by asymptotic theorems we have

$$F(s) = L\left\{\frac{c}{\sqrt{t}}\right\} = \frac{c\Gamma(1/2)}{\sqrt{s}} = \frac{c\sqrt{\pi}}{\sqrt{s}} \qquad \text{as} \quad s \to 0\ .$$

since $F(s) \sim \frac{1}{\sqrt{s}}$ as $s \to 0$ then $c = \frac{1}{\sqrt{\pi}}$.

Thus $L^{-1}\left\{\frac{e^{-a\sqrt{s}}}{\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi t}}\, e^{-\frac{a^2}{4t}}$.

Example (1-17):

If $F(s) = \frac{(\sqrt{s^2+1} - s)^n}{\sqrt{s^2+1}}$, then find $f(t) = L^{-1}\{F(s)\}$.

Solution:

By differentiating twice $F(s)$ one readily obtains.

$$(s^2+1)F''+3sF'+(1-n^2)F=0.$$

From the table A-4 we also have

$$t^2 f''(t)+t f'(t)+(t^2-n^2)f(t)=0.$$

But this is Bessel's equation $J_n(t)$ of the first kind of nth order i.e.

$$f_n(t)=\sum_{r=0}^{\infty}\frac{(-1)^r\left(\frac{t}{2}\right)^{n+2r}}{r!\,(n+r)!}$$

Now;

$$F(s)=\frac{(\sqrt{s^2+1}-s)^n}{\sqrt{s^2+1}}=\frac{1}{\sqrt{s^2+1}\,(\sqrt{s^2+1}+s)^n}=\frac{1}{2^n\,s^{n+1}}\text{ as } s\to\infty.$$

We should choose $f(t)$ such that

$$f(t)=\frac{t^n}{2^n\,n!}\qquad\text{as}\qquad t\to 0.$$

Thus $f_n(t)=J_n(t)$.

Method II:

If $F(s)$ can be expanded in the power series of the form

$$F(s)=\sum_{n=1}^{\infty}a_n s^{-n},\ |s|>R,\ \text{ then } f(t)=\sum_{n=1}^{\infty}\frac{a_n t^{n-1}}{(n-1)!}.$$

Examples (1-18):

If $F(s) = \frac{1}{s-1}$ then find $f(t) = L^{-1}\{F(s)\}$

Solution:

$$\frac{1}{s-1} = \sum_{n=1}^{\infty} \frac{1}{s^n} \qquad \forall |s| > 1 \quad ,$$

then by method II $f(t) = \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} = e^t$.

Example (1-19):

If $F(s) = \frac{1}{s^2-1}$ then find $f(t) = L^{-1}\{F(s)\}$.

Solution:

$$\frac{1}{s^2-1} = \sum_{n=1}^{\infty} \frac{1}{s^{2n}} \quad \text{for } |s| > 1 \ ,$$

Then $f(t) = \sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)!} = \sin t$.

1-10 Applications of the Laplace transforms [7, 8, 9, 10, 13, 15, 16, 17, 18]:

The Laplace transforms have a wide range of applications almost include all the science branches. A lot of the physical problems lead to differential equations with initial conditions, these equations can be solved simply if we use Laplace transform.

We can also solve a lot of boundary value problems which arise in the science and engineering and we are going to explain there applications in the following examples.

1-10-1 Ordinary differential equations with constant coefficients:

Example (1-20):

a) Solve $a_2\, y''(t) + a_1\, y' + a_0\, y = f(t),\ \ y'(0) = b_1,\ y(0) = b_0$

b) Solve $y'' - y = 4t \quad y(0) = 1 \quad y'(0) = 3$

where a_0, a_1, a_2, b_0 and b_1 are constants.

Solution:

a) Taking the Laplace transform of both sides of the differential equation and using the given initial conditions, we have

$$a_2 L\{y''(t)\} + a_1 L\{y'(t)\} + a_0 L\{y(t)\} = L\{f(t)\}$$

$$a_2(s^2 Y(s) - s\,y(0) - y'(0)) + a_1(sY(s) - y(0)) + a_0 Y(s) = L\{f(t)\}$$

$$a_2(s^2 Y(s) - s\,b_0 - b_1) + a_1(sY(s) - b_0) + a_0 Y(s) = L\{f(t)\}$$

$$Y(s) = \frac{L\{f(t)\} + a_2 s b_0 + a_2 b_1 + a_1 b_0}{a_2 s^2 + a_1 s + a_0}$$

thus $y(t) = L^{-1}\left\{\dfrac{L\{f(t)\} + a_2 s b_0 + a_2 b_1 + a_1 b_0}{a_2 s^2 + a_1 s + a_0}\right\}$

b) Similarly

$$y(t) = L^{-1}\left(\frac{4}{s^2(s^2-1)}\right) + L^{-1}\left(\frac{s+3}{s^2+1}\right)$$

Now; $L^{-1}\left\{\dfrac{4}{s^2(s^2-1)}\right\} = \displaystyle\int_0^t f(t-\tau)\, g(\tau)\, d\tau$ by theorem (1-15)

$$F(s) = \frac{4}{s^2} \text{ then } f(t) = 4t \quad \text{then} \quad f(t-\tau) = 4(t-\tau)$$

$$G(s) = \frac{1}{s^2-1} \qquad \text{then} \quad g(t) = \sin ht$$

So;

$$L^{-1}\left\{\frac{4}{s^2(s^2-1)}\right\} = \int_0^t 4(t-\tau)\ \sin h\tau\ d\tau$$

$$= 4(t-\tau)\cos h\tau\,\Big|_0^t + 4\sin h\tau\,\Big|_0^t = -4t + 4\ \sin ht$$

And $L^{-1}\left\{\frac{s+3}{s^2-1}\right\} = L^{-1}\left\{\frac{s}{s^2-1}\right\} + L^{-1}\left\{\frac{3}{s^2-1}\right\} = \cos ht + 3\sin\ ht$.

Therefore $\quad y(t) = -4t + 4\sin ht + \cos ht + 3\sin ht$

$$= -4t + 7\sin ht + \cos\ ht$$

1-10-2: Ordinary differential equation with variable coefficients:

Example (1-21):

Solve $\quad y'' + 2t\,y' - 4y = 2 \qquad y(0) = y'(0) = 0$

Solution:

Taking Laplace transform of both sides of the equation then we have

$$s^2 Y(s) - s\,y(0) - y'(0) + 2\left[-\frac{d(sY(s))}{ds}\right] - 4Y(s) = \frac{2}{s}$$

where $L\,(y(t)) = Y(s)$

$$s^2 Y(s) + 2\,(-sY'(s) - Y(s)) - 4Y(s) = \frac{2}{s}$$

$$(s^2 - 6)\,Y(s) - 2s\,Y'(s) = \frac{2}{s}$$

$$Y'(s) + \left(\frac{3}{s} - \frac{s}{2}\right)Y(s) = \frac{-1}{s^2}.$$

Therefore

$$Y(s)=\frac{-e^{\frac{s^2}{4}}}{s^3}\int s\,e^{-\frac{s^2}{4}}\,ds+\frac{c\,e^{\frac{s^2}{4}}}{s^3}$$

$$=\frac{2e^{\frac{s^2}{4}}}{s^3}\int\frac{-s}{2}\,e^{-\frac{s^2}{4}}\,ds+\frac{c\,e^{\frac{s^2}{4}}}{s^3}$$

$$=\frac{2e^{\frac{s^2}{4}}}{s^3}\cdot e^{-\frac{s^2}{4}}+\frac{c\,e^{\frac{s^2}{4}}}{s^3}=\frac{2}{s^3}+\frac{c\,e^{\frac{s^2}{4}}}{s^2},$$

Since the Laplace transform is a convergent operator then $Y(s)\to 0$ as $s\to\infty$

therefore the constant c must be equal to zero.

Then $Y(s)=\frac{2}{s^3}$ and therefore

$$y(t)=L^{-1}(y(s))=t^2.$$

1-10-3 Integral equation:

Example (1-22):

Solve the integral equation

$$y(t)=t+\frac{1}{6}\int_0^t (t-\tau)^3\,y(\tau)\,d\tau$$

Solution:

The equation can be written $y(t)=t+\frac{1}{6}(t^3 * y(t))$ taking the Laplace transform of both sides we have

$$Y(s)=\frac{1}{s^2}+\frac{1}{6}\,\frac{6Y(s)}{s^4}$$

$$Y(s)-\frac{Y(s)}{s^4}=\frac{1}{s^2}$$

$$Y(s)\left(\frac{s^4-1}{s^4}\right)=\frac{1}{s^2}$$

$$Y(s)=\frac{s^2}{s^4-1}=\frac{s^2}{(s^2-1)(s^2+1)}.$$

Using partial fraction, we have

$$Y(s)=\frac{\frac{1}{2}}{s^2+1}+\frac{\frac{1}{2}}{s^2-1}.$$

Therefore $\quad y(t)=\frac{1}{2}\,[\sin t+\sin ht]$

1-10-4 Application to boundary- value problems:

Example (1-23):

Solve $\dfrac{\partial u}{\partial t}=2\dfrac{\partial^2 u}{\partial x^2}$

$u(0,t)=0 \quad u(5,t)=0$

$u(x,0)=10\sin 4\pi x$

Solution:

Taking the Laplace transform of both sides of $\dfrac{\partial u}{\partial t}=2\dfrac{\partial^2 u}{\partial x^2}$.

$$SU(x,s)-u(x,0)=2\,\frac{d^2U(x,s)}{dx^2}$$

or

$$2\frac{d^2U(x,s)}{dx^2}-SU(x,s)=-10\sin 4\pi x. \qquad (1\text{-}13)$$

The general solution of (1-13) is

$$U(x,s) = c_1\, e^{\sqrt{\frac{s}{2}}\,x} + c_2\, e^{-\sqrt{\frac{s}{2}}\,x} + \frac{10}{s+32\pi^2}\sin 4\pi x\,.$$

Using the condition $u(0,t)=0$, we have $c_1 + c_2 = 0$ or $c_1 = -c_2$.

Also using the condition $u(5,t)=0$, we find $c_2 = 0$ and $c_1 = 0$.

Therefore $U(x,s) = \dfrac{10}{s+32\pi^2}\sin 4\pi x$, and consequently

$$u(x,t) = 10\, e^{-32\pi^2 t}\,\sin 4\pi x\,.$$

Example (1-24):

A semi-infinite solid rod $x \ge 0$ has its initial temperature equal to zero. A constant heat flux A is applied at the face $x = 0$ so that $-k\,u_x(0,t) = A$.

Find the temperature at the face after time t.

Solution:

The boundary value problems is the one dimential heat equation.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad x>0 \quad t>0$$

$$u(x,0)=0, \quad u_x(0,t) = \frac{-A}{k}$$

$$|u(x,t)| < M\,.$$

Taking Laplace transform of the heat equation, we find that:

$$sU(x,s) - u(x,0) = k\,\frac{d^2U(x,s)}{dx^2},$$

or $$\frac{d^2U(x,s)}{dx^2} - \frac{sU(x,s)}{k} = 0 \qquad (1\text{-}14)$$

Now;

$$L\{u_x(0,t)\} = \frac{-A}{sk} \quad \text{or} \quad U_x(0,s) = \frac{-A}{sk}, \quad \text{then we have}$$

$$\frac{d^2U(x,s)}{dx^2} - \frac{sU(x,s)}{k} = 0$$

$$U_x(0,s) = \frac{-A}{sk}, \quad |U(x,s)| < \frac{m}{s} = N. \qquad (1\text{-}15)$$

Solving (1-15) we find

$$U(x,s) = c_1 e^{\sqrt{\frac{s}{k}}\,x} + c_2 e^{-\sqrt{\frac{s}{k}}\,x}.$$

Then we choose $c_1 = 0$ so that $U(x,s)$ is bounded as $x \to \infty$,

we have $U(x,s) = c_2 e^{-\sqrt{\frac{s}{k}}\,x}$

$$U_x(x,s) = -c_2\,\frac{\sqrt{s}}{k}\,e^{-\sqrt{\frac{s}{k}}\,x} = \frac{-A}{sk},$$

then $c_2 = \dfrac{A\sqrt{k}}{ks\sqrt{s}}$, therefore $U(x,s) = \dfrac{A}{\sqrt{k}}\,\dfrac{e^{-\sqrt{\frac{s}{k}}\,x}}{s\sqrt{s}}$, by example (1-16) we have

$$L^{-1}\left[\frac{e^{-\sqrt{\frac{s}{k}}x}}{\sqrt{s}}\right]=\frac{1}{\sqrt{\pi t}}e^{\frac{-x^2}{4kt}}$$

and by theorem (1-17) we have

$$L^{-1}\left[\frac{e^{-\sqrt{\frac{s}{k}}x}}{s\sqrt{s}}\right]=\int_0^t\frac{1}{\sqrt{\pi\tau}}e^{\frac{-x^2}{4k\tau}}d\tau$$

thus

$$u(x,t)=\frac{A}{\sqrt{k}}\int_0^t\frac{1}{\sqrt{\pi\tau}}e^{\frac{-x^2}{4k\tau}}\,d\tau$$

Now, at the face $x = 0$ the temperature is

$$u(0,t)=\frac{A}{\sqrt{k}}\int_0^t\frac{1}{\sqrt{\pi\tau}}\,d\tau .$$

or $\quad u(0,t)=\frac{A}{k}\sqrt{\frac{kt}{\pi}}$.

1-10-5 Application to Mechanics:

Example (1-25):

Suppose a mass m, attached to a flexible spring fixed at 0, is free to move on a friction less plane PQ see fig. (1-9).

If the mass m has a force $f(t)$, $t>0$ acting on it but on damping forces are present then find x at any time if $f(t) = F_0$ (a constant) for $t>0$.

Solution:

Let the mass starts from rest at a distance $x = a$ then the displacement x at any time $t > 0$ can be determined from the equation of motion.

$$m\,X'' + k\,x = f(t)\,,\quad x(0) = a\quad x'(0) = 0.$$

Taking the Laplace transform of both sides of the equation we get:

$$m\left[s^2 X(s) - s\,x(0) - x'(0)\right] + k\,X(s) = \frac{F_0}{s}$$

$$X(s) = \left(\frac{F_0 + s^2 m a}{s(ms^2 + k)}\right)$$

Using partial fraction we have

$$X(s) = \frac{F_0}{ks} + \frac{mas}{ms^2 + k} - \frac{\frac{F_0 m}{k}s}{ms^2 + k},$$

therefore $x(t) = \frac{F_0}{k}\left(1 - \cos\sqrt{\frac{k}{m}}\,t\right) + a\cos\sqrt{\frac{k}{m}}\,t.$

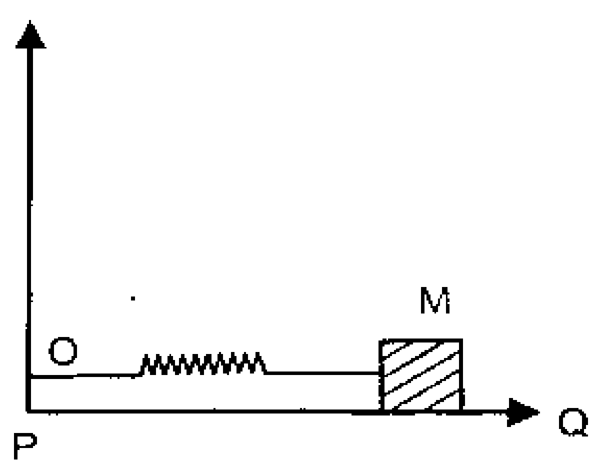

Figure (1-9)

1-10-6 Applications to electrical circuits:

Example (1-26):

Find the solution of

$$\frac{d^2q}{dt^2} + 2\lambda \frac{dq}{dt} + w^2 q = \frac{E_0}{L}\ , \quad q(0) = q'(0) = 0$$

Where

$$\lambda = \frac{R}{2L}\ , \quad w^2 = \frac{1}{Lc} \quad \text{and} \quad E_0 \text{ is constant.}$$

Solution:

Let $L\{q(t)\} = Q(s)$

Taking the Laplace transform of both sides of the differential equation, we have

$$\left[s^2 Q(s) - sq(0) - q'(0)\right] + 2\lambda\left[sQ(s) - q(0)\right] + w^2 Q(s) = \frac{E_0}{Ls}$$

$$Q(s) = \frac{E_0}{Ls}\left(\frac{1}{s^2 + 2\lambda s + w^2}\right)$$

1) If $\lambda > w$ then $Q(s) = \dfrac{E_0}{Ls\,[s^2 + 2\lambda s + \lambda^2 - (\lambda^2 - w^2)]}$,

or $\quad Q(s) = \dfrac{E_0}{Ls}\left[\dfrac{1}{(s+\lambda)^2 - (\lambda^2 - w^2)}\right].$

Let $\quad B = \sqrt{\lambda^2 - w^2}$, therefore

$$q(t) = \frac{E}{LB}\int_0^t e^{-\lambda\tau} \sinh B\tau \, d\tau$$

$$= \frac{E_0}{LBw^2}\left[e^{-\lambda\tau}(-\lambda \sinh B\tau - B\cosh B\tau)\Big|_0^t\right]$$

$$q(t)=\frac{E_0}{LBw}\left[e^{-\lambda t}\,(\lambda \sin hBt - B\cosh Bt)+B\right]$$

$$q(t)=E_0c\left[e^{-\lambda t}\,(1-e^{-\lambda t}(\cosh Bt+B\sin h\,Bt)\right].$$

2) If $\lambda = w$ then we have

$$Q(s)=\frac{E_0}{Ls}\left[\frac{1}{(s+\lambda)^2}\right]$$

$$q(t)=\frac{E_0}{L}\int_0^t \tau\, e^{-\lambda\tau}\,d\tau$$

$$q(t)=E_0c\,[1-e^{-\lambda t}(1+\lambda t)].$$

3) If $\lambda < w$ then $Q(s)=\frac{E_0}{Ls}\left(\frac{1}{(s+\lambda)^2+(w-\lambda^2)}\right).$

Let $A=\sqrt{w^2-\lambda^2}$, therefore $q(t)=\frac{E_0}{LA}\int_0^t e^{-\lambda\tau}\sin A\tau\, d\tau$

$$q(t)=[e^{-\lambda\tau}(-\lambda\sin A\tau - A\cos A\tau)\ |_0^t\ ]$$

$$=\frac{E_0}{LAw^2}[\,e^{-\lambda t}(-\lambda\sin At - A\cos At + A)\,]$$

$$q(t)=E_0c\,[\,1-e^{-\lambda t}(\cos At+\frac{\lambda}{A}\sin At)\,].$$

1-10-7 Systems of linear differential equations:

In this part we show that Laplace transform combined with the Leverrier- Faddeev method of finding solution of the matrix differential equation

$$\frac{dX(t)}{dt}=AX(t),\ X(0)=I,\quad \text{or}\quad X'(t)=AX(t),\ X(0)=I \qquad (1\text{-}16)$$

where $X'(t)$, A, $X(t)$ are nxn matrices and I is the nxn identity matrix.

Also we can solve the homogeneous differential equation,

$$X_1'(t) = AX_1(t), \quad X_1(0) = B$$

by $X_1(t) = B\,X(t)$ where $X(t)$ solution to (1-16) and $X_1(t)$, B are nxn matrices.

Also, we can solve the non homogeneous initial value problem,

$$X_2'(t) = AX_2(t) + V(t), X_2(0) = B.$$

by $$X_2(t) = \left[B + \int_0^t X(-s)\,V(s)\,ds\right] X(t)$$

where $X(t)$ is the solution to (1-16) and $V(t)$ is an $(n \times n)$ - matrix.

Remark 1-5:

We consider only initial value problems with $t_0 = 0$ otherwise we replace t by $t - t_0$.

The Leverrier – Faddeer Method:

Consider the special matrix differential equation

$$X'(t) = AX(t), \quad X(0) = I.$$

Taking the Laplace transform of both sides we have

$$SY(s) - I = AY(s), \text{ where } \quad L\{X(t)\} = Y(s)$$

i.e $$Y(s) = (sI - A)^{-1}.$$

The Leverrier- Faddeer procedure gives an easy way to compute the solution of special matrix differential by the determinant formula for the inverse of a matrix.

Now, let $\det(sI - A) = s^n + c_1 s^{n-1} + \dots + c_n$, be the characteristic polynomial of A.

Our goal is to compute the coefficients $c_1, \dots, c_n$ by comparing the coefficients in $(sI - A)\,C(s) = \det(sI - A)I$.

One sees immediately that the matrices

$Q_1, \dots, Q_{n-1}$ are given by

$$Q_1 = c_1 I + A$$

$$Q_2 = c_2 I + AQ_1$$

$$Q_{n-1} = c_{n-1} I + AQ_{n-2} \qquad (1\text{-}17)$$

If we compute the sequence $\{c_k\}$ first, then we could easily compute the sequence of coefficient matrices $\{Q_k\}$ from (1-17).

But there is a better way. Leverrier showed

$$c_1 = -T_r A$$

$$2c_2 = -T_r AQ_1$$

$$\vdots$$

$nc_n = -T_r AQ_{n-1}$ where $T_r A$ is the trace of matrix A.

Hence $Y(s)$ we can write

$$Y(s) = (sI - A)^{-1} = \frac{C(s)}{\det(sI - A)}$$

where

$$C(s) = s^{n-1} I + s^{n-2} Q_1 + \ldots .. + Q_{n-1}$$

Example (1-27):

Solve $X'(t) = AX(t)$, with $X(0) = I$ for $A = \begin{bmatrix} 4 & 3 \\ -2 & -1 \end{bmatrix}$

Solution:

$$c_1 = -T_r A = -(4-1) = -3$$

$$Q_1 = c_1 I + A = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -2 & -4 \end{bmatrix}$$

$$AQ_1 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix},$$

$$2\,c_2 = -\,Tr\,AQ_1 \quad \text{or} \quad 2\,c_2 = 4 \quad \text{or} \quad c_2 = 2$$

Hence $det\,(sI - A) = s^2 - 3s + 2 = (s-1)(s-2)$

And $C(s) = sI + Q_1$

or $C(s) = \begin{bmatrix} s+1 & 3 \\ -2 & s-4 \end{bmatrix}$

In order to find $X(t) = L^{-1}\, y(s)$ we use partial fractions,

$$\frac{C(s)}{\det(sI - A)} = \frac{P}{(s-1)} + \frac{Q}{s-2}, \quad \text{or} \quad C(s) = (s-2)\,P + (s-1)\,Q$$

$$C(1) = -P,\; C(2) = Q.$$

Thus $X(t) = e^{t} \begin{bmatrix} -2 & -3 \\ 2 & 3 \end{bmatrix} + e^{2t} \begin{bmatrix} 3 & 3 \\ -2 & -2 \end{bmatrix}$

Example (1-28):

Solve $X_1'(t) = AX_1(t)$, with $X(0) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ for $A = \begin{bmatrix} 4 & 3 \\ -2 & -1 \end{bmatrix}$.

Solution:

Comparing the solution of this problem with the solution of example (1-27), the solution of this example will be as following:

$$X_1(t) = B\,X(t) = \begin{bmatrix} -e^{t}+3e^{2t} & -3e^{t}+3e^{2t} \\ 2e^{t}-2e^{2t} & 3e^{t}-2e^{2t} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$X_1(t) = \begin{bmatrix} -4e^{t}+6e^{2t} & -9e^{t}+9e^{2t} \\ 4e^{t}-4e^{2t} & 9e^{t}-6e^{2t} \end{bmatrix}.$$

Example (1-29):

Solve $X_2'(t) = AX_2(t) + V(t)$, with $X(0) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$,

for $A = \begin{bmatrix} 4 & 3 \\ -2 & -1 \end{bmatrix}$, $V(t) = \begin{bmatrix} t & 0 \\ 0 & 0 \end{bmatrix}$

Solution:

Comparing with the solution of example (1-27), the solution of this example will be as following:

$$X_2(t) = X(t)\left[B + \int_0^t X(-s)\,V(s)\,ds \right]$$

Now; $\int_0^t X(-s)\,V(s)\,ds = \begin{bmatrix} (2t+2)e^{-t} - (\frac{3}{2}t + \frac{3}{4})\,e^{-2t} - \frac{5}{4} & 0 \\ -(2t+2)e^{-t} + (t + \frac{1}{2})\,e^{-2t} + \frac{3}{2} & 0 \end{bmatrix}$, therefore $X_2(t) = X(t)\left[B + \int_0^t X(-s)\,V(s)\,ds\right]$

$$= \begin{bmatrix} -6e^t + \frac{27}{4}e^{2t} + \frac{1}{2}t + \frac{5}{4} & -9e^t + 9e^{2t} \\ 6e^t - \frac{9}{2}e^{2t} - t - \frac{3}{2} & 9e^t - 6e^{2t} \end{bmatrix}.$$

Example (1-30):

Solve $X'(t) = AX(t)$, with $X(0) = I$ for

$$A = \begin{bmatrix} 2 & -1 & -1 \\ 2 & -1 & -2 \\ -1 & 1 & 2 \end{bmatrix}$$

Solution:

$$c_1 = -Tr\,A = -3,\ \ Q_1 = c_1 I + A = \begin{bmatrix} -1 & -1 & -1 \\ 2 & -4 & -2 \\ -1 & 1 & -1 \end{bmatrix} \quad c_2 = \frac{-1}{2}\,Tr\,A\,Q_1 = 3$$

$$Q_2 = c_2 I + AQ_1 = \begin{bmatrix} 0 & 1 & 1 \\ -2 & 3 & 2 \\ 1 & -1 & 0 \end{bmatrix}, \quad c_3 = \frac{-1}{3}\,Tr\,A\,Q_2 = -1$$

Thus, $\det(sI - A) = s^3 - 3s^2 + 3s - 1 = (s-1)^3$ and

$$C(s) = \begin{bmatrix} s^2 - s & -s+1 & -s+1 \\ 2s-2 & s^2-4s+3 & -2s+2 \\ -s+1 & s-1 & s^2-s \end{bmatrix}$$

we use partial fractions,

$$\frac{C(s)}{(s-1)^3} = \frac{P}{(s-1)} + \frac{Q}{(s-1)^2} + \frac{R}{(s-1)^3} \qquad \text{we get}$$

$$C(s) = (s-1)^2 P + (s-1)Q + R \ .$$

Now, $C(1) = R = 0$ Next $C'(s) = 2(s-1)P + Q$

$C'(1) = Q$ finally $C''(s) = 2I = 2P$, $P = I$.

Then the solution is $X(t) = e^t \, [I + tC'(1)]$.

Example (1-31):

Solve $X'(t) = AX(t)$ with $X(7) = I$, for $A = \begin{bmatrix} 4 & 3 \\ -2 & -1 \end{bmatrix}$.

Solution:

Comparing with the solution of example (1-27), the solution of this example will be as following:

$$X(t) = \begin{bmatrix} -2e^{t-7} + 3e^{2t-14} & -3e^{t-7} + 3e^{2t-14} \\ 2e^{t-7} - 2e^{2t-14} & 3e^{t-7} - 2e^{2t-14} \end{bmatrix}.$$

1-10-8 Difference Equation:

Example (1-32):

a. Show that the function $f(t) = [t]$ for $t > 0$ has Laplace transform

$$L\{f(t)\} = \frac{e^{-s}}{s(1-e^{-s})} \text{ where } [t] \text{ is the greatest integer} \le t.$$

b. Show that the solution to $y(t+1) - y(t) = 1$ $y(t) = 0$, for $t < 1$ is given by the function in part (a).

Solution:

a. $L\{[t]\} = \int_0^{\infty} [t] e^{-st}\, dt = \int_0^1 o e^{-st}\, dt + \int_1^2 e^{-st}\, dt + \int_2^3 2e^{-st}\, dt + ...$

$$= \frac{e^{-s} - e^{-2s}}{s} + 2\,\frac{e^{-2s} - e^{-3s}}{s} + 3\,\frac{e^{-3s} - e^{-4s}}{s} + ...$$

$$= \frac{e^{-s} - e^{-2s}}{s} + 2\,\frac{e^{-2s} - e^{-3s}}{s} + 3\,\frac{e^{-3s} - e^{-4s}}{s} + ...$$

$$= \frac{e^{-s}}{s}\left[1 + e^{-s} + e^{-2s} + e^{-3s} + e^{-4s} + ...\right]$$

$$= \frac{e^{-s}}{s}\left(\frac{1}{1 - e^{-s}}\right).$$

b. $L\{y(t+1)\} = \int_0^{\infty} e^{-st}\, y(t+1)\, dt$, Let $\tau = t + 1$

$$= \int_1^{\infty} e^{-s(\tau - 1)}\, y(\tau)\, d\tau$$

$$= e^{s} \int_0^{\infty} e^{-s\tau}\, y(\tau)\, d\tau - e^{s} \int_0^{1} e^{-s\tau}\, y(\tau)\, d\tau,$$

then $L\{y(t+1)\} = e^{s}\, L\{y(t)\}$.

Thus the difference equation becomes $e^{s} L\{y(t)\} - L\{y(t)\} = \frac{1}{s}$

$$L\{y(t)\} = \frac{1}{s(e^{s} - 1)} = \frac{e^{-s}}{s(1 - e^{-s})}.$$

From (a) $y(t) = [t]$.

Example (1-33):

Solve for a_n if, $a_{n+2} - 7a_{n+1} + 12a_n = 2^n$, $a_0 = 0$ $a_1 = -1$.

Solution:

To treat this sort of problem, let us define

$$y(t) = a_n \qquad n \le t < n+1 \qquad n = 0, 1, 2, \ldots,$$

then our difference equation becomes $y(t+2) - 7y(t+1) + 12y(t) = 2^{[t]}$

taking the Laplace transform, we first have:

$$L\{y(t+2)\} = \int_0^\infty e^{-st}\, y(t+2)\, dt \qquad \text{Let } \tau = t+2$$

$$= \int_2^\infty e^{-s(\tau-2)}\, y(\tau)\, d\tau = e^{2s} \int_0^\infty e^{-s\tau}\, y(\tau)\, d\tau -$$

$$- e^{2s} \int_0^1 e^{-s\tau}\, a_0\, d\tau - e^{2s} \int_1^2 e^{-s\tau}\, a_1\, d\tau$$

$$= e^{2s}\, L\{y(t)\} + \frac{e^s}{s}(1 - e^{-s}).$$

Similary $L\{y(t+1)\} = \int_0^\infty e^{-st}\, y(t+1)\, dt$, $\tau = t+1$

$$= e^s \int_1^\infty e^{-s\tau}\, y(\tau)\, d\tau = e^s \int_0^\infty e^{-s\tau}\, y(\tau)\, d\tau -$$

$$- e^s \int_0^1 e^{-s\tau}\, a_0\, d\tau = e^s\, L\{y(t)\}.$$

Thus the transform is

$$e^{2s} \; L\{y(t)\} + \frac{e^s}{s}(1-e^{-s}) - 7e^s \; L\{y(t)\} + 12 \; L\{y(t)\}$$

$$= L\{2^{[t]}\}$$

$$(e^{2s} - 7e^s + 12) \; L\{y(t)\} = L\{2^{[t]}\} - \frac{e^s}{s}(1-e^{-s}) \; ,$$

or $$L\{y(t)\} = \frac{L\{2^{[t]}\}}{e^{2s} - 7e^s + 12} - \frac{e^s(1-e^{-s})}{s\,(e^{2s} - 7e^s + 12)} \; .$$

Now;

$$\frac{e^s(1-e^{-s})}{s}\left(\frac{1}{(e^s-4)(e^s-3)}\right) = \frac{e^s(1-e^{-s})}{s}\left(\frac{1}{e^s-4} - \frac{1}{e^s-3}\right)$$

$$= \frac{1-e^{-s}}{s}\left(\frac{1}{1-4e^{-s}} - \frac{1}{1-3e^{-s}}\right) = \frac{1-e^s}{s}\left(\frac{1}{1-4e^{-s}} - \frac{1}{1-3e^{-s}}\right)$$

$$= L\{4^{[t]}\} - L\{3^{[t]}\}, \tag{1-18}$$

and

$$\frac{L\{2^{[t]}\}}{(e^s-4)(e^s-3)} = \frac{1-e^{-s}}{s(1-2e^{-s})}\left(\frac{1}{(e^s-4)(e^s-3)}\right)$$

$$= \frac{e^s-1}{s}\left(\frac{1}{(e^s-2)(e^s-3)(e^s-4)}\right)$$

$$= \left(\frac{1-e^{-s}}{s}\right)\left(\frac{1/2}{e^s-2} - \frac{1}{1-3e^{-s}} + \frac{1/2}{1-4e^{-s}}\right)$$

$$= \frac{1}{2}\left(\frac{1-e^{-s}}{s(1-2e^{-s})}\right) - \frac{1-e^{-s}}{s(1-3e^{-s})} + \frac{1}{2}\,\frac{1-e^{-s}}{s(1-4e^{-s})}$$

$$= \frac{1}{2} L\{2^{[t]}\} - L\{3^{[t]}\} + \frac{1}{2} L\{4^{[t]}\} \; . \tag{1-19}$$

From (1-18) and (1-19) we have

$$L\{y(t)\} = \frac{1}{2} L\{2^{[t]}\} - \frac{1}{2} L\{4^{[t]}\}, \text{ then } y(t) = \frac{1}{2}(2^{[t]} - 4^{[t]}).$$

Thus $a_n = \frac{1}{2}(2^n - 4^n)$.

1-10-9 State Equation:

Definition (1-3):

Consider a differential equation of the form

$$c_r \frac{d^r y(t)}{dt^r} + c_{r-1} \frac{d^{r-1} y(t)}{dt^{r-1}} + \dots + c_0 y(t) = E_0 x(t), \text{ where } E_0 \text{ is constant.}$$

If we define the state variables to be

$$\lambda_1(t) = y(t)$$

$$\lambda_2(t) = \frac{dy(t)}{dt}$$

$$\dots \quad \dots \quad \dots$$

$$\lambda_r(t) = \frac{d^{r-1} y(t)}{dt^{r-1}},$$

then the state equations are

$$\frac{d\lambda_1(t)}{dt} = \lambda_2(t)$$

$$\frac{d\lambda_2(t)}{dt} = \lambda_3(t)$$

$$\dots \quad \dots \quad \dots$$

$$\frac{d\lambda_{r-1}(t)}{dt^{r-1}} = \lambda_r(t)$$

$$\frac{d\lambda_r(t)}{dt} = \frac{-1}{c_r}\left[c_0 \lambda_1(t) + c_1 \lambda_2(t) + \dots + c_{r-1} \lambda_r(t) - E_0 x(t)\right]$$

Example (1-34):

Solve

$$\frac{d\lambda_1(t)}{dt} = 2\lambda_1(t) - \lambda_2(t) + x(t)$$

$$\frac{d\lambda_2(t)}{dt} = -4\lambda_1(t) + 5\lambda_2(t) \ ,$$

where $x(t) = \sin 2t$ and $\lambda_1(0) = 0 \quad \lambda_2(0) = 2$.

Solution:

Let $L\{\lambda_1(t)\} = \mu_1(s),\ L\{\lambda_2(t)\} = \mu_2(s)$,

From the state equations, we obtain

$$s\mu_1(s) = 2\mu_1(s) - \mu_2(s) + \frac{2}{s^2+4}$$

$$s\mu_2(s) - 2 = -4\mu_1(s) + 5\mu_2(s) \ ,$$

or

$$(s-2)\,\mu_1(s) + \mu_2(s) = \frac{2}{s^2+4}$$

$$4\mu_1(s) + (s-5)\,\mu_2(s) = 2 \ .$$

Solving these equations, we get:

$$\mu_1(s) = \frac{-\frac{33}{100}s - \frac{19}{50}}{s^2+4} + \frac{\frac{18}{25}}{s-1} - \frac{\frac{39}{100}}{s-6}$$

$$\mu_2(s) = \frac{\frac{18}{25}}{s-1} + \frac{\frac{39}{25}}{s-6} + \frac{\frac{7}{25}s - \frac{2}{25}}{s^2-4} .$$

Thus $\lambda_1(t) = \frac{18}{25}e^{t} - \frac{59}{100}e^{t} - \frac{33}{100}\cos 2t - \frac{19}{100}\sin 2t$

$$\lambda_2(t) = \frac{18}{25}e^{t} + \frac{39}{25}e^{6t} - \frac{7}{25}\cos 2t - \frac{1}{25}\sin 2t \ ; \quad t \ge 0$$

Chapter Two

Z-Transform

2-1 Introduction:

Z-transform, like the Laplace transform, is an indispensable mathematical tool for the designing, analysing and monitoring of systems. The Z-transform is used to transform discrete sequence domain into complex variable domain.

The Z-transform plays a similar role to that of the Laplace transform in the continuous function domain.

Before we start to define the Z-transform, we should firstly study Dirac delta function because, the Dirac delta function is very important in explaining the relation between the Laplace transform and the Z-transform.

2-2 Dirac delta function [9, 10, 16]:

Definition (2-1):

The Dirac delta function its defined by $\int_{t_1}^{t_2} f(t)\,\delta(t)\,dt = f(0)$;

whenever $t_1 < 0 < t_2$

and provided that $f(t)$ is continuous at $t = 0$ and it has the following properties.

1) If $t = 0$ then $\delta(t) \to \infty$.

2) If $t \neq 0$ then $\delta(t) \to 0$.

3) $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

4) $\delta(t)$ is an even function, that is $\delta(-t) = \delta(t)\ \forall t \in R$.

Remark (2-1):

The Dirac delta function is not a usual function but it is a generalized function and it can be obtained as the limit of usual functions, that is,

$$\delta(t) = \lim_{n\to\infty} \delta_n(t) \quad \text{or} \quad \delta(t) = \lim_{\varepsilon\to 0^+} \delta_\varepsilon(t)$$

Example (2-1):

Show that the function $\delta_n = \begin{cases} n(tn+1); & \frac{-1}{n} < t < 0 \\ n(1-tn); & 0 < t < \frac{1}{n} \\ 0; & other\ wise \end{cases}$

represents delta function as $n \to \infty$.

Solution:

1) $\lim_{n\to\infty} \delta_n(0) = \lim_{n\to\infty} \frac{n+n}{2} = \infty$, then $\delta(0) = \lim_{n\to\infty} \delta_n(0) = \infty$

2) For $t \neq 0$ $\delta(t) = \lim_{n\to\infty} \delta_n(t) = \begin{cases} \infty, & 0 < t < 0 \\ \infty, & 0 < t < 0 \\ 0, & other\ wise \end{cases} = \begin{cases} \infty, & t = 0 \\ 0, & otherwise \end{cases}$

Since $\delta(t) = \lim_{n\to\infty} \delta_n(t) = 0,\ \forall t \neq 0$, then $\delta(t) = 0$ for all $t \neq 0$

3) $$\int_{-\infty}^{\infty} \lim_{n\to\infty} \delta_n(t)\, dt = \lim_{n\to\infty} \int_{\frac{-1}{n}}^{0} n(tn+1)\, dt + \int_{0}^{\frac{1}{n}} n(1-tn)\, dt$$

$$= n(\frac{t^2 n}{2} + t)\Big|_{-\frac{1}{n}}^{0} + n(t - \frac{t^2}{2}n\Big|_{0}^{\frac{1}{n}}$$

$$= 0 - n(\frac{\frac{1}{n}}{2} - \frac{1}{n}) + n(\frac{1}{n} - \frac{\frac{1}{n}}{2}) - 0$$

$$= -\frac{1}{2} + 1 + 1 - \frac{1}{2} = 1$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = \int_{-\infty}^{\infty} \lim_{n\to\infty} \delta_n(t)\, dt = 1$$

4) Clearly $\delta_n(-t) = \delta_n(t)$, then $\delta(t) = \delta(-t)$, $\forall t \in R$

Therefore $\delta_n(t)$ represents the delta function $\delta(t)$ since $\delta(t) = \lim_{n\to\infty} \delta_n(t)$

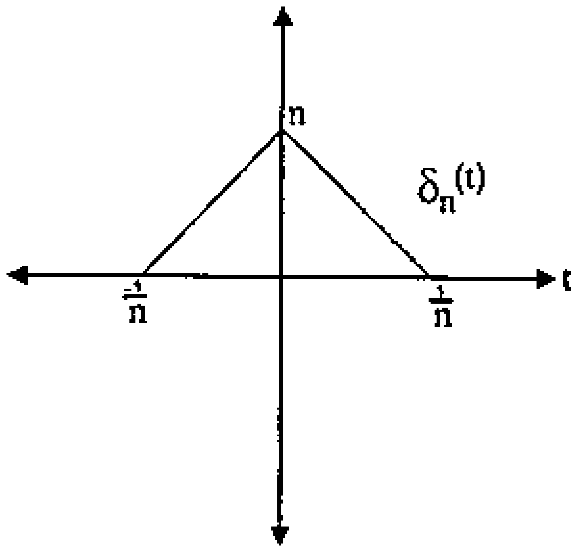

Figure (2-1)

2-3 Properties of the dirac delta function [9, 10, 16]:

1- Shiting Property:

$$\int_{t_1}^{t_2} f(t)\,\delta(t-t_0)\, dt = \begin{cases} f(t_0), & t_1 < t_0 < t_2 \\ 0, & t_0 < t_1 \text{ or } t_0 > t_2 \\ \frac{1}{2} f(t_0), & t_0 = t_1 \text{ or } t_0 = t_2 \end{cases}$$

Proof:

i) If $t_1 < t_0 < t_2$,

Then let $\tau = t - t_0$,

$$\int_{t=t_1}^{t_2} f(t)\,\delta(t-t_0)\,dt = \int_{\tau=t_1-t_0}^{t_2-t_0} f(\tau+t_0)\,\delta(\tau)\,d\tau$$

whenever $t_1 - t_0 < 0 < t_2 - t_0$

$$= f(0+t_0) = f(t_0)$$

ii) $t_0 < t_1$ or $t_0 > t_2$

a) Let $t_0 < t_1$ then we have $t_0 < t_1 < t_2$ and

$$\int_{-\infty}^{t_2} f(t)\,\delta(t-t_0)\,dt = f(t_0)\,, \qquad \text{by (i)}$$

but $\int_{-\infty}^{t_2} f(t)\,\delta(t-t_0)\,dt =$

$$\int_{-\infty}^{t_1} f(t)\,\delta(t-t_0)\,dt + \int_{t_1}^{t_2} f(t)\,\delta(t-t_0)\,dt = f(t_0),$$

or $$f(t_0) + \int_{t_1}^{t_2} f(t)\,\delta(t-t_0)\,dt = f(t_0).$$

Thus

$$\int_{t_1}^{t_2} f(t)\,\delta(t-t_0)\,dt = 0$$

b) Let $t_0 > t_2$ then we have $t_1 < t_2 < t_0$ and

$$\int_{t_1}^{\infty} f(t)\,\delta(t-t_0)\,dt = f(t_0) \qquad \text{by (i)},$$

but $\int_{t_1}^{\infty} f(t)\,\delta(t-t_0)\,dt =$

$$\int_{t_1}^{t_2} f(t)\,\delta(t-t_0)\,dt + \int_{t_2}^{\infty} f(t)\,\delta(t-t_0)\,dt = f(t_0)\,,$$

or $\int_{t_1}^{t_2} f(t)\,\delta(t-t_0)\,dt + f(t_0) = f(t_0)$, since $t_0 \in (t_2, \infty)$

thus

$$\int_{t_1}^{t_2} f(t)\,\delta(t-t_0)\,dt = 0$$

iii) a) Let $t_0 = t_1$ then $\delta(t_1 - t_0) = \delta(0)$,

therefore $\delta(t) \to \infty$,

that is t_0 is a point of discontinuity of $\delta(t)$.

Thus the value of the integral is given by $\dfrac{f(t_0^+) + f(t_0^-)}{2}$,

therefore $\displaystyle\int_{t_1}^{t_2} f(t)\,\delta(t-t_0)\,dt = \frac{0 + f(t_0)}{2} = \frac{1}{2} f(t_0)$.

b) Similarly if $t_0 = t_2$ then $\displaystyle\int_{t_1}^{t_2} f(t)\,\delta(t-t_0)\,dt = \frac{1}{2} f(t_0)$ □.

2- Sampling property:

$$f(t)\,\delta(t-t_0) \equiv f(t_0)\,\delta(t-t_0)$$

Proof:

Since

$$\int_{t_1}^{t_2} f(t)\,\delta(t-t_0)\,dt = f(t_0) \quad t_1 < t_0 < t_2 \quad \text{(by shifting property)} \quad (2\text{-}1)$$

$$\int_{t_1}^{t_2} f(t_0)\,\delta(t-t_0)\,dt = f(t_0)\int_{t_1}^{t_2} \delta(t-t_0)\,dt = f(t_0)\,.\,1 = f(t_0)\ , \qquad (2\text{-}2)$$

from (2-1) and (2-2) $f(t)\,\delta(t-t_0) \equiv f(t_0)\,\delta(t-t_0)$ □.

Note that: The two functions $f_1(\delta(t))$ and $f_2(\delta(t))$ are said to be equivalent, if they are integrated over the same interval (t_1, t_2) with respect to a continuous function $g(t)$ then

$$\int_{t_1}^{t_2} g(t)\, f_1(\delta(t))\, dt = \int_{t_1}^{t_2} g(t)\, f_2(\delta(t))\, dt .$$

3- Scaling Property:

$$\delta(at+b) = \frac{1}{|a|}\,\delta\,(t+\frac{b}{a})$$

Proof:

1) When $a>0$ let $\tau = at+b$ then we have

$$\int_{t_1}^{t_2} f(t)\,\delta(at+b)\,dt = \frac{1}{a}\int_{at_1+b}^{at_2+b} f(\frac{\tau-b}{a})\,\delta(\tau)\,d\tau = \frac{1}{a}\,f(\frac{-b}{a}).$$

$$at_1+b<0<at_2+b \tag{2-3}$$

Now; $\frac{1}{a}\int_{t_1}^{t_2} f(t)\,\delta(t+\frac{b}{a})\,dt = \frac{1}{a}\,f(\frac{-b}{a})$. $\quad t_1 < \frac{-b}{a} < t_2$ (2-4)

From (2-3) and (2-4) we find

$$\delta\,(at+b) \equiv \frac{1}{a}\,\delta\,(t+\frac{b}{a})$$

2) When $a<0$ let $\tau = -|a|\,t+b$ then we have

$$\int_{t_1}^{t_2} f(t)\,\delta(at+b)\,dt = \frac{-1}{|a|}\int_{-|a|t_1+b}^{-|a|t_2+b} f(\frac{\tau-b}{-|a|})\,\delta(\tau)\,d\tau = \frac{-1}{|a|}\,f(\frac{b}{a}). \tag{2-5}$$

Now; $\frac{-1}{|a|}\int_{t_1}^{t_2} f(t)\,\delta(t-\frac{b}{a})\,dt = \frac{-1}{|a|} f(\frac{b}{a})$. (2-6)

From (2-5) and (2-6) we find

$$\delta(at+b) = \frac{1}{|a|}\,\delta(t+\frac{b}{a}) \;\square .$$

4- $\int_{-\infty}^{t} \delta(\tau)\,d\tau = \begin{cases} 1 & , t>0 \\ 0 & , t<0 \end{cases}$

$= u(t)$

2-4 Derivatives of delta function:

$$\int_{t_1}^{t_2} f(t)\,\delta'(t-t_0)\,dt = -f'(t_0), \qquad t_1 < t_0 < t_2 .$$

Proof:

Integration by parts gives:

$$\int_{t_1}^{t_2} f(t)\,\delta'(t-t_0)\,dt = f(t)\,\delta(t-t_0)\,\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} f'(t)\,\delta(t-t_0)\,dt$$

$$= 0 - f'(t_0) = -f'(t_0) \;\square .$$

2-5 Properties of the derivatives of delta function:

1) $f(t)\,\delta'(t-t_0) = f(t_0)\,\delta'(t-t_0) - f'(t_0)\,\delta(t-t_0)$.

Proof:

$$\int_{t_1}^{t_2} f(t)\,\delta'(t-t_0)\,dt = -f'(t_0). \tag{2-7}$$

Now;

$$\int_{t_1}^{t_2} f(t_0)\,\delta'(t-t_0)\,dt = f(t_0)\int_{t_1}^{t_2}\delta'(t-t_0)\,dt$$

$$= f(t_0)\,0 = 0\,, \qquad t_1 < t_0 < t_2 \tag{2-8}$$

and

$$\int_{t_1}^{t_2} f'(t_0)\,\delta(t-t_0)\,dt = f'(t_0). \tag{2-9}$$

From (2-7), (2-8) and (2-9) we have

$$f(t)\,\delta'(t-t_0) = f(t_0)\,\delta'(t-t_0) - f'(t_0)\,\delta(t-t_0) \;\square.$$

2) $\delta'(at+b) = \frac{1}{|a|}\,\delta'(t+\frac{b}{a})$.(proof is similar to that of scaling property).

3) $\frac{d}{dt}\left[f(t)\,u(t)\right] = f(t)\,\delta(t) + f'(t)\,u(t)$.

4) $\frac{d}{dt}\left[f(t)\,\delta(t)\right] = \frac{d}{dt}\left[f(0)\,\delta(t)\right]$.

5) $\int_{t_1}^{t_2} f(u)\,\delta^{(n)}(t-t_0)\,dt = (-1)^n f^{(n)}(t_0)$, $t_1 < t_0 < t_2$, provided $f^{(n)}(t_0)$ exist.

2-6 Derivation and definition [9, 10, 14]:

The Z-transform is the discrete sequence counterpart of the Laplace transform.

In this section we derive the Z-transform from the bilateral Laplace transform of a discrete sequence. In order to do so; we consider the discrete sequence $f(n)$; from which we define its continuous counterpart $f(t)$ such that

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\ \delta(t-n)$$

Now; take the bilateral Laplace transform of $f(t)$, then we have

$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} f(n)\ \delta(t-n) \right] e^{-st}\ dt$$

$$= \sum_{-\infty}^{\infty} f(n) \int_{-\infty}^{\infty} e^{-st}\ \delta(t-n)\, dt = \sum_{n=-\infty}^{\infty} f(n)\, e^{-sn} \qquad (2\text{-}10)$$

therefore

$$\sum_{-\infty}^{\infty} f(n)\, e^{-sn} = \sum_{n=-\infty}^{\infty} f(n)\,(e^{s})^{-n} = Z\left[e^{s}\right],$$

that is $F(s) = Z\left[e^{s}\right]$, where $Z\left[e^{s}\right]$ denotes the Z-transform of $f(n)$.

Now; replacing e^{s} by z we get

$$Z[z] = \sum_{n=-\infty}^{\infty} f(n)\, z^{-n}, \qquad (2\text{-}11)$$

where z is a complex variable.

Also; consider the following line integral which is similar to the inverse Laplace transform

$$\frac{-i}{2\pi}\int_{\sigma-i\pi}^{\sigma+i\pi} F(s)\,e^{sn}\,ds = \frac{-i}{2\pi}\int_{\sigma-i\pi}^{\sigma+i\pi}\left[\sum_{k=-\infty}^{\infty} f(k)\,e^{-sk}\right]e^{sn}\,ds$$

$$= \frac{-i}{2\pi}\sum_{k=-\infty}^{\infty}\int_{\sigma-i\pi}^{\sigma+i\pi} f(k)\,e^{s(n-k)}\,ds\ ,$$

where $\sigma = \mathrm{Re}(s)$,

since $\int_{\sigma-i\pi}^{\sigma+i\pi} e^{s(n-k)}\,ds = \begin{cases} 0 & , \ n \neq k \\ 2\pi i\,, & n = k \end{cases}$,

then $\frac{-i}{2\pi}\int_{\sigma-i\pi}^{\sigma+i\pi} F(s)\,e^{sn}\,ds = f(n),$ (2-12)

let $z = e^s$ then $dz = e^s\,ds$ or $z^{-1}\,dz = ds$,

thus (2-12) becomes, $f(n) = Z^{-1}[z] = \frac{-i}{2\pi}\oint_c Z[z]\,z^{n-1}dz$. (2-13)

As defined in (2-11), $Z[z]$ is known as the bilateral Z-transform of the discrete sequence $f(n)$ and denoted by $Z[f(n)]$ or $Z[z]$ and (2-13) is the corresponding inverse transformation formula, and denoted by $Z^{-1}[F(z)]$ or $Z^{-1}[z]$. On the other-hand the unilateral Z-transform is defined as

$$Z[z] = \sum_{n=0}^{\infty} f(n)\,z^{-n} \qquad (2\text{-}14)$$

2-7 The region of convergence (R. O. C) [9, 10, 14]:

Since the Z-transform is an infinite power series in z^{-1}, it exists only for those values of the variable z for which the series converges to a finite sum.

The region of convergence of $Z[z]$ is the set of all the values of z for which $Z[z]$ attains a finite computable value.

Now; we want to determine the values of z for which $Z[z]$ exists.
In order to do so we represent z in polar coordinates, that is, $z = re^{i\theta}$, $r \geq 0$.

Then $Z[z] = \sum_{-\infty}^{\infty} f(n)\,(re^{i\theta})^{-n} = \sum_{n=-\infty}^{\infty} f(n)\, r^{-n}\, e^{-in\theta}$ (2-15-a)

(2-15-a) can be written as

$$Z[z] = \sum_{n=-\infty}^{\infty} \left(f(n)\, u(-n-1) + f(n)\, u(n) \right) r^{-n}\, e^{-in\theta}$$

$$= \sum_{n=-\infty}^{-1} f(n)\, r^{-n}\, e^{-in\theta} + \sum_{n=0}^{\infty} f(n)\, r^{-n}\, e^{-in\theta} \qquad \text{(2-15-b)}$$

Let $f_+(n) = \begin{cases} f(n) & , n \geq 0 \\ 0 & , n < 0 \end{cases} = f(n)\, u(n)$

and $f_-(n) = \begin{cases} 0 & , n \geq 0 \\ f(n) & , n < 0 \end{cases} = f(n)\, u(-n-1)$

then if we let $n = -m$ in the first sum then (2-15-b) becomes:

$$Z[z] = \sum_{m=1}^{\infty} f_-(-m)\, r^{m}\, e^{im\theta} + \sum_{n=0}^{\infty} f_+(n)\, r^{-n}\, e^{-in\theta}$$

$$|Z[z]| \le \sum_{n=1}^{\infty} |f_-(-n)|\, r^{n} + \sum_{n=0}^{\infty} |f_+(n)|\, r^{-n} . \qquad (2\text{-}16)$$

In order for $Z[z]$ to exist, each of two sums in (2-16) must be finite. Suppose there exists constants M, N, R_-, R_+ such that

$$|f_-(-n)| < M R_-^{n} \quad \text{for} \quad n < 0,$$

and, $|f_+(n)| < N R_+^{n}$ for $n \ge 0$.

then (2-16) becomes, $|Z[z]| \le M \sum_{n=1}^{\infty} R_-^{-n} r^{n} + N \sum_{n=0}^{\infty} R_+^{n} r^{-n}$

for the convergence of the first sum we must have, $\left|\frac{r}{R_-}\right| < 1$, or $r < R_-$.

and for the convergence of the second sum we mast have, $\left|\frac{R_+}{r}\right| < 1$, or

$r > R_+$,

hence the region of convgence of $Z[z]$ must be, $R_+ < r < R_-$, or $R_+ < |z| < R_-$

Remarks (2-2):

a) If $f(n) = 0$ for $n < 0$ then $Z[z]$ is a series of negative powers of z only, and the region of convergence of $f(n)$ is the area outside a circle centered at the origin of the complex plane with a radius R_+ and the function $Z[z]$ has poles inside the contour C (see figure 2-1).

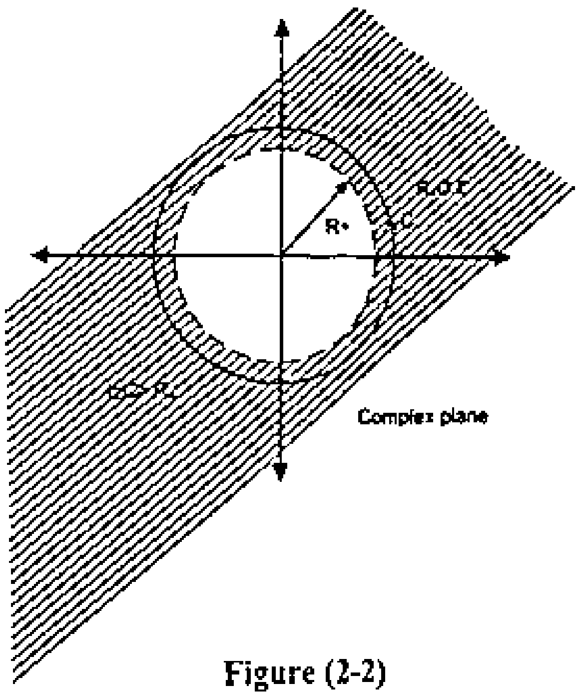

Figure (2-2)

b) If $f(n)=0$ for $n \geq 0$ then $Z[z]$ is a series of positive powers of z only, and the region of convergence of $f(n)$ is the area inside a circle centered at the origin of the complex plane with a radius R_+ and the function $Z[z]$ has poles outside the contour C (see figure 2-2).

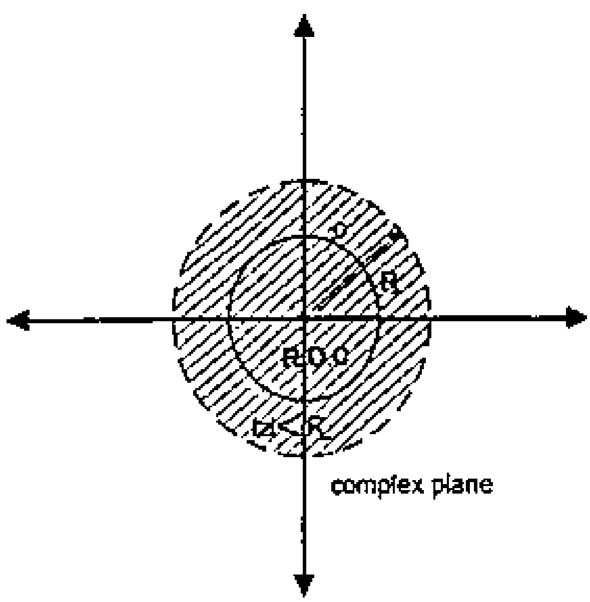

Figure (2-3)

c) If $R_- < R_+$ then $Z[z]$ does not exist.

Example (2-2):

Determine the Z-transform and its region of convergence for the following sequences.

a) $f(n)=\begin{cases}(\frac{1}{2})^n & n\geq 0\\ 0 & n<0\end{cases}$, b) $g(n)=\begin{cases}-(\frac{1}{2})^n & n<0\\ 0 & n\geq 0\end{cases}$

solution:

a) Let $Z_1[z]=Z[f(n)] = Z\left[f(n)\right]$

$$Z_1[z]=\sum_{n=0}^{\infty}(\frac{1}{2z})^n=\frac{1}{1-\frac{1}{2z}}, \qquad \left|\frac{1}{2z}\right|<1$$

then $Z_1[z]=\dfrac{z}{z-\frac{1}{2}}$, $|z|>\frac{1}{2}$

b) Let $Z_2[z]=Z[g(n)] = Z\left[g(n)\right]$

$$Z_2[z]=-\sum_{n=-\infty}^{-1}(\frac{1}{2})^n z^{-n}=-\sum_{n=-\infty}^{-1}(\frac{1}{2z})^n$$

Let $n=-m$,

thus $Z_2[z]=-\sum_{m=1}^{\infty}(\frac{1}{2z})^{-m}=-\sum_{m=1}^{\infty}(2z)^m=\dfrac{-2z}{1-2z}$, $\quad |2z|<1.$

So, $Z_2[z]=\dfrac{z}{z-\frac{1}{2}}$, $|z|<\frac{1}{2}$

Remark (2-3):

It can be noted from example (2-1), the importance of region of convergence in Z-transform. As it is possible for two sequences to be different but both have the same transform.

Therefore, the region of conversion plays an important role in determining the sequence for a given Z-transform.

Example (2-3):

Find the Z-transform of $f(n) = \cos\omega n,\ n \geq 0$

Solution:

$$Z[f(n)] = Z[\cos \omega n] = Z\left[\frac{e^{i\omega n} + e^{-i\omega n}}{2}\right]$$

$$= \frac{1}{2} Z[e^{i\omega n}] + \frac{1}{2} Z[e^{-i\omega n}] = \frac{1}{2}\frac{z}{z - e^{i\omega}} + \frac{1}{2}\frac{z}{z - e^{-i\omega}}$$

$$= \frac{z(z - \cos\omega)}{z^2 - 2z\cos\omega + 1}, \quad |z| > |e^{iw}| = 1$$

2-8 Properties of Z-transform [3, 9, 10, 14]:

The properties of the Z-transform are similar to those of Laplace transform. We state below some of these the properties.

1- Linearity:

Theorem (2-1):

If $\quad Z_1[z] = Z[f_1(n)]$ and $Z_2[z] = Z[f_2(n)]$,

then $Z[a f_1(n) + b f_2(n)] = a Z_1[z] + b Z_2[z]$, for all constants a and b.

Proof:

Easy to prove.

2- Shifting theorem on n:

Theorem (2-2):

Let $Z_1[z]$ be the Z-unilatieral transform of $f(n)$ and $n \geq 0$,

then $Z[f(n+n_0)] = z^{n_0}\left[Z[f(n)] - \sum_{n=n_0}^{n_0-1} f(n)\, z^{-n}\right]$

and $Z[f(n-n_0)] = z^{-n_0}\left[Z[f(n)] + \sum_{n=-n_0}^{-1} f(n)\, z^{-n}\right]$ $n_0 \in N$.

Proof:

$$Z[f(n+n_0)] = \sum_{n=0}^{\infty} f(n+n_0)\, z^{-n}$$

Let $n+n_0 = m$ then $n = m - n_0$ and,

$$\sum_{n=0}^{\infty} f(n+n_0)\, z^{-n} = \sum_{m=n_0}^{\infty} f(m)\, z^{-m+n_0} = z^{n_0} \sum_{m=n_0}^{\infty} f(m)\, z^{-m}$$

$$= z^{n_0}\left[\sum_{m=0}^{\infty} f(m)\, z^{-m} - \sum_{m=0}^{n_0-1} f(m)\, z^{-m}\right] = z^{n_0}\left[Z[f(n)] - \sum_{n=0}^{n_0-1} f(n)\, z^{-n}\right]$$

Similarly, $Z[f(n-n_0)] = z^{-n_0}\left[Z[f(n)] + \sum_{n=-n_0}^{-1} f(n)\, z^{-n}\right]$ □.

Corollary (2-1):

If $Z[z]$ is a Z-bilateral transform of $f(n)$, then $Z[f(n-n_0)] = z^{-n_0}\, Z[f(n)]$ and $Z[f(n+n_0)] = z^{n_0}\, Z[f(n)]$

Proof:

$$Z[f(n-n_0)] = \sum_{n=-\infty}^{\infty} f(n-n_0)\, z^{-n} = \sum_{m=-\infty}^{\infty} f(m)\, z^{-(n_0+m)} = z^{-n_0}\, Z[f(n)].$$

Similarly, $Z[f(n+n_0)] = z^{n_0}\, Z[f(n)]$ □.

3- Complex Translation:

Theorem (2-3):

If $Z[z]$ is the Z-unilateral transform of $f(n)$ for $n \geq 0$,

then $Z[e^{-an} f(n)] = Z[e^{a} z]$, and $Z[e^{an} f(n)] = Z[e^{-a} z]$ $\quad a \in R$.

Proof:

$$Z[e^{-an} f(n)] = \sum_{n=0}^{\infty} f(n)\, e^{-an}\, z^{-n} = \sum_{n=0}^{\infty} f(n)\, (e^{a} z)^{-n} = Z[e^{a} z]$$

Similarly, $Z[a^{an} f(n)] = \sum_{n=0}^{\infty} f(n)\, e^{an} z^{-n} = \sum_{n=0}^{\infty} f(n)\, (e^{-a} z)^{-n} = Z[e^{-a} z]$ □.

4-Differentiation:

Theorem (2-4):

If $\quad Z[z] = Z[f(n)]$, $n \geq 0$,

then $\quad Z[n^{k} f(n)] = (-z\frac{d}{dz})^{k}\, Z[f(n)]$, $k = 1, 2, \ldots$

Proof:

We shall prove the theorem by induction.

For $k = 1$ we have

$$\frac{d}{dz} Z[f(n)] = \frac{d}{dz} \sum_{n=0}^{\infty} f(n)\, z^{-n} = \sum_{n=0}^{\infty} -n\, f(n)\, z^{-n-1}$$

$$= -z^{-1} \sum_{n=0}^{\infty} [n f(n)]\, z^{-n} = -z^{-1}\, Z[n f(n)].$$

Therefore $Z[n\, f(n)] = -z \frac{d}{dz} Z[f(n)].$

We assume it is true for the integer $k = m$,

that is $Z[n^m f(n)] = (-z\frac{d}{dz})^m\, Z[f(n)]$ is true, we need to prove that statement must be true for $k = m + 1$

$$Z[n^{m+1} f(n)] = Z[n\, n^m f(n)] = (-z\frac{d}{dz})\, Z[n^m f(n)]$$

$$= (-z\frac{d}{dz})\,(-z\frac{d}{dz})^m\, Z[f(n)] = (-z\frac{d}{dz})^{m+1}\, Z[f(n)]\ \square .$$

Example (2-4):

If $f(n) = n^2 u(n)$ then find $Z[f(n)]$.

Solution:

$$Z[f(n)] = Z[n^2 u(n)] = (-z\frac{d}{dz})^2\, Z[u(n)] = (-z\frac{d}{dz})^2 \left(\frac{z}{z-1}\right)$$

$$= -z \frac{d}{dz}\left(\frac{z}{(z-1)^2}\right) = \frac{z(z+1)}{(z-1)^3}$$

5- Real convolution:

Definition (2-2):

The discrete convolution of $f(n)$ with $g(n)$ is defined by

$$f(n) * g(n) = \sum_{k=-\infty}^{\infty} f(k)\, g(n-k)$$

Theorem (2-5):

If $\quad y(n) = f(n) * g(n)$, then $Z[y(n)] = Z[f(n)].Z[g(n)]$.

Proof:

$$Z[y(n)] = \sum_{n=-\infty}^{\infty} (f(n) * g(n))\, z^{-n} = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} f(k)\, g(n-k) \right] z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} f(k) \left[\sum_{n=-\infty}^{\infty} g(n-k)\, z^{-n} \right]$$

Let $\quad n - k = m$, then $n = k + m$ and

$$Z[y(n)] = \sum_{k=-\infty}^{\infty} f(k) \left[\sum_{m=-\infty}^{\infty} g(m)\, z^{-m-k} \right] = \left[\sum_{k=-\infty}^{\infty} f(n)\, z^{-k} \right] \cdot \left[\sum_{m=-\infty}^{\infty} g(m)\, z^{-m} \right]$$

$$= Z[f(n)].Z[g(n)]$$

6- Frequency convolution:

Theorem (2-6):

If $\quad Z[f(n)] = Z_1[\alpha]$ and $Z[g(n)] = Z_2[z]$,

then $\quad Z[f(n).g(n)] = \dfrac{-i}{2\pi} \oint_c \dfrac{Z_1[\alpha]\, Z_2[z\alpha^{-1}]}{\alpha}\, d\alpha$

where α is a complex number and C is a closed contour in the region of convergence.

Proof:

$$Z[f(n)\, g(n)] = \sum_{n=0}^{\infty} f(n)\, g(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \left[\frac{-i}{2\pi} \oint_c Z_1[\alpha]\, \alpha^{n-1}\, d\alpha \right] g(n)\, z^{-n}$$

$$= \frac{-i}{2\pi} \oint_c Z_1[\alpha] \sum_{n=0}^{\infty} g(n)\, \alpha^{n-1}\, z^{-n}\, d\alpha$$

$$= \frac{-i}{2\pi} \oint_c Z_1[\alpha] \frac{1}{\alpha} \sum_{n=0}^{\infty} \alpha^{n} g(n)\, z^{-n}\, d\alpha = \frac{-i}{2\pi} \oint_c \frac{Z_1[\alpha]\, Z_2[z\alpha^{-1}]}{\alpha}\, d\alpha$$

7- Conjugate Sequence:

Theorem (2-7):

If $\quad Z[f(n)] = Z[z]$ then $Z[\bar{f}(-n)] = \bar{Z}\left[\frac{1}{\bar{z}}\right]$

Proof:

$$Z[\bar{f}(-n)] = \sum_{n=-\infty}^{\infty} \bar{f}(-n)\, z^{-n} = \sum_{m=-\infty}^{\infty} \bar{f}(m)\, (z^{-1})^{-m} = \overline{\sum_{m=-\infty}^{\infty} f(m)\, (\frac{1}{\bar{z}})^{-m}} = \bar{Z}\left[\frac{1}{\bar{z}}\right]$$

by letting $m = -n$.

Example (2-5):

If $\quad f(n) = \begin{cases} e^{-in}, & n \geq 0 \\ 0, & n < 0 \end{cases}$, then find $Z[\bar{f}(-n)]$.

Solution:

Since $Z[f(n)] = \sum_{n=0}^{\infty} (e^{-i})^{n} z^{-n} = \frac{z}{z - e^{-i}}$, $|z| > 1$

Then; by theorem (2-7) we have

$$Z[\bar{f}(-n)] = \bar{Z}\left[\frac{1}{\bar{z}}\right] = \overline{\frac{1/\bar{z}}{1/\bar{z} - e^{-i}}} = \frac{1/z}{1/z - e^{i}} = \frac{\bar{e}^{i}}{\bar{e}^{i} - z} \quad , \ |z| > 1$$

where $\bar{f}(-n) = \begin{cases} e^{-in}, & n \le 0 \\ 0, & n > 0 \end{cases}$

Corollary 2-2:

a) If the sequence $f(n)$ is real, then $\overline{Z}[\frac{1}{\bar{z}}] = Z[\frac{1}{z}]$.

b) If $Z[z]$ convergences in the ring $r_1 < |z| < r_2$

then $\bar{Z}\left[\frac{1}{\bar{z}}\right]$ converges in the ring $\frac{1}{r_2} < |z| < \frac{1}{r_2}$

8- Initial value theorem:

Theorem (2-8):

If $Z[z] = Z[f(n)]$, and the $\lim_{z\to\infty} Z[z]$ exists,

then $\lim_{z\to\infty} Z[z] = f(0)$.

Proof:

By definition of the Z-transform

$$Z[z] = Z[f(n)] = \sum_{n=0}^{\infty} f(n)\, z^{-n} = f(0) + f(1)z^{-1} + f(2)z^{-2} + \dots \qquad (2\text{-}17)$$

If we let $z \to \infty$ in (2-17), then we obtain

$$\lim_{Z\to\infty} Z[z] = f(0) + f(1)\,.\,0 + f(2)\,.\,0 + \dots = f(0) \qquad \square\,.$$

9- Transform of the difference equations:

Theorem (2-9):

If $\Delta\left[f(n-1)\right]=f(n)-f(n-1)$,

then $Z\left[\Delta f(n-1)\right]=(1-z^{-1})\,Z[z]$, where $Z[z]=Z[f(n)]$.

Proof:

By theorem (2-2) we have

$$Z[f(n)-f(n-1)]=Z[z]-z^{-1}Z[z]=(1-z^{-1})\,Z[z],$$

therefore $Z[\Delta f(n-1)]=(1-z^{-1})\,Z[z]$ $\square$.

Example (2-6):

Let $f(n)=\begin{cases}(\frac{1}{2})^n, & n\geq 0\\ 0, & n<0\end{cases}$, then find the Z-transform of $\Delta f(n-1)$

Solution:

$\Delta f(n-1)=f(n)-f(n-1)$ where $f(n-1)=\begin{cases}(\frac{1}{2})^{n-1}, & n\geq 1\\ 0, & n<1\end{cases}$

By theorem (2-9) we have

$$Z\left[\Delta f(n-1)\right]=(1-z^{-1})\,Z[f(n)]$$

Now; from example (2-1) we have $Z[f(n)] = \frac{z}{z-\frac{1}{2}}$, $|z| > \frac{1}{2}$,

thus $Z[\Delta f(n-1)] = (1 - z^{-1}) \frac{z}{z-\frac{1}{2}} = \frac{z-1}{z-\frac{1}{2}}$, $|z| > \frac{1}{2}$

10- Final Value Theorem:

Theorem (2-10):

If $\lim_{n\to\infty} f(n)$ exists, then $\lim_{n\to\infty} f(n) = \lim_{z\to 1} (z-1)\, Z[z]$

where $Z[f(n)] = Z[z]$.

Proof:

$$Z[f(n+1) - f(n)] = \lim_{n\to\infty} \sum_{k=0}^{n} [f(k+1) - f(k)]\, z^{-k}$$

$$z\,[Z[z] - f(0)] - Z[z] = \lim_{n\to\infty} \sum_{k=0}^{n} [f(k+1) - f(k)]\, z^{-k}$$

Now, take the limit as $z \to 1$ on both sides

$$\lim_{z\to 1} \{(z-1)\, Z[z] - z f(0)\} = \lim_{z\to 1} \left\{ \lim_{n\to\infty} \sum_{k=0}^{n} [f(k+1) - f(k)]\, z^{-k} \right\}$$

$$\lim_{z\to 1} (z-1)\, Z[z] - f(0) = \lim_{n\to\infty} \sum_{k=0}^{\infty} [f(k+1) - f(k)]$$

but $\lim_{n\to\infty} \sum_{k=0}^{n} [f(k+1) - f(k)]$

$$= \lim_{n\to\infty} [f(1) - f(0) + f(2) - f(1) + \dots + f(n+1) - f(n)]$$

$$= \lim_{n \to \infty} [f(n+1) - f(0)] = \lim_{n \to \infty} [f(n)] - f(0) ,$$

that is, $\lim_{z \to 1} [(z-1)\, Z(z)] - f(0) = \lim_{n \to \infty} f(n) - f(0)$

hence, $\lim_{z \to 1} (z-1)\, Z[z] = \lim_{n \to \infty} f(n)$ □.

Remark (2-4):

It must be stressed that this theorem only applies if $\lim_{n \to \infty} f(n)$ exists. It is possible that $\lim_{z \to 1} (z-1)\, Z[z]$ exists, but $\lim_{n \to \infty} f(n)$ does not.

For example, if $Z[z] = \dfrac{z \sin(\omega)}{z^2 - 2z \cos(\omega) + 1}$,

then $\lim_{z \to 1} (z-1)\, Z[z] = \lim_{z \to 1} (z-1) \dfrac{z \sin(\omega)}{z^2 - 2z \cos(\omega) + 1} = 0$.

But the inverse Z-transform $f(n) = \sin(\omega n)$ and $\lim_{n \to \infty} f(n)$ does not exist.

11- Upsampling property:

Theorem (2-11):

Let $f(n)$ and $g(n)$ be two sequences with Z-transform $F[z]$ and $G[z]$ respectively and

a) If $f(n) = \begin{cases} g(\frac{n}{2}) , & \text{if } n \text{ is even} \\ 0 , & \text{otherwise} \end{cases}$, then $F[z] = G[z^2]$.

b) In general if

$$f(n)=\begin{cases} g(\frac{n}{m}), & \text{if } n \text{ is a multiple of } m \\ 0 & , \text{ otherwise} \end{cases}, \text{ then } F[z]=G[z^m].$$

Proof:

a) Let $k=\frac{n}{2}$

then $F[z]=\sum_{n=-\infty}^{\infty} f(n)\,z^{-n}=\sum_{k=-\infty}^{\infty} f(2k)\,z^{-2k}=\sum_{k=-\infty}^{\infty} g(k)\,(z^2)^{-k}$

$=G[z^2]$

b) In general, we let $k=\frac{n}{m}$

$$F[z]=\sum_{n=-\infty}^{\infty} f(n)\,z^{-n}=\sum_{k=-\infty}^{\infty} f(mk)\,z^{-mk}=\sum_{k=-\infty}^{\infty} g(k)\,(z^m)^{-k}$$

$=G[z^m]$ □.

Example (2-7):

Find the Z-transform of $f(n)=\cos\frac{\omega n}{2}$ where n is even, and $n \geq 0$.

Solution:

From example (2-2) we have

$$Z[\cos\omega n]=\frac{z(z-\cos\omega)}{z^2-2z\cos\omega+1}, \quad |z|>|e^{j\omega}|=1.$$

By theorem (2-11) we find, $Z\left[\cos\frac{\omega n}{2}\right]=\frac{z^2(z^2-\cos\omega)}{z^4-2z^2\cos\omega+1}$.

Example (2-8):

If $f(n)=\begin{cases}(\frac{1}{2})^n, & n\geq 0\\ 0, & n<0\end{cases}$, then find the Z-transform of $\sum_{k=0}^{n} f(k)$.

Solution:

Since $\sum_{k=0}^{n} f(k)$ can be written as

$$\sum_{k=0}^{n} f(k)=\sum_{k=-\infty}^{\infty} u(k)\, f(n-k),\ \text{where } u(k)=\begin{cases}1, & k\geq 0\\ 0, & k<0\end{cases}$$

then $\sum_{k=0}^{n} f(k) = u(n) * f(n)$.

Thus, $Z\left[\sum_{k=0}^{n} f(k)\right]=Z\left[u(n)*f(n)\right]=Z\left[u(n)\right].Z\left[f(n)\right]$

$$=\frac{z}{z-1}\cdot\frac{z}{z-\frac{1}{2}}=\frac{z^2}{(z-1)(z-\frac{1}{2})},\ |z|>1$$

Remark (2-5):

If $f(n)=0$ for $n<0$,

then $Z\left[\sum_{k=0}^{n} f(k)\right]=Z\left[u(n)*f(n)\right]=Z\left[u(n)\right].Z\left[f(n)\right]$

Example (2-9):

Find Z-transform of $f(n)=\frac{(n+m)(n+m-1)+......+(n-1)}{m!}a^n$

Where $n\geq 0$ and $m\in N$

Solution:

Since, $Z\left[a^{n+m}\right]=\sum_{n=0}^{\infty} a^{n+m} z^{-n} = a^m \sum_{n=0}^{\infty} a^n z^{-n} = a^m \frac{z}{z-a}$,

then $Z\left[a^{n+m}\right]=(a^m z)(z-a)^{-1}$.

Differentiate m-times both sides of $\sum_{n=0}^{\infty} a^{n+m} z^{-n} = (a^m z)(z-a)^{-1}$ with respect to a, that is,

$$\frac{d^m}{da^m}\sum_{n=0}^{\infty} a^{n+m} z^{-n} = \frac{d^m}{da^m}\left[(a^m z)(z-a)^{-1}\right]$$

then; we use Liebniz rule for differentiation on the right hand side of the above equation to get:

$$\frac{d^m}{da^m}\left[(a^m z)(z-a)^{-1}\right] = \sum_{k=0}^{m} \binom{m}{k}\left(\frac{d^k}{da^k} a^m z\right)\left(\frac{d^{m-k}}{da^{m-K}}(z-a)^{-1}\right)$$

$$=\binom{m}{0}(a^m z)\left(\frac{d^m}{da^m}(z-a)^{-1}\right)+\binom{m}{1}\left(\frac{d}{da}a^m z\right)\left(\frac{d^{m-1}}{da^{m-1}}(z-a)^{-1}\right)+$$

$$+\binom{m}{2}\left(\frac{d^2}{da^2}a^m z\right)\left(\frac{d^{m-2}}{da^{m-2}}(z-1)^{-1}\right)+\dots\dots+\binom{m}{m}\left(\frac{d^m}{da^m}a^m z\right)(z-a)^{-1}$$

$$= a^m z\frac{m!}{(z-a)^{m+1}} + m\,(m\,a^{m-1} z)\frac{(m-1)!}{(z-a)^m} + \frac{m\,(m-1)}{2!}\,(m\,(m-1)\,a^{m-2} z)\frac{(m-2)!}{(z-a)^{m-1}}$$

$$+\frac{m\,(m-1)\,(m-2)}{3!}\,(m\,(m-1)\,(m-2)\,a^{m-3} z)\frac{(m-3)!}{(z-a)^{m-2}} + \dots\dots$$

$$+\,(m\,(m-1)\dots\dots(m-(m-1))\,z\frac{1}{(z-a)^{-1}}$$

$$=\frac{m!\,z}{(z-a)^{m+1}}\left[a^m + \binom{m}{1}a^{m-1}(z-a)+\binom{m}{2}a^{m-2}(z-a)^2 + \right.$$

$$+ \binom{m}{3} a^{m-3} (z-a)^3 + \ldots\ldots + (z-1)^m \Bigg]$$

$$= \frac{m!\,z}{(z-a)^{m+1}} (a + (z-a))^m = \frac{m!\,z^{m+1}}{(z-a)^{m+1}}.$$

Thus, $\sum_{n=0}^{\infty} (n+m)(n+m-1)\ldots\ldots\ldots(n+1)\, a^n z^{-n} = \frac{m!\,z^{m+1}}{(z-a)^{m+1}}$,

or,

$$Z[f(n)] = Z\left[\frac{(n+m)(n+m-1)\ldots(n+1)\,a^n}{m!}\right] = \frac{z^{m+1}}{(z-a)^{m+1}}$$

Table (A-5) Basic Z-transform:

Sequence	Transform	R. O. C
$u(n)$	$\frac{z}{z-1}$	$\lvert z\rvert > 1$
$a^n\, u(n)$	$\frac{z}{z-a}$	$\lvert z\rvert > a$
$u(n)\cos(\omega n)$	$\frac{1 - z^{-1}\cos\omega}{1 - z^{-1}\, 2\cos\omega + z^{-2}}$	$\lvert z\rvert > 1$
$u(n)\sin(\omega n)$	$\frac{1 - z^{-1}\sin\omega}{1 - z^{-1}\, 2\cos\omega + z^{-2}}$	$\lvert z\rvert > 1$
$u(-n-1)$	$\frac{1}{1 - z^{-1}}$	$\lvert z\rvert < 1$
$-a^n\, u(-n-1)$	$\frac{1}{1 - a z^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$[a^n \cos(wn)]\, u(n)$	$\frac{1 - z^{-1}\, a\cos w}{1 - z^{-1}\, 2a\cos w + a^2 z^{-2}}$	$\lvert z\rvert > a$
$[a^n \sin(wn)]\, u(n)$	$\frac{z^{-1}\, a\sin w}{1 - z^{-1}\, 2a\cos w + a^2 z^{-2}}$	$\lvert z\rvert > a$

2-9 The inverse Z-transform [9, 10, 14]:

We saw in section (2-1) how a discrete sequence $f(n)$ can be represented by its Z-transform $Z[z]$. Conversely, we can find the discrete sequence $f(n)$ by the following methods.

Method 1:

Since $Z[z] = Z[f(n)] = \sum_{n=0}^{\infty} f(n)\, z^{-n}$ multiply both sides by z^{k-1} and integrate along the closed contour C in a counter clockwise direction of the complex plan, that is

$$\oint_c Z[z]\, z^{k-1}\, dz = \oint_c \sum_{n=0}^{\infty} f(n)\, z^{k-n-1}\, dz \quad = \sum_{n=0}^{\infty} f(n) \left[\oint_c z^{k-n-1}\, dz \right]$$

By Cushy theorem we have $\oint_c z^{k-n-1}\, dz = \begin{cases} 2\pi i\,, & k = n \\ 0 \quad, & k \neq n \end{cases}$.

So that, $\oint_c Z[z]\, z^{k-1}\, dz = 2\pi i\, f(k)$.

From which it follows that, $f(k) = \dfrac{1}{2\pi i} \oint_c Z[z]\, z^{k-1}\, dz$ (2-18)

The contour integral in (2-18) can be evaluated by the method of residues, it can be shown that

$$f(n) = \begin{cases} \text{sum of residues of } Z[z]\, z^{n-1} \text{ at } |z| > R_+ \text{ for } n \geq 0 \\ -(\text{sum}) \text{ of residues of } Z[z]\, z^{n-1} \text{ at } |z| < R_- \text{ for } n < 0 \end{cases} \qquad (2\text{-}19)$$

where the residue of the function $Z[z]\,z^{n-1}$ at the simple pole z_0 is equal to $(z - z_0)\, Z[z]\, z^{n-1}\big|_{z=z_0}$, and the residue of the function $Z[z] z^{n-1}$ at $k-th$ order pole z_0 is equal to

$$\frac{1}{(k-1)!}\frac{d^{k-1}}{d\,z^{k-1}}\left[(z - z_0)^k\, Z[z]\, z^{n-1}\right]\bigg|_{z=z_0}$$

Example (2-10):

If $\quad Z[z] = \dfrac{5z}{(2-z)(3z-1)}, \; \dfrac{1}{3} < |z| < 2$, then find $f(n)$.

Solution:

Since the (R. O. C.) is outside the circle $|z| = \frac{1}{3}$, the sequence $f(n)$ is equal to zero for $n < 0$, and the function $Z[z] = \dfrac{5z}{(2-z)(3z-1)}$ has a pole $z = \dfrac{1}{3}$ inside the contour C (see Fig. (2-4)).

then by (2-19) we have

$$f(n) = Re(s)\frac{5z\, z^{n-1}}{(2-z)(3z-1)}\bigg|_{z=\frac{1}{3}} = (z - \frac{1}{3})\frac{5z\, z^{n-1}}{(2-z)(3z-1)}\bigg|_{z=\frac{1}{3}}$$

$$= (\frac{1}{3})^n \qquad (2\text{-}20)$$

Similarly, since the (R. O. C) is inside of the circle $|z| = 2$ of radius 2 the sequence is equal to zero for $n \geq 0$ and the function $Z[z] = \dfrac{5z}{(2-z)(3z-1)}$ has a pole $Z = 2$, outside the contour C (see Fig. (2-4)).

then by (2-19) we have

$$f(n) = -\,\mathrm{Re}(s)\,\frac{5z\,z^{n-1}}{(2-z)(3z-1)}\Bigg|_{z=2} = -(z-2)\,\frac{5z\,z^{n-1}}{(2-z)(3z-1)}\Bigg|_{z=2},$$

$$= 2^n \qquad\qquad (2\text{-}21)$$

From (2-20) and (2-21) we have $f(n) = \begin{cases} (\frac{1}{3})^n, & n \geq 0 \\ 2^n, & n < 0 \end{cases}$

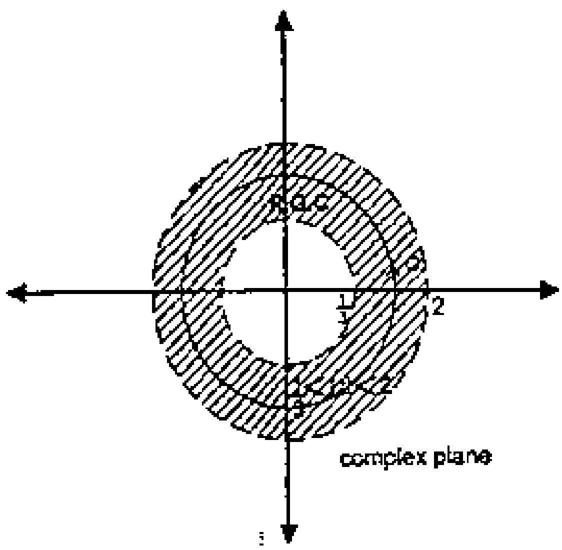

Figure (2-4)

Remark (2-6):

The region of convergence must be given for a given Z-transform.

Example (2-11):

The same expression in example (2-6) but a different region of convergence $|z| > 2$.

Solution:

Since the (R. O. C) is outside the circle $|z| = 2$, the sequence $f(n) = 0$ for $n < 0$, and the function $Z[z] = \dfrac{5z}{(2-z)(3z-1)}$ has two poles in side the contour C (see Fig. (2-5)).

$z = 2$ and $z = \frac{1}{3}$ for $n \geq 0$.

then by (2-19) we have

$$f(n) = (z - \frac{1}{3})\left(\frac{5z\, z^{n-1}}{(2 - z)(3z - 1)}\right)\Bigg|_{z=\frac{1}{3}} +$$

$$+ (z - 2)\frac{5z\, z^{n-1}}{(2 - z)(3z - 1)}\Bigg|_{z=2}$$

$$f(n) = (\frac{1}{3})^n - 2^n.$$

and $f(n) = 0$ for $n < 0$

Thus $f(n) = \begin{cases} (\frac{1}{3})^n - 2^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$

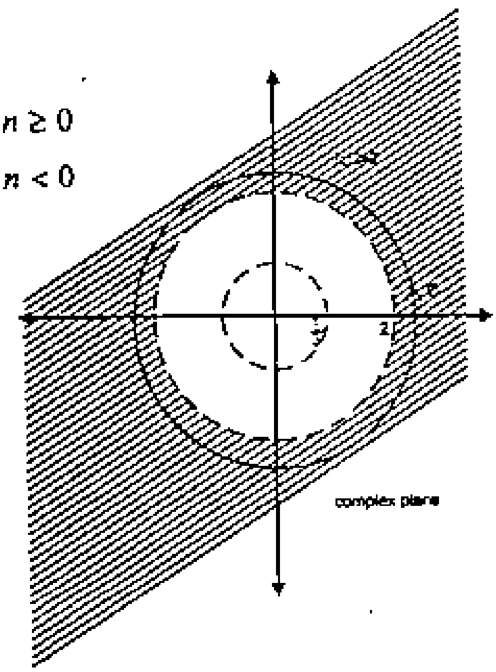

Figure (2-5)

Example (2-12):

Find $Z^{-1}[z]$ of $Z[z] = \frac{z^3}{(z-1)(z-2)^2}$, if $1 < |z| < 2$

Solution:

Let $Z^{-1}[z] = f(n)$.

Since the (R. O. C) is the outside the circle $|z| = 1$, (see Fig. (2-6)), then by (2-19) we have

$$f(n) = \frac{z^3 \, z^{n-1}}{(z-2)^2}\bigg|_{z=1} = 1$$

Also; the (R. O. C) is inside the circle $|z| = 1$, (see Fig. (2-6)), then by (2-19) we have

$$f(n) = -\frac{d}{dz}\left(\frac{z^{n+2}}{z-1}\right)\bigg|_{z=2} = -n2^{n+1}$$

Thus, $f(n) = \begin{cases} 1 & , \quad n \geq 0 \\ -n2^{n+1} & , \quad n < 0 \end{cases}$.

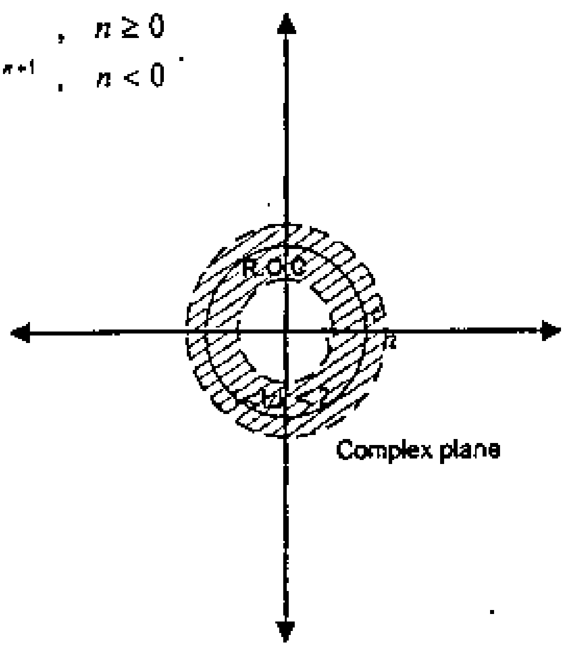

Figure (2-6)

Method II (Power Series Expansion):

If the Z-transform is given as a power series in the form

$$Z[z] = \sum_{n=-\infty}^{\infty} f(n)\, z^{-n}$$

$$= \ldots\ldots + f(-1)z + f(0) + f(1)\, z^{-1} + \ldots$$

Then any value in the sequence $f(n)$ can be found by identifying the coefficients of the appropriate power of z^{-1}.

Example (2-13):

Find $Z^{-1}[z]$ of $Z[z]=z^2(1-\frac{1}{2}z^{-1})(1+z^{-1})(1-z^{-1})$.

Solution:

We multiplied out the factors in $Z[z]$ to get

$$Z[z]=z^2-\frac{1}{2}z-1+\frac{1}{2}z^{-1}$$

then $z^2-\frac{1}{2}z-1+\frac{1}{2}z^{-1}$

$$=\dots+f(-2)\,z^2+f(-1)\,z+f(0)+f(1)\,z^{-1}+f(2)\,z^{-2}+\dots$$

Therefore

$$f(n)=\begin{cases}1\ , & n=-2\\ \frac{-1}{2}\ , & n=-1\\ -1\ , & n=0\\ \frac{1}{2}\ , & n=1\\ 0\ , & otherwise\end{cases}$$

Example (2-14):

If $Z[z]=\ln(1+az^{-1})$, $|z|>|a|$, then find $f(n)$.

Solution:

Using the power series expansion for

$\ln(1 + a z^{-1})$ with $|z| > |a|$ that is $\ln(1 + a z^{-1}) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (az^{-1})^n$

$$Z[z] = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n} ,$$

comparing with $Z[z] = \sum_{n=-\infty}^{\infty} f(n) z^{-n}$

therefore, $f(n) = \begin{cases} (-1)^{n-1} \dfrac{a^n}{n}, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$

Example (2-15):

If $Z[z] = \frac{1}{1 - a z^{-1}}$, $|z| > |a|$

Solution:

Since the (R. O. C.) is outside the circle $|z| = |a|$, then the sequence is equal to zero for $n < 0$.

Now; by long division for $\frac{1}{1 - a z^{-1}}$ we have

$$\begin{array}{r|l} & 1 + a z^{-1} + a^2 z^{-2} + \ldots\ldots \\ \hline 1 - a z^{-1} & 1 \\ & 1 - a z^{-1} \\ \hline & a z^{-1} \\ & a z^{-1} - a^2 z^{-2} \\ \hline & a z^{-2} \\ & \vdots \end{array}$$

thus $\dfrac{1}{1 - a z^{-1}} = 1 + a z^{-1} + a^2 z^{-2} + ...$

therefore $f(n) = a^n u[n]$

Example (2-16):

If $Z[z] = \dfrac{1}{1 - a z^{-1}}$, $|z| < |a|$, then find $f(n)$.

Solution:

Because the (R. O. C.) is inside the circle $|z| = |a|$, then the sequence is equal to zero for $n \geq 0$.

Now; by long division we have

$$
\begin{array}{r|l}
 & -a^{-1} z - a^{-2} z^2 \; ...... \\
\hline
-a + z & z \\
 & z - a^{-1} z^2 \\
\cline{2-2}
 & a z^{-1} \\
 & \vdots
\end{array}
$$

therefore, $Z[z] = -a^{-1} z - a^{-2} z^2 \; ....$

and $f(n) = -a^n u[-n-1]$.

Method III (Partial fractions):

For rational functions we can obtain partial fractions expansion, and we can identify the Z-transform of each term. Assume that $Z[z]$ is expressed as a ratio of polynomials in z^{-1}, that is, if

$$Z[z] = \frac{\sum_{k=0}^{m} b_k z^{-k}}{\sum_{k=0}^{n} a_k z^{-k}}$$

then we may factor $\sum_{k=0}^{m} b_k z^{-k}$ and $\sum_{k=0}^{n} a_k z^{-k}$, that is;

$$Z[z] = \frac{b_0 \prod_{k=1}^{m} (1 - c_k z^{-1})}{a_0 \prod_{k=1}^{n} (1 - d_k z^{-1})}$$

where the c_k and d_k are the zeros and poles of $Z[z]$ respectively.

Case 1:

If $m < n$ and the poles are all first order, then $Z[z]$ can be expressed as, $Z[z] = \sum_{k=1}^{n} \frac{A_k}{1 - d_k z^{-1}}$ in this case the coefficients A_k are given by $A_k = (1 - d_k z^{-1}) Z[z] \big|_{z=d_k}$.

Case 2:

a) If $m \geq n$ and the poles are all first order then an expansion of the form

$$Z[z] = \sum_{r=0}^{m-n} B_r z^{-r} + \sum_{k=1}^{n} \frac{A_k}{1 - d_k z^{-1}}$$

can be used, and the B_r be obtained by long division and A_k can be obtained using the same equation as for $m < n$.

Example (2-17):

If $Z[z] = \dfrac{1 + 2z^{-1} + z^{-2}}{1 + \frac{3}{2} z^{-1} + \frac{1}{2} z^{-2}}$, $\quad |z| > 1$

then find $f(n)$.

Solution:

$$Z[z]=\frac{(1+z^{-1})^2}{(1-\frac{1}{2}z^{-1})(1-z^{-1})} = \frac{(z+1)^2}{(z-\frac{1}{2})(z-1)}$$

Since $m=n=2$ then $Z[z]$ can be expressed in a sum of partial fraction as follows:

$$Z[z]=B_0+\frac{A_1}{1-\frac{1}{2}z^{-1}}+\frac{A_2}{1-z^{-1}}.$$

The value B_0 can be found by long division of the numerator $(z+1)^2$ by the denominator $(z-\frac{1}{2})(z-1)$, that is $B_0=2$.

Also, the coefficients A_1 and A_2 can be found using the fact that

$$A_k=(1-d_k z^{-1})\,Z[z]\big|_{z^{-1}=d_k}.$$

So, $A_1=\frac{1+2z^{-1}+z^{-2}}{1-z^{-1}}\Big|_{z^{-1}=2}=-9$,

and $A_2=\frac{1+2z^{-1}+z^{-2}}{1-\frac{1}{2}z^{-1}}\Big|_{z^{-1}=1}=8$, so $Z[z]=2-\frac{9}{1-\frac{1}{2}z^{-1}}+\frac{8}{1-z^{-1}}$:

Using the fact the (R. O. C) is $|z|>1$ the terms can be inverted one at a time by inspection to give

$$f(n)=2\,\delta(n)-9(\frac{1}{2})^n\,u(n)+8u(n).$$

b) If $m \geq n$ and the function $Z[z]$ has multiple- order poles

$$Z[z] = \sum_{r=0}^{m-n} B_r z^{-r} + \sum_{k=1}^{n} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1 - d_i z^{-1})^m}$$

where s is the highest power of multiple poles.

Example (2-18):

If $\quad Z[z] = \dfrac{3z^4 - 12z^3 + 13z^2 + 2z + 2}{(z^2 - 2z - 3)(z^2 - 2z + 1)}, \quad |z| > 3$

then find $f(n)$.

Solution:

Since $m = n = 4$ then $Z[z]$ can be expressed in a sum of partial fraction as follows:

$$Z[z] = 3 - \frac{\frac{7}{3}}{(z+1)} + \frac{\frac{11}{4}}{(z-3)} - \frac{2}{(z-1)} - \frac{1}{(z-1)^2}$$

using the fact the (R. O. C.) is $|z| > 3$ the terms can be inverted one at a time by inspection to give

$$f(n) = 3\,\delta(n) - \frac{7}{4}(-1)^n\, u(n) + \frac{11}{4}(3)^n\, u(n) - 2u(n) - (n-1)$$

Method IV (using the convolution theorem(2-5)):

Example (2-19):

Find the inverse Z-transform of $Z[z]=\frac{z}{(z-1)^2}$ for $|z|>1$.

Solution:

Let $Z^{-1}[z]=f(n)$,

Now; $Z[z]=\frac{z}{(z-1)^2}=\frac{z}{(z-1)}\,\frac{z^{-1}z}{z-1}$

$Z[z]=Z_1[z]\,.\,Z_2[z]$,

where, $Z_1[z]=\frac{z}{z-1}$ and $Z_2[z]=\frac{z^{-1}z}{z-1}$.

Using the fact $Z_1^{-1}[z]=f_1(n)=u(n)$, and $Z_2^{-1}[z]=f_2(n)=u(n-1)$,

and by theorem (2-5) we have

$$f(n)=f_1(n)*f_2(n)=\sum_{m=-\infty}^{\infty}u(m)\,u(n-1-m)$$

or

$$f(n)=\dots+u(-1)\,u(n)+u(0)\,u(n-1)+u(1)\,u(n-2)+\dots$$
$$+u(n-1)\,u(n-1-(n-1))+u(n)\,u(n-1-n)+\dots$$

since $u(n)=0$ for $n<0$,

then, $f(n)=u(0)\,u(n-1)+u(1)\,u(n-2)+\dots+u(n-1)\,u(n-1-(n-1))$

$$f(n) = 1 + 1 + \dots + 1 = n \cdot 1 = n.$$

Thus $f(n) = n$.

2-10 Applications of the Z-transform [9, 10, 14]:

1- State- variable and Z-transform:

Consider the equations

$$q_1(n+1) = -a_1 q_1(n) - a_2 q_2(n) + X(n) \quad (2\text{-}22)$$

$$q_2(n+1) = q_1(n) \quad (2\text{-}23)$$

and

$$y(n) = X(n) - a_1 q_1(n) - a_2 q_2(n) + b_1 q_1(n) + b_2 q_2(n) \quad (2\text{-}24)$$

where a_1, a_2, b_1 and b_2 are constants.

We write (2-22) and (2-23) in matrix form as

$$\begin{bmatrix} q_1(n+1) \\ q_2(n+1) \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q_1(n) \\ q_2(n) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} X(n), \quad (2\text{-}25)$$

while (2-24) is expressed as

$$y(n) = \begin{bmatrix} b_1 - a_1 & b_2 - a_2 \end{bmatrix} \begin{bmatrix} q_1(n) \\ q_2(n) \end{bmatrix} + [1]\, X(n) \quad (2\text{-}26)$$

If we define the state vectors as

$$q(n) = \begin{bmatrix} q_1(n) \\ q_2(n) \end{bmatrix},$$

then we can rewrite (2-25) and (2-26) as

$$q(n+1) = Aq(n) + B\,X(n) \tag{2-27}$$

$$y(n) = Cq(n) + DX(n), \tag{2-28}$$

where the matrix A, vectors B and C, and Scalar D are given by

$$A = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} b_1 - a_1 & b_2 - a_2 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 \end{bmatrix}$$

Equations (2-27) and (2-28) are the general form a state- variable equations corresponding to a discrete sequence system.

Now; taking the Z-transform both sides of (2-27)

$$z\,Z[q(n)] = A\,Z[q(n)] + B\,Z[X(n)], \tag{2-29}$$

where we have defined the Z-transform of state- variable as the vector containing the Z-transform of each state variables, that is

$$Z[q(n)] = \begin{bmatrix} Z_1[z] \\ Z_2[z] \\ \vdots \\ Z_n[z] \end{bmatrix}$$

where the ith-entry in $Z[q(n)]$ is the Z-transform of the ith-state variable, $Z_i[z] = Z[q_i(n)]$, and we may rewrite (2-29) as

$$(z\,I - A)\;Z[q(n)] = B\,Z[X(n)]$$

where I is the $(2x2)$ identity matrix

Now; take the Z-transform of (2-28), obtaining

$$Z[y(n)] = CZ[q(n)] + DZ[X(n)]$$

Substitute the expression for $Z[q(n)]$ in this Z-transform $Z[y(n)]$ yielding

$$Z[y(n)] = \left[C(zI-A)^{-1}B+D\right]Z[x(n)]. \qquad (2\text{-}30)$$

Hence the systems transfer function $H(z) = \dfrac{Z[y(n)]}{Z[X(n)]}$ is then given by

$$H(z) = C(zI-A)^{-1}B + D \qquad (2\text{-}31)$$

Example (2-20):

Determine the transfer function for a linear system with state-variable matrices:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 3 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \end{bmatrix}$$

Solution:

We begin with

$$zI - A = \begin{bmatrix} z & -1 \\ 1 & z-1 \end{bmatrix},$$ and this implies that

$$(zI-A)^{-1} = \frac{1}{z^2-z+1}\begin{bmatrix} z-1 & 1 \\ -1 & z \end{bmatrix}.$$

Hence (2-31), yields

$$H(z)=\frac{1}{z^2-z+1}\begin{bmatrix}3 & 0\end{bmatrix}\begin{bmatrix}z-1 & 1\\ -1 & z\end{bmatrix}\begin{bmatrix}0\\ 2\end{bmatrix}+0=\frac{6}{z^2-z+1}.$$

2- Linear difference equations and Z-transform:

The Z-transform bears arelationship to difference equations an alogous to the relationship of the Laplace tranasform to differential equations.

A linear differential equation with initial conditions can be converted by Laplace transform into an algebraic equation.

The solution is then found in the Laplace domain and is inverse Lapalce transformed to find continuous function domain solution.

A linear difference equation with initial conditions can be converted by Z-transform into an equation.

Then it is solved, and the solution in the discrete sequence domain is found by an inverse Z-transform.

Now; we recall that some basic facts about the Z-transform.

a) The Z-transform is linear.

b) $Z[1]=\frac{z}{z-1}$

c) $Z[a^n]=\frac{z}{z-a}$

d) If $f(n)=g(n+1)$ then the Z-transform of $g(n)$ is related to that of $f(n)$, much as the Laplace transform of $y'(t)$ is related to that of $y(t)$.

$$Z[g(n+1)]=z\,Z[g(n)]-zg(0)$$

e) The rules (a) and (b) can be used to compute the transform of the sequence n.

$$Z[n] = \frac{z}{(z-1)^2}, \quad |z| > 1$$

since:

$$\frac{z}{z-1} = Z[1] = Z[(n+1)-n], \text{ or } Z[(n+1)-n] = \frac{z}{z-1}$$

or $\quad Z[n+1] - Z[n] = \frac{z}{z-1} \quad$ or $\quad z\,Z[n] - Z[n] = \frac{z}{z-1}$

So that, $Z[n] = \frac{z}{(z-1)^2}$.

f) An argument similar to (e) shows that

$$Z[n\,a^n] = \frac{a\,z}{(z-a)^2} \qquad \text{for} \quad |Z| > |a|$$

Proof:

By (d) we have,

$$Z[(n+1)\ a^{n+1}] = z\,Z[n\,a^n]$$

Now; $Z[(n+1)a^{n+1}] = Z[na^{n+1} + a^{n+1}] = Z[n\,a^{n+1}] + Z[a^{n+1}]$

or $\quad z\,Z[n\,a^n] = a\,Z[n\,a^n] + a\,Z[a^n] \quad$ or $\quad (z-a)\,Z[n\,a^n] = \frac{a\,z}{z-a}$

so that, $Z[n\,a^n] = \frac{a\,z}{(z-a)^2}$

Now; we apply these formulas to solve a second order difference equation.

Example (2-21):

Solve the difference equation

$$y(n+2) - 3\,y(n+1) - 4\,y(n) = 12, \text{ with } y(0)=1 \text{ and } y(1)=2$$

Solution:

We compute the Z-transform of both sides of the difference equation

$$(z^2\,Z\left[y(n)\right] - z^2 - 2z) - z(3z\left[y(n)\right] - z) - 4\,Z\left[y(n)\right] = \frac{12z}{z-1}$$

$$(z^2 - 3z - 4)\,Z\left[y(n)\right] = z^2 - z + \frac{12z}{z-1}$$

$$Z\left[y(n)\right] = \frac{z^2 - z}{z^2 - 3z - 4} + \frac{12z}{(z-1)(z^2 - 3z - 4)}.$$

Now; we can use partial fractions to evaluate the right hand side, we observe that all of our Z-transform formulas (a) through (f) give fractions with a z in the numerator,

Therefore, $$Z\left[y(n)\right] = z\left(\frac{0.4}{z+1} + \frac{0.6}{z-4} + \frac{1.2}{z+1} + \frac{0.8}{z-4} - \frac{2}{z-1}\right)$$

$$= z\left(\frac{1.6}{z+1} + \frac{1.4}{z-4} - \frac{2}{z-1}\right)$$

So that, $y(n) = 1.6\,(-1)^n + 1.4\,(4)^n - 2$

Chapter Three

Unification and Extension

3-1 Unification and extension of Laplace transform and Z-transform [1, 2, 5]:

In this chapter we introduce the Laplace transform for an arbitrary t-variable. Two particular choices of the variable t, namely the continuous variable and the discrete variable which yield the concepts of the classical Laplace transform and of the classical Z-transform, other choices of t-variable yield new concepts of our Laplace transform, which can be applied to find solution of higher order linear dynamic equation with constant coefficients. We present several useful properties of our Laplace transform of many elementary functions, among the results is the convolution of t-variable, which is introduced in this chapter as well.

Definition (3-1):

Let $f(t): T \to C$ be a function, where T is closed subset of the reals R. Such that $T \neq \phi$ and $0 \in T$. We define the forward operator σ by:

$$\sigma(t): T \to T, \quad \sigma(t) = \inf\{u \in T : u > t\},$$

and the backward operator ρ by $\rho(t): T \to T$, $\rho(t) = \sup\{u \in T : u < T\}$.

It is convenient to define some useful operators $\mu, \eta : T \to [0, \infty)$ as $\mu(t) = \sigma(t) - t$ and $\eta(t) = t - \rho(t)$.

Definition (3-2):

The delta derivative of $f(t)$ denoted by $f^{\Delta}(t)$ with $f^{\Delta}(t): T \to C$; is a number, it exists if for all $\varepsilon > 0$ there is a neighbohood $N(t)$ such that

$$\left| f(\sigma(t)) - f(u) - f^{\Delta}(t)\,(\sigma(t) - u) \right| \leq \varepsilon \left| \sigma(t) - u \right|$$

for all $u \in N(t)$.

Definition (3-3):

The nable derivative of $f(t)$ denoted by $f^{\nabla}(t)$ with $f^{\Delta}(t): T \to C$; is a number, it exists if all $\varepsilon > 0$ there is a neighbohood $N(t)$ such that $\left| f(\rho(t)) - f(u) - f^{\nabla}(t)(\rho(t) - u) \right| \le \varepsilon \left| \rho(t) - u \right|$ for all $u \in N(t)$.

Remarks (3-1):

1. If $T = R$ then $f^{\Delta}(t)$ and $f^{\nabla}(t)$ becomes the usual derivative, that is $f^{\Delta}(t) = f^{\nabla}(t) = f'(t)$. (see appendix II).
2. If $T = Z$ then $f^{\Delta}(n)$ and $f^{\nabla}(n)$ becomes the usual difference, that is $f^{\Delta}(n) = f(n+1) - f(n)$ and $f^{\nabla}(n) = f(n) - f(n-1)$. (see appendix II).
3. If $f(t)$ is delta differentiable at t, then
   $f^{\sigma}(t) = f(t) + \mu(t) f^{\Delta}(t)$, where $f^{\sigma}(t) = f(\sigma(t))$.
4. If $f(t)$ is nable differentiable at t, then
   $f^{\rho}(t) = f(t) + \eta(t) f^{\nabla}(t)$, where $f^{\rho}(t) = f(\rho(t))$.

Now;

We have some important formulas about the delta and nable derivatives. (see appendix II).

1. $(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t)$ and $(f+g)^{\nabla}(t) = f^{\nabla}(t) + g^{\nabla}(t)$.
2. $(f(t)\, g(t))^{\Delta} = f^{\Delta}(t)\, g(t) + f^{\sigma}(t)\, g^{\Delta}(t)$ and
   $(f(t)\, g(t))^{\nabla} = f^{\nabla}(t)\, g(t) + f^{\rho}(t)\, g^{\nabla}(t)$.
3. $\left[\dfrac{f(t)}{g(t)}\right]^{\Delta} = \dfrac{f^{\Delta}(t)\, g(t) - f(t)\, g^{\Delta}(t)}{g(t)\, g^{\sigma}(t)}$ and

   $\left[\dfrac{f(t)}{g(t)}\right]^{\nabla} = \dfrac{f^{\nabla}(t)\, g(t) - f(t)\, g^{\nabla}(t)}{g(t)\, g^{\rho}(t)}$.

Definition (3-4):

A function $f(t)$ defined on T is called regulated if its right-sided limit exist and finite at points $t \in T$ with $\sigma(t) = t$ and, if its left- sided limit exist and finite at points $t \in T$, with $\rho(t) = t$.

Definition (3-5):

A function $f(t)$ defined on T is called regressive

if $1 + \mu(t)\, f(t) \neq 0$ for all $t \in T$, and if $1 + \eta(t)\, f(t) \neq 0$ for all $t \in T$.

Definition (3-6) (Delta integral):

Let $f(t): T \to C$ be a function and $a, b \in T$

If there exists a function $F(t): T \to C$ such that $F^{\Delta}(t) = f(t)$ for all $t \in T$.

then $F(t)$ is said to be a delta ant derivative of $f(t)$ and in this case the integral called delta integral, and is given by

$$\int_a^b f(\tau)\, \Delta\tau = F(b) - F(a) \quad \text{for } a, b \in T.$$

Definition (3-7) (Nable integral):

Let $f(t): T \to C$ be a function, and $a, b \in T$

If there exists a function $F(t): T \to C$ such that $F^{\nabla}(t) = f(t)$ for all $t \in T$, then $F(t)$ is said to be nable ant derivative of $f(t)$, and in this case the integral called nable integral, and is given by

$$\int_a^b f(\tau)\, \nabla\tau = F(b) - F(a) \quad \text{for } a, b \in T.$$

Theorem (3-1) (see appendix II):

If $f(t)$ and $f^{\Delta}(t)$ are continuous, then

$$\left[\int_s^t f(t,\tau)\,\Delta\tau\right]^{\Delta} = f(\sigma(t),t) + \int_s^t f^{\Delta}(t,\tau)\,\Delta\tau .$$

Also, if $f(t)$ and $f^{\nabla}(t)$ are continuous, then

$$\left[\int_s^t f(t,\tau)\,\nabla\tau\right]^{\nabla} = f(\sigma(t),t) + \int_s^t f^{\nabla}(t,\tau)\,\nabla\tau .$$

Theorem (3-2):

a) The initial value problem defined by $y^{\Delta}(t) = P(t)\,y(t)$, $y(t_0) = 1$, has unique solution if $t_0 \in T$ and $P(t)$ is regulated and regressive function.

The unique solution is denoted by $K_P(t, t_0)$ or $K_P(t) = K_P(t, 0)$.

b) The initial value problem defined by $y^{\nabla}(t) = -P(t)\,y(t)$, $y(t_0) = 1$, has unique solution if $t_0 \in T$ and $P(t)$ is regulated and regressive function.

The unique solution is denoted by $R_P(t, t_0)$ or $R_P(t) = R_P(t, 0)$.

Some useful formulas can be obtained for $K_p(t)$:

1. $K_P^{\sigma}(t) = (1 + \mu\,P(t))\,K_P(t)$ (see appendix II Theorem 1 (b)) (3-1)

2. $K_P(t)\,.\,K_q(t) = K_{P\oplus q}(t)$ (3-2)

3. $\dfrac{k_P(t)}{k_q(t)} = K_{P\ominus q}(t)$ (3-3)

where $p \oplus q = p + q + \mu(t)\, pq$ and $p \ominus q = \dfrac{p - q}{1 + \mu(t) s}$ (3-4)

Remark (3-2):

We defined $\ominus s$ as $0 \ominus s = \dfrac{0 - s}{1 + \mu(t)\, s} = \dfrac{-s}{1 + \mu(t)\, s}$

Now if $t \in R$ then $\mu(t) = 0$ and $\ominus s = -s$

and if $t \in Z$ then $\mu(t) = 1$ and $\ominus s = \dfrac{-s}{1 + s}$

Definition (3-8):

The solution $y(t)$ of the initial value problem.

$$(y^{\Delta})^{n}(t) + \sum_{i=1}^{n} P_i(t)\,(y^{\Delta})^{n-i}(t) = 0,\ (y^{\Delta})^{i}(\sigma(t), t) = 0, \quad 0 \le i \le n-2,$$

$(y^{\Delta})^{n-1}(\sigma(t), t) = 1$, for each fixed $t \in T$, is called Cauchy function.

where $P_i(t)$ is continuous function.

Example (3-1):

Show that $y(t) = \int_{0}^{t} k_a(t, \sigma(\tau))\; g(\tau)\, \Delta\tau$ Is a solution of the initial value problem

$$y^{\Delta}(t) - a\, y(t) = g(t), \qquad y(0) = 0$$

Solution:

By theorem (3-1) and the properties of the Cauchy function we have

$$y^{\Delta}(t) = \int_0^t k_\alpha^{\Delta}(t, \sigma(\tau))\, g(\tau)\, \Delta\tau + k_\alpha(\sigma(t), \sigma(t))\, g(t)$$

$$= \int_0^t \alpha\, k_\alpha(t, \sigma(\tau)\, g(\tau)\, \Delta\tau + g(t),$$

thus

$$y^{\Delta}(t) - \alpha y(t) = \alpha \int_0^t k_\alpha(t, \sigma(\tau))\, g(\tau)\, \Delta\tau - \alpha \int_0^t k_\alpha(t, \sigma(\tau))\, g(\tau)\, \Delta\tau + g(t) = g(t)$$

Example (3-2):

Show that $y(t) = \int_s^t h_n(t, \sigma(\tau))\, g(\tau)\, \Delta\tau$, is a solution of the initial value problem

$$(y^{\Delta})^{n+1}(t) = g(t), \quad (y^{\Delta})^{i}(t) = 0 \qquad 0 \le i \le n$$

for each fixed $s \in T$.

Solution:

Clearly $y(s) = 0$, also, by theorem (3-1) we have

$$y^{\Delta}(t) = \int_s^t h_n^{\Delta}(t, \sigma(\tau))\, g(\tau)\, \Delta\tau + h_n(\sigma(t), \sigma(t))\, g(t)$$

$$= \int_s^t h_n^{\Delta}(t, \sigma(\tau))\, g(\tau)\, \Delta\tau.$$

Note that $y^{\Delta}(s) = 0$ and

$$(y^{\Delta})^{2}(t) = \int_{s}^{t} (h_{n}^{\Delta})^{2}(t, \sigma(\tau))\, g(\tau)\, \Delta\tau + h_{n}(\sigma(t), \sigma(t))\, g(\tau)$$

$$= \int_{s}^{t} (h_{n}^{\Delta})^{2}(t, \sigma(\tau))\, g(\tau)\, \Delta\tau, \text{ (by the same theorem (3-1))}$$

Note that $(y^{\Delta})^{2}(s) = 0$,

then we obtain, $(y^{\Delta})^{i}(t) = \int_{s}^{t} (h_{n}^{\Delta})^{i}(t, \sigma(\tau))\, g(\tau)\, \Delta\tau$, for $0 \le i \le n$,

and $(y^{\Delta})^{i}(s) = 0 \quad 0 \le i \le n$.

Finally;

$$(y^{\Delta})^{n+1}(t) = \int_{s}^{t} (h_{n}^{\Delta})^{n+1}(t, \sigma(\tau))\, g(\tau)\, \Delta\tau + h_{n}(\sigma(t), \sigma(t))\, g(t) = g(t)$$

after this introduction we can shown the particularity unification and extension of the Laplace transform and Z-transform.

Definition (3-9):

a) We denote the right Laplace transform of a regulated function $f(t): T \to C$ by $L^{+}\{f(t)\}$, defined by

$$L^{+}\{f(t)\} = \int_{s}^{\infty} f(t)\, K_{\ominus s}(\sigma(t))\, \Delta t = F(s) \quad \text{for} \quad s \in D\{f(t)\}.$$

where $D\{f(t)\}$ consists of all $s \in C$ for which the improper integral exists and $1 + \mu(t)s \neq 0$ for all $t \in T$.

b) We denote the left Laplace transform of a regulated function $f(t): T \to C$ by $L^{-}\{f(t)\}$, defined by

$$L^{-}\{f(t)\} = \int_{-\infty}^{0} f(t)\, R_{\ominus s}(\rho(t))\, \nabla t = H(s) \quad \text{for} \quad s \in D\{f(t)\},$$

where $D\{f(t)\}$ consists of all $s \in C$ for which the improper integral exists and $1 + \eta(t)\, s \neq 0$ for all $t \in T$.

c) We denote the bilateral Laplace transform of a regulated function $f(t): T \to C$ by $L_B\{f(t)\}$ defined by

$$L_B\{f(t)\} = L^{-}\{f(t)\} + L^{+}\{f(t)\}.$$

The definition (2-9) explain the extension property of our Laplace transform and the Z-transform.

Remark (3-3):

1. If $T = R$ then

a) $\sigma(t) = t,\ \mu(t) = 0, \ominus s = -s,\ k_\alpha(t) = e^{\alpha t}$ and $\int_0^{\infty} f(t)\, \Delta t = \int_0^{\infty} f(t)\, dt.$

b) $\rho(t) = t,\ \eta(t) = 0, \ominus s = s,\ R_\alpha(t) = e^{\alpha t}$ and $\int_{-\infty}^{0} f(t)\, \nabla t = \int_{-\infty}^{0} f(t)\, dt.$

2. If $T = Z$ then

a) $\sigma(t) = t + 1,\ \mu(t) = 1, \ominus s = \dfrac{-s}{1+s},\ K_\alpha(t) = (1+\alpha)^t$ and

$$\int_0^{\infty} f(t)\, \Delta t = \sum_{t=0}^{\infty} f(t).$$

b) $\rho(t)=t-1,\ \eta(t)=1,\ \ominus s=s,\ R_\alpha(t)=(1+\alpha)^{-t}$ and

$$\int_{-\infty}^{0} f(t)\,\nabla t=\sum_{n=-\infty}^{0} f(t).$$

Hence, clearly if $T=R$, our Laplace transform is the classical Laplace transform while if $T=Z$ then:

a) $$L^{*}\{f(t)\}=\sum_{n=0}^{\infty} f(n)\,(1-\frac{s}{1+s})^{n+1}=\sum_{n=0}^{\infty}\frac{f(n)}{(s+1)^{n+1}}$$

$$=\frac{Z[f(n)]}{s+1}=\frac{Z[s+1]}{s+1}.$$

b) $$L^{*}\{f(t)\}=\sum_{-\infty}^{0} f(n)\,(1+s)^{-n+1}=\sum_{n=-\infty}^{0}\frac{f(n)\,(1+s)^{-n}}{(1+s)^{-1}}$$

If we let $m=-n$ then we have

$$\sum_{m=0}^{\infty}\frac{f(-m)\,(1+s)^{m}}{(1+s)^{-1}}=\frac{Z[f(-m)]}{(1+s)^{-1}}=\frac{Z[1+s]}{(1+s)^{-1}}.$$

These remarks explain the unification property of our Laplace transform and the Z-transform.

3-2 Right Laplace transform (Theory and Applications) [5]:

After having studied the preceding section (2-7), we understand the linearity of right Laplace transform

$$L^{*}\{\alpha\ f(t)+\beta\,g(t)\}=\alpha\ L^{*}\{f(t)\}+\beta\,L^{*}\{g(t)\},$$

it is now obvious, to derive further results on the right Laplace transforms.

The following auxiliary results are needed.

Lemma (3-1):

If $s \in C$ and $1+\mu(t)s \neq 0$, then $K^{\sigma}_{\Theta s}(t) = \dfrac{K_{\Theta s}(t)}{1+\mu(t)\,s}$

Proof:

By the property (3-1) we have

$$K^{\sigma}_{\Theta s}(t) = \left[1+\mu(\Theta s)\right] K_{\Theta s}(t)$$

$$= \left[1 - \frac{\mu(t)s}{1+\mu(t)s}\right] K_{\Theta s}(t) = \frac{K_{\Theta s}(t)}{1+\mu(t)s} \quad \square.$$

Example (3-3):

Find the right Laplace transform of $f(t)=1$.

Solution:

Let $s \neq 0$

$$L^{*}\{1\} = \int_0^{\infty} k_{\Theta s}(\sigma(t))\,\Delta t = \int_0^{\infty} k^{\sigma}_{\Theta s}(t)\,\Delta t = \int_0^{\infty} \frac{k_{\Theta s}(t)}{1+\mu(t)\,s}\,\Delta t$$

by remark (3-2) we have

$$L^{*}\{1\} = \frac{-1}{s}\int_0^{\infty} \Theta s\, k_{\Theta s}(t)\,\Delta t = \frac{-1}{s}\int_0^{\infty} k^{\Delta}_{\Theta s}(t)\,\Delta t = \frac{-1}{s}\, k_{\Theta s}(t)\Big|_0^{\infty}$$

$$= \frac{-1}{s}(0-1) = \frac{1}{s}$$

Provided $\lim_{t\to\infty} k_{\Theta s}(t) = 0$ for all $s \in D\{1\}$ (e.g., if there exists $c = \mathrm{Re}(s) < 0$ with $1+\mu(t)\,c > 0$ and $\Theta s\,(t) \leq c$ for all $t \in T$).

Theorem (3-3):

If $f(t): T \to C$ is such that $f^{\Delta}(t)$ is regulated, then

$$L^{*}\left\{f^{\Delta}(t)\right\} = s\, L^{*}\left\{f(t)\right\} - f(0)$$

for all $s \in D\left\{f(t)\right\}$ such that $\lim_{t\to\infty}\left\{f(t)\, k_{\ominus s}(t)\right\} = 0$

Proof:

$$L^{*}\left\{f^{\Delta}(t)\right\} = \int_{0}^{\infty} f^{\Delta}(t)\; k_{\ominus s}(\sigma(t))\; \Delta t$$

$$= \int_{0}^{\infty} \left\{\left[\, f(t)\; k_{\ominus s}(t)\right]^{\Delta} - f(t)\, k_{\ominus s}^{\Delta}\right\} \Delta t$$

$$= -f(0) - \int_{0}^{\infty} f(t)\,(\ominus s)\, k_{\ominus s}(t)\; \Delta t$$

$$= -f(0) + s\int_{0}^{\infty} f(t)\, \frac{k_{\ominus s}(t)}{1+\mu(t)\, s}\; \Delta t$$

$$= -f(0) + s\int_{0}^{\infty} f(t)\; k_{\ominus s}(\sigma(t))\; \Delta t = s\, L^{*}\left\{f(t)\right\} - f(0) \square .$$

Theorem (3-4):

If $f(t): T \to C$ is such that $f^{\Delta\Delta}(t)$ is regulated then

$$L^{*}\left\{f^{\Delta\Delta}(t)\right\} = s^{2}\, L^{*}\left\{f(t)\right\} - s f(0) - f^{\Delta}(0)$$

for all $s \in D\,(f(t))$ with $\lim_{t\to\infty}\left\{f(t)\, k_{\ominus s}(t)\right\} = 0$ and $\lim_{t\to\infty}\left\{f^{\Delta}(t)\, k_{\ominus s}(t)\right\} = 0$

Theorem (3-5):

If $f(t): T \to C$ is regulated and

$$F(t) = \int_0^t f(t)\,\Delta t \text{ for } t \in T$$

Then $L^{*}\{F(t)\} = \frac{1}{s}\,L^{*}\{f(t)\} = 0$ for all $s \in D - \{0\}$ Such that $\lim_{t\to\infty}\{f(t)\;k_{\ominus s}(t)\} = 0$

Proof:

Let $F^{\Delta}(t) = f(t)$, then by theorem (3-4) we have

$$L^{*}\{F^{\Delta}(t)\} = s\,L^{*}\{F(t)\} - F(0)\,,\quad F(0) = 0$$

that is $L^{*}\{f(t)\} = s\,L^{*}\{F(t)\}$

or $\quad L^{*}\{F(t)\} = \frac{1}{s}\,L^{*}\{f(t)\}$

Hence

$$L^{*}\left\{\int_0^t f(t)\,\Delta t\right\} = \frac{1}{s}\,L^{*}\{f(t)\} \qquad \square.$$

Example (3-4):

Find the right Laplace transform $K_{\alpha}(t)$

Solution:

Assuming $s \neq \alpha$

$$L^{*}\{k_{\alpha}(t)\} = \int_0^{\infty} k_{\alpha}(t)\,k_{\ominus s}(\sigma(t))\,\Delta t = \int_0^{\infty} k_{\alpha}(t)\,\frac{k_{\ominus s}(t)}{1+\mu(t)\,s}\,\Delta t$$

$$= \int_0^{\infty} \frac{k_{\alpha\ominus s}(t)}{1+\mu(t)s}\,\Delta t = \frac{1}{\alpha - s}\int_0^{\infty} \frac{\alpha - s}{1+\mu(t)\,s}\,k_{\alpha\ominus s}(t)\,\Delta t$$

by formula (3-4) we have

$$\frac{1}{\alpha - s}\int_0^\infty (\alpha \Theta s)\, k_{\alpha\Theta s}(t)\, \Delta t = \frac{1}{\alpha - s}\int_0^\infty k^{\Delta}_{\alpha\Theta}(t)\, \Delta t$$

Thus

$$L^{*}\{k_\alpha(t)\} = \frac{1}{s-\alpha} \qquad \text{provided} \quad \lim_{t\to\infty} k_{\alpha\Theta s}(t) = 0$$

Definition (3-10):

The hyperbolic function is defined by

$$\cos h_\alpha(t) = \frac{k_\alpha(t) + k_{-\alpha}(t)}{2} \quad \text{and} \quad \sin h_\alpha(t) = \frac{k_\alpha(t) - k_{-\alpha}(t)}{2}$$

Definition (3-11):

The trigonometric is defined by

$$\cos_\alpha(t) = \frac{k_{i\alpha}(t) + k_{-i\alpha}(t)}{2} \quad \text{and} \quad \sin_\alpha(t) = \frac{k_{i\alpha}(t) - k_{-i\alpha}(t)}{2i}$$

Note that: the above functions satisfy $\cos^2_\alpha(t) + \sin^2_\alpha(t) = \cos h^2_\alpha(t) - \sin h^2_\alpha(t) = 1$ iff $\mu(t) = 0$

Example (3-5):

Find the right Laplace transform of $f(t) = \cos h_\alpha(t)$

Solution:

$$L^{*}\{\cos h_\alpha(t)\} = \frac{\frac{1}{s-\alpha} + \frac{1}{s+\alpha}}{2} = \frac{s}{s^2 - \alpha^2}$$

Definition (3-12):

The generalized function is defined by

$$h_k(t,s) = \begin{cases} 1, & k=0 \\ \int_s^t h_{k-1}(\tau,s)\,\Delta\tau, & k=1,2,\ldots \end{cases} \qquad \text{for all } s,t \in T,$$

and; if we let $h_{k+1}^{\Delta}(t,s)$ denote, the derivatives of $h_{k+1}(t,s)$ with respect to t then for each fixed $s \in T$.

and $h_{k+1}^{\Delta}(t,s) = h_k(t,s)$ for $k = \{0, 1, 2, 3, \ldots\}$

Note that $h_k(t,0) = h_k(t)$

Example (3-6):

Show that $L^{*}\{h_k(t)\} = \frac{1}{s^{k+1}}$

Proof (by induction):

1. If $k=0$ then $L^{*}\{h_0(t)\} = \frac{1}{s}$, (by Example (3-3)).
2. We assume it is true for the integer $k = n$.

   That is $L^{*}\{h_n(t)\} = \frac{1}{s^{n+1}}$

3. We need to prove that statement is true for $k = n+1$.

$$L^{*}\{h_{n+1}(t)\} = L^{*}\left\{\int_0^t h_n(\tau)\,\Delta\tau\right\} = \frac{1}{s} L^{*}\{h_n(t)\}$$

$$= \frac{1}{s}\left(\frac{1}{s^{n+1}}\right) = \frac{1}{s^{n+2}}$$

Hence from 1, 2 and 3 we find that

$$L\{h_k(t)\} = \frac{1}{s^{k+1}},$$

provided $s \neq 0$ and $\lim_{t\to\infty} h_k(t)\, k_{\ominus s}(t) = 0$.

Example (3-7):

Find the right Laplace transform of $f(t) = u_a(t)$

where $u_a(t) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$ for all $t \in T$

Solution:

$$L^{*}\{u_a(t)\} = \int_0^{\infty} u_a(t)\, k_{\Theta_s}(\sigma(t))\, \Delta t = \int_a^{\infty} k_{\Theta_s}(\sigma(t))\, \Delta t$$

$$= \frac{k_{\Theta_s}(a)}{s}, \qquad \text{provided } \lim_{t\to\infty} K_{\Theta_s}(t) = 0.$$

Examples (3-8):

. Solve the intial value problems

$$y^{\Delta\Delta\Delta}(t) + y^{\Delta}(t) = k_1(t), \; y(0) = y^{\Delta}(0) = y^{\Delta\Delta}(0) = 0 \qquad (3\text{-}5)$$

Solution:

Take the right Laplace transform of both sides of (3-5) then we have

$s^3 L^{*}\{y(t)\} + sL^{*}\{y(t)\} = \frac{1}{s-1}$ and hence

$$L^{+}\{y(t)\} = \frac{1}{(s^3+s)(s-1)}$$

using partial fractions;

we have $\quad L^{*}\{y(t)\} = \frac{-1}{s} + \frac{\frac{1}{2}}{s-1} + \frac{\frac{1}{2}s - \frac{1}{2}}{s^2+1}$

So that

$$y(t) = -1 + \frac{1}{2}k_1(t) + \frac{1}{2}\cos_1(t) - \frac{1}{2}\sin_1(t)$$

Theorem (3-6) (Convolution property):

Let $f(t): T \to C$ be a generalized exponential, hyperbolic, trigonometric, or polynomial function and let $g(t): T \to C$ be regulated function then;

$$L^{+}\{f(t) * g(t)\} = L^{+}\{f(t)\} \; L^{+}\{g(t)\}, \text{ on } D\{f * g\}$$

Proof:

Case 1:

If $f(t) = k_{\alpha}(t, \sigma(t))$

then the convolution of $k_{\alpha}(t, \sigma(t))$ and $g(t)$ is

$$k_{\alpha}(t, \sigma(t)) * g(t) = \int_{0}^{t} k_{\alpha}(t, \sigma(\tau)) \, g(\tau) \, \Delta\tau = y(t) \qquad (3\text{-}6)$$

Now, since the integral (3-6) is the unique solution of the initial value problem

$$y^{\Delta}(t) - \alpha \, y(t) = g(t), \qquad y(0) = 0$$

for regressive $\alpha \in C$ and regulated function $g(t): T \to C$ then we have

$$L^{+}\{g(t)\} = L^{+}\{y^{\Delta}(t) - \alpha \, y(t)\} = s \, L^{+}\{y(t)\} - \alpha L^{+}\{y(t)\}$$

or $\quad L^{+}\{g(t)\} = (s - \alpha) \, L^{+}\{y(t)\}$

or $\quad L^{+}\{y(t)\} = \dfrac{1}{s - \alpha} \, L^{+}\{g(t)\}$

that is

$$L^{*}\{k_{\alpha}(t,\sigma(t)) * g(t)\} = L^{*}\{k_{\alpha}(t,\sigma(t))\} \ L^{*}\{g(t)\} \ \square .$$

Case 2:

If $f(t) = \sin h_{\alpha}(t, \sigma(t))$

Then we define the convolution of $\sin h_{\alpha}(t.\sigma(t))$ and $g(t)$ by

$$\sin h_{\alpha}(t,\sigma(t)) * g(t) = \int_{0}^{t} \sin h_{\alpha}(t.\sigma(\tau))\, g(\tau)\, \Delta\tau$$

$$L^{+}\left\{ \int_{0}^{t} \sin h_{\alpha}(t.\sigma(\tau))\, g(\tau)\, \Delta\tau \right\} =$$

$$L^{*}\left\{ \int_{0}^{t}\left[\frac{k_{\alpha}(t,\sigma(\tau)) - k_{-\alpha}(t,\sigma(\tau))}{2} \ g(\tau) \right] \Delta\tau \right\} =$$

$$L^{+}\left\{ \int_{0}^{t} \frac{k_{\alpha}(t,\sigma(\tau))}{2} \ g(\tau)\,\Delta\tau \right\} - L^{+}\left\{ \int_{0}^{t} \frac{k_{-\alpha}(t,\sigma(\tau))}{2} \ g(\tau)\,\Delta\tau \right\} =$$

$$\frac{1}{2} L^{*}\{k_{\alpha}(t,\sigma(t))\} . L^{*}\{g(t)\} - \frac{1}{2} L^{*}\{k_{-\alpha}(t,\sigma(t))\} L^{*}\{g(t)\} =$$

$$L^{*}\left\{ \frac{k_{\alpha}(t.\sigma(t) - k_{-\alpha}(t,\sigma(t)}{2} \right\} \ L^{*}\{g(t)\} =$$

$$L^{*}\{\sin h_{\alpha}(t,\sigma(t))\} \ L^{*}\{g(t)\} \qquad \square .$$

Similarly:

If $f(t) = \cos h_{\alpha}(t,\sigma(t))$, $f(t) = \cos_{\alpha}(t.\sigma(t))$ or $f(t) = \sin_{\alpha}(t,\sigma(t))$

then $L^{*}\{f(t) * g(t)\} = L^{*}\{f(t)\} \ L^{*}\{g(t)\}$

Case 3:

If $f(t) = h_{n}(t,\sigma(t))$

Then we define the convolution of $h_n(t, \sigma(t))$ and $g(t)$ to be

$$h_n(t, \sigma(t)) * g(t) = \int_0^t h_n(t, \sigma(\tau))\, g(\tau)\, \Delta\tau = y(t) \qquad (3\text{-}7)$$

since the integral (3-7) is the unique solution of the initial value problem

$$(y^{\Delta})^{n+1}(t) = g(t),\ (y^{\Delta})^{i}(r) = 0,\ \ 0 \le i \le n,\ \ r \in T$$

then we have

$$L^{+}\{g(t)\} = L^{+}\left\{(y^{\Delta})^{n+1}(t)\right\} = s^{n+1}\, L^{+}\{y(t)\}$$

or

$$L^{+}\{y(t)\} = \frac{1}{s^{n+1}}\, L^{+}\{g(t)\}$$

that is

$$L^{+}\{h_n(t, \sigma(t)) * g(t)\} = L^{+}\{h_n(t, \sigma(t))\}\ L^{+}\{g(t)\} \quad \square .$$

3-3 The generalized inverse of Laplace transform

After having studied the preceding sections (1-8), (2-9), (3-1) and (3-2), the generalized inverse of Laplace transform denoted by $L_{\kappa}^{-1}\{F(s)\}$ can be defined by following integrals.

1- For $t > 0$

$$L_{\kappa}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} F(s)\ K_t(t)\, ds \qquad (3\text{-}8)$$

2- For $t < 0$

$$L_{\kappa}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} F(s)\ R_{\Theta_{\kappa}}(t)\, ds \qquad (3\text{-}9)$$

Remarks (3-4):

1- If $T = R$ then the integrals (3-8) and (3-9) become

$$f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)\, e^{st}\, ds$$

2- If $T = Z$ then the integral (3-8) becomes

$$f(n) = \frac{1}{2\pi i}\oint_c F(s)\,(1+s)^n\, ds$$

and the integral (3-9) becomes

$$f(n) = \frac{1}{2\pi i}\oint_c F(s)\,(1+s)^{-n}\, ds$$

Note that: The generalized inverse of Laplace transform is used to take the function $F(s)$ from a complex variable domain into T-domain, therefore the integrals (3-8) and (3-9) can be evaluated by the method of residues.

Example (3-9):

If $\quad F(s) = \frac{1}{s-\alpha}$, $\mathrm{Re}(s) > \alpha$, then find $L_t^{-1}\{F(s)\}$

Solution:

$$L_t^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{1}{s-\alpha}\, k_s(t)\, ds$$

$$= \mathrm{Res}\, \frac{K_s(t)}{s-\alpha}\bigg|_{s=\alpha} = \lim_{s\to\alpha}(s-\alpha)\,\frac{K_s(t)}{s-\alpha} = K_\alpha(t)$$

Example (3-10):

If $\quad F(s) = \frac{\alpha}{s-\alpha^2}$, $\mathrm{Re}(s) > 0$, then find $L_t^{-1}\{F(s)\}$

Solution:

$$L_t^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\alpha}{s-\alpha^2}\, K_s(t)\, ds$$

$$= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{1}{2i} \left(\frac{1}{s-\alpha i} - \frac{1}{s+\alpha i} \right) K_s(t)\, ds$$

$$= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{1}{2i} \frac{K_s(t)}{s-\alpha i}\, ds - \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{1}{2i} \frac{K_s(t)}{s+\alpha i}\, ds$$

thus, $f(t) = \frac{1}{2i} [K_{\alpha i}(t) - K_{-\alpha i}(t)] = \sin_\alpha(t)$

Example (3-11):

If $\quad F(s) = \frac{1}{s^{n+1}}$, $\quad \mathrm{Re}(s) > 0$, then find $L_g^{-1}\{F(s)\}$.

Solution:

If $\quad n = 0$ then $\quad f(t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{1}{s} K_s(t)\, ds$

$$= \mathrm{Res}\, \frac{K_s(t)}{s}\bigg|_{s=0} = K_0(t) = 1 = h_0(t). \qquad \text{(see appendix II)}$$

Also;

If $\quad n = 1$ then $\quad f(t) = \mathrm{Res}\, \frac{K_s(t)}{s^2}\bigg|_{s=0} = \lim_{s\to 0} \left\{ \frac{d}{ds} \left[s^2 \frac{K_s(t)}{s^2} \right] \right\}$

$$= \lim_{s\to 0} t\, k_s(t) = t = \int_0^t h_0(t)\ \Delta t = h_1(t)$$

We obtain for any $n \geq 2$

$$f(t) = \mathrm{Res}\, \frac{K_s(t)}{s^{n+1}}\bigg|_{s=0} = \lim_{s\to 0} \left\{ \frac{1}{n!} \frac{d^n}{ds^n} \left[s^{n+1} \frac{K_s(t)}{s^{n+1}} \right] \right\}$$

$$= \lim_{s\to 0} \frac{1}{n!} t^n\ K_s(t) = \int_0^t h_{n-1}(t)\ \Delta t = h_n(t)$$

thus, $L_g^{-1} \left\{ \frac{1}{s^{n+1}} \right\} = h_n(t)$

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APPENDIX I

Definition 1:

A function $f(t)$ has discontinuity at point a if

$$f(a^+) = \lim_{t \to a^+} f(t) \neq f(a^-) = \lim_{t \to a^-} f(t)$$

Definition 2:

A function $f(t)$ is piece- wise continuous on the interval $[a,b]$ if

(i) Both limits $\lim_{t \to a^+} f(t)$ and $\lim_{t \to b^-} f(t)$ exist

(ii) $f(t)$ is continuous on (a,b) except possibly at a finite numbers $a_0, a_1, \ldots, a_n$ in (a,b).

Definition 3:

A function $f(t)$ has exponential order α if there exists contants $M > 0$ and α such that

$$|f(t)| \leq M e^{\alpha t}$$

Dirichlet conditions

1. $f(t)$ is periodic function with period T .
2. $f(t)$ and $f'(t)$ are piece- wise continuous on the intral $(\alpha , \alpha+T)$.
3. $f(t)$ is absolutely integerable on $(\alpha , \alpha+T)$ that is $\int_{\alpha}^{\alpha+T} |f(t)| \, dt < \infty$.
4. At any point of discontinuity a in $(\alpha , \alpha+T)$, $f(a) = \dfrac{f(a^+) + f(a^-)}{2}$.

Definition 4:

We say that a complex function $f(z)$ defined on a domain D is differentiable at a point $z_0 \in D$ if the limit

$$\left.\frac{d f(z)}{d z}\right|_{z=z_0} = f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists.}$$

If $f(z)$ is differentiable at all points of some neighborhood $|z - z_0| < r$, then $f(z)$ is said to be analytic at z_0.

If $f(z)$ is analytic at each point of a domain D, then $f(t)$ is anatytic in D.

Definition 5:

If $f(z) = u(x,y) + iv(x,y)$ is defined in domain D and the partial derivatives.

$\frac{\partial u}{\partial x}, \frac{\partial u}{d y}, \frac{\partial v}{\partial x}, \frac{\partial v}{d y}$ are continuous and satisfy the cauchy- Riemann equation that is

$$\frac{\partial u}{\partial x} = \frac{\partial v}{d y} \quad , \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

Then $f(z)$ is analytic in D.

Definition 6:

a. A curve is closed if its end points coincide.

b. A closed curve is simple closed curve if it does not intersect it self.

c. A region is called simple connected region if any simple closed curve in region can be shrunk to a point of the same region.

Cauchy's integral formula:

If $f(z)$ is analytic within and on a simple closed curve C and a is any point interior to C then

$$f(a)=\frac{1}{2\pi i}\oint_{c}\frac{f(z)}{z-a}\,dz$$

where C is traversed in the positive direction (counter clock wise).

The nth derivative of $f(z)$ at $z=a$ is given by

$$f^{(n)}(a)=\frac{n!}{2\pi i}\oint_{c}\frac{f(z)}{(z-a)^{n+1}}\,dz$$

Taylor's series:

Let $f(t)$ be analytic inside and on a circle having its center at $z=a$ then for all points z in the circle we have the Taylor series representation of $f(z)$ given by

$$f(z)=f(a)+f'(a)(z-a)+\frac{f''(a)}{2!}(z-a)^2+\ldots\ldots$$

$$=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(z-a)^n$$

Definition 7:

A singular point of function $f(z)$ is a value of z at which $f(z)$ fails to be analytic.

If $f(z)$ is analytic everywhere in some region except at an interior point $z=a$ we call $z=a$ and isolated singularity of $f(z)$.

Definition 8:

If $f(z)=\frac{\phi(z)}{(z-a)^n}$, $\phi(a)\neq 0$ where $\phi(z)$ is analytic everywhere in a region including $z=a$ and if n is a positive integer then $f(z)$ has an isolated singularity at $z=a$ which is called a pole of order n, if $n=1$ then it is called a simple pole.

Leurent's series:

If $f(z)$ has a pole of order n at $z=z_0$ but is analytic at every other point inside and on a circle with center at z_0 then $(z-a)^n\ f(z)$ is analytic at all points inside and on C and has a Taylor series about $z=z_0$, it is also called Laurent series in this case.

So that Laurent series for $f(z)$ is:

$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_{-n}}{(z-z_0)^n} \tag{I-1}$$

where

$$a_n=\frac{1}{2\pi i}\oint_c \frac{f(z)\,dz}{(z-z_0)^{n+1}}, \qquad (n=0,1,2,\dots)$$

and

$$b_{-n}=\frac{1}{2\pi i}\oint_c \frac{f(z)\,dz}{(z-z_0)^{-n+1}}; \qquad (n=1,2,\dots)$$

Laurent series (I-1) is often written

$$f(z)=\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$

where

$$C_n = \frac{1}{2\pi i}\oint \frac{f(z)\,dz}{(z-z_0)^{n+1}} \quad , \qquad (n = 0, \pm 1, \pm 2, \ldots\ldots)$$

Residue theorem:

Let C be a positively oriented simple closed contour.

If a function $f(z)$ is analytic inside and on C except for a finite number of singular points z_k, $(k = 1, 2, \ldots, n)$ inside C, then

$$\oint_c f(z)\,dz = 2\pi i \sum_{k=1}^{n} \text{Res}\, f(z).$$

where $\text{Res}\, f(z)$ denote the residues of function $f(z)$.

APPENDIX II

Theorem 1:

a) If $f(t)$ is continuous at t and $\sigma(t) = t$,

then $f^{\Delta}(t) = \lim_{u \to t} \dfrac{f(t) - f(u)}{t - u}$

b) If $f(t)$ is continuous at t and $\sigma(t) \neq t$,

then $f^{\Delta}(t) = \dfrac{f(\sigma(t)) - f(t)}{\mu(t)}$

Proof:

a) By definition (3-2) we have

$$\left| \frac{f(\sigma(t)) - f(u)}{\sigma(t) - u} - f^{\Delta}(t) \right| \leq \varepsilon \qquad . \ \forall\, u \in N(t) \text{ (neighbohood)}$$

or

$$f^{\Delta}(t) = \lim_{u \to t} \frac{f(\sigma(t)) - f(u)}{\sigma(t) - u}$$

Since $\sigma(t) = t$, then $f^{\Delta}(t) = \lim_{u \to t} \dfrac{f(t) - f(u)}{t - u}$ □. (II-1)

b) Similarly, $f^{\Delta}(t) = \lim_{u \to t} \dfrac{f(\sigma(t)) - f(u)}{\sigma(t) - u}$

Since $\sigma(t) \neq t$, then $f^{\Delta}(t) = \dfrac{f(\sigma(t)) - f(t)}{\mu(t)}$ □. (II-2)

Remarks 1:

a) If $T = R$ then $f^{\Delta}(t) = \lim_{u \to t} \dfrac{f(t) - f(u)}{t - u} = f'(t)$

Remarks 2:

a) If $T = R$, then $f^{\nabla}(t) = f'(t)$

b) If $T = Z$, then $f^{\nabla}(n) = \dfrac{f(n) - f(n-1)}{1} = f(n) - f(n-1)$

Theorem 3:

If $f(t), g(t)$ are delta (Δ) differentiable at t then

a) $[f(t) + g(t)]^{\Delta} = f^{\Delta}(t) + g^{\Delta}(t)$

b) $[f(t)\ g(t)]^{\Delta} = f(t)\, g^{\Delta}(t) + f^{\Delta}(t)\, g(\sigma(t))$

c) $\left[\dfrac{f(t)}{g(t)}\right]^{\Delta} = \dfrac{f^{\Delta}(t)\, g(t) - f(t)\, g^{\Delta}(t)}{g(t)\ g(\sigma(t))}$, provided $g(t)\, g(\sigma(t)) \neq 0$

d) If $f^{\Delta}(t)$ is continuous, then

$$\left[\int_{s}^{t} f(t, \tau)\ \Delta\tau\right]^{\Delta} = f(\sigma(t), t) + \int_{s}^{t} f^{\Delta}(t, \tau)\ \Delta\tau$$

Proof:

Case 1: If $\sigma(t) \neq t$, then by (II-2) we have

a) $$[f(t) + g(t)]^{\Delta} = \frac{f(\sigma(t)) + g(\sigma(t)) - [f(t) + g(t)]}{\mu(t)}$$

$$= \frac{f(\sigma(t)) - f(t)}{\mu(t)} + \frac{g(\sigma(t)) - g(t)}{\mu(t)}$$

$$= f^{\Delta}(t) + g^{\Delta}(t) \qquad \square .$$

Case 2: If $\sigma(t) = t$, then by (II-1) we have

a) $$[f(t) + g(t)]^{\Delta} = \lim_{u \to t} \frac{f(t) + g(t) - [f(u) + g(u)]}{t - u}$$

$$= f^{\Delta}(t) + g^{\Delta}(t) \quad \square .$$

b) $$[f(t)\, g(t)]^{\Delta} = \lim_{u \to t} \frac{g(t)\, f(t) - f(u)\, g(u)}{t - u}$$

$$= f(t)\, g^{\Delta}(t) + g(t)\, f^{\Delta}(t) \quad \square .$$

c) $$\left[\frac{f(t)}{g(t)}\right]^{\Delta} = \lim_{u \to t} \frac{f(t)/g(t) - f(u)/g(u)}{t - u}$$

$$= \frac{g(t)\, f^{\Delta}(t) - f(t)\, g^{\Delta}(t)}{(g(t))^2} \quad \square .$$

d) $$\left[\int_s^t f(t, \tau)\Delta\tau\right]^{\Delta} = \lim_{u \to t} \frac{\int_s^t f(t,\tau)\,\Delta\tau - \int_s^u f(u,\tau)\,\Delta\tau}{t - u}$$

$$= \int_s^t f^{\Delta}(t,\tau)\,\Delta t \quad \square .$$

Theorem 4:

If $f(t)$ and $g(t)$ are nable (∇) differentiable at t then

a) $[f(t) + g(t)]^{\nabla} = f^{\nabla}(t) + g^{\nabla}(t)$

b) $[f(t)\; g(t)]^{\nabla} = f^{\nabla}(t)\; g(t) + f(\rho(t))\, g^{\nabla}(t)$

c) $\left[\dfrac{f(t)}{g(t)}\right]^{\nabla} = \dfrac{f^{\nabla}(t)\, g(t) - f(t)\, g^{\nabla}(t)}{g(t)\; g(\rho(t))}$ provided $g(t)\, g(\rho(t)) \neq 0$

d) If $f^{\nabla}(t)$ is continuous then $\left[\int_s^t f(t, \tau)\,\nabla\tau\right]^{\nabla} = f(\rho(t), t) + \int_s^t f^{\nabla}(t,\tau)\,\nabla\tau$

Proof:

Case 1: If $\rho(t) \neq t$, then by (II-4) we have

a) $$[f(t)+g(t)]^{\nabla} = \frac{f(t)+g(t) - [f(\rho(t))+g(\rho(t))]}{\eta(t)}$$

$$= f^{\nabla}(t) + g^{\nabla}(t) \quad \Box .$$

b) $$[f(t)\,g(t)]^{\nabla} = \frac{f(t)\,g(t) - [f(\rho(t))\,g(\rho(t))]}{\eta(t)}$$

$$= f^{\nabla}(t)\,g(t) + f(\rho(t))\,g^{\nabla}(t) \quad \Box .$$

c) $$\left[\frac{f(t)}{g(t)}\right]^{\nabla} = \frac{f(t)/g(t) - f(\rho(t))/g(\rho(t))}{\eta(t)}$$

$$= \frac{f^{\nabla}(t)\,g(t) - f(t)\,g^{\nabla}(t)}{(g(t)\,g(\rho(t))} \quad \Box .$$

d) $$\left[\int_s^t f(t,\tau)\,\nabla\tau\right]^{\nabla} = \frac{\int_s^t f(t,\tau)\,\nabla\tau - \int_s^{\rho(t)} f(\rho(t),\tau)\,\nabla\tau}{\eta(t)}$$

$$= \frac{\int_s^t f(t,\tau)\,\nabla\tau - \left[\int_s^t f(\rho(t),\tau)\,\nabla\tau - \int_{\rho(t)}^t f(\rho(t),\tau)\,\nabla\tau\right]}{\eta(t)}$$

$$= \frac{\int_s^t [f(t,\tau) - f(\rho(t),\tau)]\,\nabla\tau}{\eta(t)} + \frac{\int_{\rho(t)}^t f(\rho(t),\tau)\,\nabla\tau}{\eta(t)}$$

$$= \int_s^t f^{\nabla}(t,\tau)\,\nabla\tau + f(\rho(t),t) \quad \Box .$$

Case 2: If $\rho(t) = t$, then by (II-3) we have

a) $$[f(t) + g(t)]^{\nabla} = \lim_{u \to t} \frac{f(t) + g(t) - [f(u) + g(u)]}{t - u}$$

$$= f^{\nabla}(t) + g^{\nabla}(t) \qquad \square .$$

b) $$[f(t)\ g(t)]^{\nabla} = \lim_{u \to t} \frac{f(t)\,g(t) - f(u)\ g(u)}{t - u}$$

$$= f^{\nabla}(t)\ g(t) + f(t)\ g^{\nabla}(t) \qquad \square .$$

c) $$\left[\frac{f(t)}{g(t)}\right]^{\nabla} = \lim_{u \to t} \frac{f(t)/g(t) - f(u)/g(u)}{t - u}$$

$$= \frac{g(t)\ f^{\nabla}(t) - f(t)\ g^{\nabla}(t)}{(g(t))^2} \qquad \square .$$

d) $$\left[\int_s^t f(t, \tau)\ \nabla\tau\right]^{\nabla} = \lim_{u \to t} \frac{\int_s^t f(t,\tau)\ \nabla\tau - \int_s^u f(u, \tau)\ \nabla\tau}{t - u}$$

$$= \int_s^t f^{\nabla}(t, \tau)\ \nabla\tau \qquad \square .$$

Remarks 3:

We can define $k_0(t)$ as $h_0(t)$ $(k_0(t) = 1 = h_0(t))$

To see this we have $k_p(t)$ is a solution of $y^{\Delta}(t) = p(t)y$, $y(0) = 1$

and if $p(t) = 0$ then, $y^{\Delta}(t) = 0$, $y(0) = 1$

Therefore $k_0(t) = 1 = h_0(t)$ •

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كليــــة العلــــوم
قسـم الرياضيات

عنـــوان البحـــث

((تحــويلات لابلاس و تحويلات Z – نظـــريات وتطبيقـــات))

سـرت / ليبيـا

مقدمـــة مــن الطـــالب

صـــلاح علـــي ضـــو حمد الفرجاني

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مقدم كجــزء من متطلبات استكمال درجة الماجستير في العلوم قسـم الرياضيات

صلاح علي ضو 032308

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