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ON HÜBLER AXIOMATIC DISCRETE GEOMETRY

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((On Hubler Axiomatic Discrete Geometry))

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بِسْمِ اللَّهِ الرَّحْمَٰنِ الرَّحِيمِ

هُوَ الَّذِي جَعَلَ الشَّمْسَ ضِيَاءً وَالْقَمَرَ نُورًا وَقَدَّرَهُ مَنَازِلَ
لِتَعْلَمُوا عَدَدَ السِّنِينَ وَالْحِسَابَ مَا خَلَقَ اللَّهُ ذَٰلِكَ إِلَّا
بِالْحَقِّ يُفَصِّلُ الْآيَاتِ لِقَوْمٍ يَعْلَمُونَ {5}

من سورة يونس عليه السلام

Dedication

To my mother
my father, and my brothers.
Also my friends.

Anees .J. Fadeel

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I am grateful to my teacher professor

Dr. Ibrahim A. Tentush

Who teach; supervise and guide me during training period.

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B.Sc

Anees .I. Fadeel

Contents

Chapter four

Affine geometry , generators and isomorphism

Abstract

In this thesis we analyze Hűbler's axiomatic discrete geometry, one of the few of its kind—perhaps the only one. The system is characterized in terms of torsion free Z-modules satisfying some so-called generator properties. The new acclimatization obtained is arguably easier to understand than the original one, and the work casts new light on different properties of Hűbler's geometries. His system turns out to be too restricted for our purposes, but the results indicate some ways in which to continue this thread of work.

Introduction

Hűbler's geometry, Hűbler has developed an axiom system with the intention to capture the essence of discrete geometry as utilized in image processing and computer graphs, "Hűbler axiomatic discrete geometry can be characterized in terms of modules over the rings of integers " [26]

This thesis is mainly about Hűbler axiomatic discrete geometry. The main intended application of this subject is image analysis and manipulation, computational geometry, and related fields.

"Currently the most commonly used approaches to these areas are continuous instead of discrete, continuous approaches are plagued by the inherent finiteness of computing hardware " [25]. In view of this we can argue that the proper framework for many algorithms is not continuous, but discrete , furthermore , " It is preferable if such a framework is axiomatically defined , so that the essential properties of the system are clearly stated and many models can share the same theory "[25].

In chapter zero we introduce an introduction to different kinds of geometry, chapter one background material of the later chapter's, chapter two we study Hűbler's work on discrete geometry, chapter three we explores some closure operator's defined on modules over integral domain and , the associated matroids and geometry, and chapter four contains the affine geometry , generators and isomorphisms .

The aim of this work is to show that matroid methods can be applied to many discrete geometrics, "namely those based on modules over integral domain, the trick is to emulate the structure of a vector space within the module there by allowing matroid methods to be used as if the module were a vector space"[17].

CHAPTER ZERO

INTRODUCTION TO DIFFERENT KINDS OF GEOMETRY

Chapter zero
"Introduction to different kinds of geometry"

Geometry existed long before the time of Euclid. In this chapter we introduce an introduction to different kinds of geometry.

The references of the following materials are [1],[9],[11], [24],[29],[32].

0-1: Euclidean geometry:

Euclid (300 BC) established five axioms for geometry, and then showed that every result in his text could be proven from those axioms.

Euclidean axioms can be stated as follows:

I. For every point P and for every point Q not equal to P there exist a unique line l that passes through P and Q .

II. For every segment of line AB and for every segment of line CD there exists a unique point E such that B is between A and E and segment CD is congruent to segment BE.

III. For every point O and every point A not equal to O there exists a circle with center O and radius OA.

IV. All right angles are congruent to each other.

Before stating Euclid's fifth axiom, it is necessary to state a formal definition.

Definition (0-1-1) (Parallel)

Two lines l and l' are **parallel** if they do not intersect, i.e. if no point lies on both of them .We denote this by $l \| l'$.

We are now ready to state Euclid's fifth axiom, sometimes referred to as the *parallel axiom*

V. For every line l and for any point P that does not lie on l there exists a unique line l' through P that is parallel to l.

0-2: Incidence geometry:

In an attempt to fill in some of the gaps in Euclid's postulates, we consider these three postulates pertaining to only *incidence*

Axioms of the incidence geometry:

1- For every point P and for every point Q not equal to P there exists a unique line *l* that passes through (is incident with) P and Q.

2- For every line *l* there exists at least two distinct point incidents with *l*.

3- There exist three distinct points with the property that no line is incident with all three of them.

The incidence axioms are the first of the set of axioms introduced by ***David Hilber***, a renowned mathematician of the early 1900s.

0-3: Neutral geometry:

Neutral geometry, sometimes called absolute geometry.

0-3-1 The Incidence Axioms

Points, lines and planes are undefined terms.

1- All lines and planes are sets of points.

2- Given any two points, there is exactly one line containing them.

3- Given any three noncollinear points, there is exactly one plane containing them.

4- If two points lie in a plane E, then the line containing the two points lies in E.

5- If two planes intersect, their intersection is a line.

6- Every line contains at least two points and there are at least three noncollinear points. Every plane contains at least three noncollinear points and there are at least four noncoplanar points.

0-3-2 The Betweenness Axioms

The notion of betweenness, as in "the point B is between the points A and C", is taken to be an undefined term. If a point B is between the points A and C, we write A − B − C.

It is assumed that betweenness satisfies the following five axioms:

1- If A − B − C, then C − B − A. (If B is between A and C, then B is between C and A.)

2- Given three collinear points A,B and C, then exactly one of the following is true: A − B − C, B − A − C, or A − C − B. (Given any three collinear points, exactly one point is between the other two.)

Definition 0-3-1(four collinear points)

Let A,B,C, and D be *four collinear points*. We write A−B− C −D provided each of the following relations hold: A−B −C, A−B −D, A − C − D, and B − C − D.

3- Any four points of a line can be named in order A,B,C, and D, in such a way that A − B − C − D.

4- If A and B are any two points, then there is a point D such that A − B − D and there is a point C such that A − C − B.

5- If A − B − C, then A,B, and C are collinear.

We are now ready to introduce the definition of line segment, ray and angle.

Definition 0-3- 2 (line segment)

If A and B are two points, the line segment between A and B is the set points between A and B together with A and B.

This line segment is usually denoted AB.

Note that: AB = {C : A − C − B} ∪{A} ∪ {B}.

Definition 0-3-3 (ray)

Let A,B be two points. The ray from A through B, denoted $\overrightarrow{AB}$, is the set of all points C on $\overleftrightarrow{AB}$ such that C − A − B does not hold. The point A is called the end point of the ray. If the point A is in a ray $\overrightarrow{AB}$ then is *closed ray*, and if the point A is not in a ray $\overrightarrow{AB}$ then is *open ray*.

Definition 0-3-4 (angle)

An angle is the union of two rays which have the same endpoint but do not lie on the same line. If the angle is the union of the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$, the angle is denoted by $\angle$BAC. The rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are called the sides of the angle and the point A is called the vertex of the angle.

Here are a couple of sample theorems which can be established using the above axioms and definitions.

Theorem 0-3-1

If A and B are two points, then $\overline{AB} = \overline{BA}$

Theorem0-3-2

Let $\overrightarrow{AB}$ be a ray. If C is a point on $\overrightarrow{AB}, C \neq A$, then $\overrightarrow{AB} = \overrightarrow{AC}$

0-3-3 The plane separation axiom

Definition0-3-5 convex

A set K is said to be **convex** provided that the line segment $\overline{AB}$ is contained in K whenever the points A and B are in K.

1- (*Plane separation axiom*) Given a line and a plane containing it, the set of all points of the plane that do not lie on the line is the union of two disjoint sets H_1 and H_2 such that each of the sets is convex and if P belongs to one of the sets and Q belongs to the other, then the segment $\overline{PQ}$ intersects the line. Each of these sets is called a *half plane*.

0-3-4 ***Angle Measurement Axioms***

We let m(∠ABC) denote the measure of the angle ∠ABC. It is assumed that the measure of an angle is a real number.

1- Given an angle ∠ A, m assigns one and only one real number to ∠A.
2- For every angle ∠ A, m(∠A) is between 0 and 180.
3- (Angle Construction Axiom) Let $\overrightarrow{AB}$ be a ray on the edge of a half plane H. For every number α between 0 and 180 there is exactly one ray $\overrightarrow{AC}$ in H such that m(∠BAC) = α.
4- (Angle Addition Postulate). If D is in the interior of ∠ABC, then m(∠ABC) = m(∠ABD) +m(∠DBC).

Definition 0-3-5(opposite pair)

If B −A− C, then the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ form an *opposite pair.*

Definition 0-3-6(linear pair)

Let $\overrightarrow{AB}$ and $\overrightarrow{AC}$ be opposite rays and D a point not on the line $\overline{CB}$. Then ∠BAD and ∠DAC form *a linear pair.*

Definition 0-3-7(supplementary and complementary angles)

If m(∠ABC) + m(∠DEF) = 180, then ∠ABC and ∠DEF are called *supplementary angles* .If m(∠ABC) + m(∠DEF)=90, then ∠ABC and ∠DEF are called *complementary angles.*

5- (Supplement Axiom) If two angles form a linear pair, then they are Supplementary.

In a metric geometry, two angles are defined to be congruent if they have the same measure.

Definition 0-3-8 (right angle)

If the angles of a linear pair are congruent, then each of them is called a *right angle.*

Definition 0-3-9 (perpendicular)

Two rays are called *perpendicular* if their union is a right angle. An angle is acute if its measure is less than 90° and obtuse if their measure is greater than 90^0.

0-4: Non- Euclidean geometry

In the previous section, we considered a geometry without the parallel postulate. Now we will consider a geometry under the negation of the parallel axiom "the five axiom of Euclid".

Under scrutiny and controversy for 2000 years.

Some tried to prove the parallel axiom from (section 0-1: I-IV) thus no need to assume it- failed .

Saccheri (1667-1733) assumed the negation of the parallel axiom and obtained strange results .

Gauss believed that negation the axiom would not lead to a contradiction. **Gauss**, **Bolyai**, and **Lobachevsky** independently developed a noncontradictory geometry in which the parallel postulate is false **Beltrami** (1900-1935) and **Klein** (1849-1925) produced models within *Euclidean Geometry* of the geometry of *Bolyai* and *Lobachevsky* – now called **Hyperbolic Geometry** Satisfied (see sec 0-1) axioms (I-IV) except V implied that proof of V was impossible ,Euclid's fifth axiom can fail if we have "too many parallel lines through a point". There exists a line *l* and a point P not on *l* ,such that at least two distinct lines parallel to *l*, pass through P - the Hyperbolic Axiom . Geometry built on I-IV and Hyperbolic axiom called **HYPERBOLIC GEOMETRY.**

Also "not enough parallel lines through a point". There exists a line *l* and a point P not on *l*, such that there are no lines parallel to *l* which pass through P - the **Elliptic Axiom** ,Leads to **ELLIPTIC GEOMETRY**

Non-Euclidean Geometry includes both *Hyperbolic* and *Elliptic Geometry*. Any theorem of Euclid not using V led to Absolute Geometry (see 0-3) both holding in Euclidean ,and Hyperbolic Geometry .Theorems based on (I, II) and V became known as **Affine Geometry**

Leonardo da Vinci (1452-1519): Problems faced by artists of projected objects onto a canvas where lengths are distorted relative to the objects around them. How can structure still be recognized?. Some geometric properties invariant under central projection – **Projective Geometry**.

Descartes (1596-1650) allowed geometries to make use of Algebra and Calculus making calculations simpler – e.g. theory of conics – **Analytic Geometry.**

Analytic geometry

Analytic geometry is concerned with the representation of internal properties of geometric objects by algebraic equations of points. For the present purposes we identify points as vectors

$$P = P_0 + X \text{ ....(1)}$$

which are bound to the specified reference point P_0 (see (1)). Furthermore the distance of the point p from the origin is now given as $|P| = |X|$,where it follows that $P_2 = (P_0 + X)^2 = X^2$ or $P_0^2 + P_0X + X P_0 = 0$, which in turn implies that $P_0^2 = 0$,and $P_0X = -XP_0$, for any vector X. In other words, P_0 is a null vector, which is orthogonal to any vector in E(N). In effect, the inclusion of P_0 expands the dimension of space to N + 1.

However, the internal properties of any geometric figure depend, not on specific points, but their difference. Since the difference $P_j - P_i$ is a free vector $X_j - X_i$, we can be formulate analytic geometry in terms of free vector variable X, and we need not concern us with the distinction between a point P and the vector P used to label it.

Riemann developed *elliptic geometry* and considered higher-dimensional Euclidean and spherical spaces leading to study of surfaces of any dimension – Riemannian Manifolds – **Differential geometry**

Used by **Wintein** in his general theory of relativity .

The topology capture essential under Riemann geometry [see [9]]

0-5: Projective geometry

In standard Euclidean geometry such as we have discussed so far, there is a rather inelegant situation. Two points always determine a line. However two lines in a 2-plane may determine a point if the lines intersect, but may not determine a point if the lines are parallel.

This somewhat cumbersome situation is resolved in projective geometry. This is done by asserting that "parallel lines meet at infinity."

Axioms of Projective Geometry

1. There exists at least one line.
2. On each line there are at least three points.
3. Not all points lie on the same line.
4. Two distinct points lie on one and only one line.
5. Two distinct lines meet in one and only one point.
6. There is a one-to-one correspondence between the real numbers and all points but one point on a line.

Theorem0-5-1 : (Fundamental Theorem of Projective Geometry):

There exists one and only one projective transformation mapping three distinct points on a line onto three distinct points on another line in a given order. The lines need not be distinct

0-6: The future work

The future work is a new geometry that is using in a technology i.e. computer screen and digital camera and image processing.... Etc, called discrete geometry or digital geometry, Hűbler has developed an axiom system with the intention to capture the essence of discrete geometry as utilized in image processing and computer graphs .This thesis is mainly about Hűbler axiomatic discrete geometry. The main intended application of this subject is image analysis and manipulation, computational geometry, and related fields...

CHAPTER ONE

BACKGROUND MATERIAL

Chapter one
"Background material"

This chapter contains background material of the later chapter's. Note that the treatment of matroids, antimatroids, and oriented matroids is somewhat nonstandard, as infinite ground set are allowed.

1-1: Notation and terminology:

The references of the following material are [25] ,[26], [27].

A few words on notation and terminology.

- If A is equal to B, then we sometimes make this explicit by writing A:=B .
- We let $A \subseteq_{fin} B$ be true iff A is a finite subset of B.
- $A \subset B$ is equivalent to ($A \subseteq B$ and $A \neq B$) , i.e. A is proper subset of B.
- The cardinality of the set A is denoted by $|A|$.
- The power set of A is denoted by $\mathcal{P}(A)$, and we let

  $\mathcal{P}_{fin}(A) := \{ B \mid B \subseteq_{fin} A \}$.
- When the thesis allows so the singleton set $\{x\}$ is often written simply as x.
- Set difference is always written using \ , never (-) , this because – is used for a different purpose .
- If $f: A \to B$ is a function then whenever there is no risk of confusion , f also denoted the function $f': \mathcal{P}(A) \to \mathcal{P}(B)$ defined by $f'(P) := \{ f(p) \mid p \in P \}$, $P \in \mathcal{P}(A)$.
- A subset $P \subseteq S$ is maximal with respect to S and some property iff the set P has the property and there is not any set $A \subseteq S$ with $P \subset A$,such that A has same property of P .
- A subset $P \subseteq S$ is minimal with respect to S and some property iff the set P has the property and there is not any set $A \subseteq S$ with $A \subset P$, such that A has same property of P .

- The set of natural numbers (non- negative integers) is denoted by $\mathbb{N}$
- The set of all integer numbers is denoted by $\mathbb{Z}$
- The set of rational numbers is denoted by $\mathbb{Q}$.
- The set of real numbers is denoted by $\mathbb{R}$.
- The notation X^+, where X is one of the above mentioned sets , stands for all positive elements of X.

1-2: Set theory and lattices

In this section we define any notations in a set theory and lattice . The references of the following materials are [3],[5],[7],[13],[28],[31]

Definition (1-2-1) (Partial- order relation)

If R is a relation on a set $M \neq \phi$ satisfy

1- reflexive (if for all $x \in M$, xRx)

2- transitive (if $x,y,z \in A$ |xRy and yRz then xRz)

3- anti-symmetric (for all $x,y \in A$, $(xRy) \wedge (yRx)$ then x=y

Then it's called *partial –order relation.*

Definition (1-2-2) (Comparable)

Any two elements x and y in a partial order are said to be *comparable* if $x \leq y$ or $y \leq x$.

Definition (1-2-3) (Total order)

If every element in a partial ordered set A are comparable then the partial ordered in A is called a *total* in A

Definition (1-2-4) (Poset)

A nonempty set with a partial order on it, is called *poset*.

Definition (1-2-5) (Linearly ordered)

A poset (P,≤) is called – *linearly ordered* set or a chain if for all x,y∈P either x≤y or y≤x .

Definition (1-2-6) (Well-ordered)

Let (A,≤) be linearly ordered . the (A,≤) is *well –ordered* iff any nonempty subset of A contains a last element.

Definition (1-2-7) (Equivalence relation ~)

If R is a relation on a set M≠ ϕ satisfy:

1- reflexive (if for all x∈M , xRx)

2- symmetric (for all x,y ∈A, (xRy) then (yRx)

3- transitive (if x,y,z∈A |xRy and yRz then xRz)

Then it's called *Equivalence relation*

Definition (1-2-8) (Equivalence class)

A equivalence class Ex of x with respect to the relation ~ is the set of all elements y of M : x~y and Ex= {y∈M: x~y}.

Definition (1-2-9) (Join)

We write a∨b (read as "a join b") in pleas sup{a,b}.

Definition (1-2-10) (Meet)

We write a∧b (read as "a meet b") in pleas inf{a,b}

Definition (1-2-11) (Birkhoff)

A lattice is a poset P any two of elements have a g.l.b "meet", denote by a∧b and a l.u.b "join" denoted by a∨b.

1-3 : Matroid And Geometries:

The references of the following material are [2] ,[4], [20], [25] , [26] ,[27].

From the viewpoint of this thesis matroids capture the essence of independence as found in e.g.(linear algebra (linear independence)) and affine geometry (affine independence).

This general treatment includes concepts such that as bases dimension , etc . Note, however , that the subject of matroids is much larger than this thesis might indicate .

Usually matroids have a finite ground set , but for treating independence this is an unnecessary assumption .

The following definitions are relatively standard .

Definition (1-3-1) (Closure operator)

A closure operator is a pair (M, cl) , where M is a set (the ground set) and cl: $\mathcal{P}(M) \to \mathcal{P}(M)$ is a function (closure operator) satisfying: for all

$A,B \subseteq M$,

1. $A \subseteq cl(A)$ (increasing) .
2. If $A \subseteq B$,then $cl(A) \subseteq cl(B)$ (monotone).
3. $cl(cl(A)) = cl(A)$ (idempotent).

Definition (1-3-2) (finitary)

A set $A \subseteq M$ is called **a finitary** if $x \in cl(A)$, then $x \in cl(B)$ for some $B \subseteq_{fin} A$.

Definition (1-3-3)(Exchange property)

The exchange property is characterized by the following property: if $y \in cl(A \cup x) \setminus cl(A)$ for some $x,y \in M$, then $x \in cl(A \cup y)$.

Definition (1-3-4)(Matroid)

A matroid is a closure space satisfying the exchange property and finitary .

Definition (1-3-5) (Simple)

If a closure space satisfying $cl(\varnothing)=\varnothing$ and $cl(x) = x$ for all $x\in M$ then it is called **a simple.**

Definition (1-3-6)(Geometry)

A geometry is a simple matroid .

Definition (1-3-7)(Flat)

The closed subset of matroid is called **subspace or flat**

Definition (1-3-8)(Projective law)

Let $A\neq\varnothing$ and $B\neq\varnothing$ be two subsets of the ground set M , then **the projective law** satisfying : $cl(A\cup B)=\cup\ \{ cl(x,y) \mid x\in cl(A), y\in cl(B) \}$

Definition (1-3-9)(Projective matroid)

A matroid satisfying the projective law is called a **projective matroid**

Definition (1-3-10)(Independent set)

Independent set I is a matroid M together with set $I\subseteq\mathcal{P}(M)$ is satisfying :

1- $I\neq\varnothing$.

2- If $B\subseteq A\in I$ then $B\in I$.

3- If $A ,B \in I$ and $|A| <|B|<\infty$ then there is some $x\in B\setminus A$ such that $A\cup x \in I$.

4- If $A\subseteq M$ and $B\in I$ for every $B\subseteq_{fin} A$ then $A\in I$.

Definition (1-3-11)(Basis)

A basis of A is B , if B is independent set and cl(B) = A ,for some A,B $\subseteq$M.

Note:

Every closed set has a basis, and all basis of closed set are equipotent .

Definition (1-3-12)(Rank function)

A matroid is a set M together with **a rank function** r: $\mathcal{P}_{fin}(M) \rightarrow \mathbb{N}$,

satisfying (for all A,B $\subseteq_{fin}$M).

1- $r(A) \leq |A|$.

2- If A$\subseteq$B then , $r(A) \leq r(B)$.

3- $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$.

Note that:

- The cardinality of any *basis* of A is the rank of A.
- $E \wedge F = E \cap F$, $E \vee F = cl(E \cup F)$
- $r(E \wedge F) + r(E \vee F) \leq r(E) + r(F)$, for any subspaces E,F .

Definition (1-3-13)(Equivalence relation ~)

The equivalent relation ~ is defined by x~y iff $cl(E \cup x) = cl(E \cup y)$,

$x,y \in M \setminus E$, E$\subseteq$M .

Definition (1-3-14)(Quotient set)

Let E be a subspace of M . Take **the quotient set** M/E consisting of the *equivalence classes* of the equivalence relation ~.

Definition (1-3-15)(Canonical projection)

The canonical projective is defined by the map $\pi: M\backslash E \to M/E$ i.e. A map from $M\backslash E$ to the quotient set M/E.

Note :

Define the closure operator on the quotient set $cl_{M/E}: \mathcal{P}(M/E) \to \mathcal{P}(M/E)$ by $Cl_{M/E}(A) = \pi\,(cl(\pi^{-1}(A) \cup E) \backslash E)$.

Definition (1-3-16)(Corank)

The corank of a subspace $E \subseteq M$ is $\bar{r}(E) := r(M \setminus E)$.

Remarks:-

- The corank satisfies $\bar{r}(E) + r(E) = r(M)$.
- The matroid M itself has corank 0.
- $\bar{r}(E) + \bar{r}(F) \le \bar{r}(E \wedge F) + \bar{r}(E \vee F)$
- $\bar{r}(E) + r(E \wedge F) \le \bar{r}(E \vee F) + r(F)$ for any subspaces E,F.

Definition (1-3-17)(hyperplane)

A hyperplane is defined as a subspace with corank 1 .

Definition (1-3-18)(Degree)

A geometry is of **degree n** if it satisfies for any subspaces E,F, $n \in \mathbb{N}$, If $r(E \wedge F) \ge n$ then $r(E \wedge F) + r(E \vee F) = r(E) + r(F)$.

Definition (1-3-19)(Modular)

A matroid of degree 0 is called **modular.**

Definition (1-3-20)(projective geometry)

A projective geometry is a set G of point together with an operator $\cdot : G\times G \to \mathcal{P}(G)$, satisfying for all a,b,c,d,p $\in$G

1- $a.a=\{a\}$

2- $a\in b\cdot a$

3- if $a\in b\cdot p$, $p\in c\cdot d$, and $a\neq c$ then $(a\cdot c)\cap(b\cdot d)\neq\varnothing$.

Definition (1-3-21)(Line)

In a geometry of degree 1 , the subspace of rank 2 is called **a line**

Definition (1-3-22)(Plane)

In a geometry of degree 1 , the subspace of rank 3 is called a **plane**

Definition (1-3-23)(Parallel)

Two lines l_1,l_2 are parallel $(l_1\|l_2)$ iff either $l_1=l_2$ or $l_1\cap l_2=\varnothing$ and $r(l_1\vee l_2)=3$.

Definition (1-3-24)(Affine geometry)

An **Affine geometry** is a geometry M of degree 1 for which, for every line $l\subseteq M$ and point $p\in M/l$ there is a unique line l'. parallel to l , with $p\in l'$.

1-4 :Antimatroid:

The references of the following material are [2] ,[4],[19] [20], [25] , [26],[27].

Antimatroids are related to matroids. Convexity is treated abstractly in two different ways.

As the following definitions show :

Definition (1-4-1)(anti-exchange property)

For some x,y ∈M and S⊆M **the anti-exchange property** is characterized by the following axiom: if x,y ∈M , x≠y , S⊆M, and y∈cl(S∪x)\cl(S), then x∉cl(S∪y).

Definition (1-4-2)(Convex set)

A set A in the plane is called **convex set** if , for any points x and y in A , the entire segment of line $\overline{xy}$ lies in A.

Remark:

For any intersection of convex sets is convex.

Definition (1-4-3)(Convex hull)

We say that minimal convex enclosing set is called **convex hull of A** and is denoted by H(A), that is

H(A)=∩{B:A⊆B and B is convex set }

Definition (1-4-4)(Antimatroid)

An antimatroid is a closure space satisfying *the anti-exchange property* and finitary .

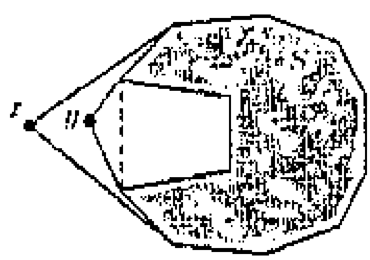


Figure: 1.4.1

Figure: 1.4.1 : Illustration of the anti-exchange property for a planar convex hull operation . The notation used in the same as in definition (1.4.1). The point y is not in the convex hull of S , but it is the convex hull of S∪x . On the other hand , x is not in the convex hull of S∪y .

The standard example of an antimtroid is a vector spaces with the standard convex closure . Figure(1.4.1) , motivates the definition of the anti-exchange property in this context .

Let us finish this section with some simple results about anitmatroid , indicating in what way anti-exchange property is related to convexity .

Definition (1-4-5)(Extreme point)

The point $e \in S$ is an **extreme point** of S if S be a subset of the antimatroid M and $e \notin cl(S \backslash e)$.

Definition (1-4-6)(Extreme set)

The set of all extreme points of S is called **the extreme set** and denoted by E(S) and $E(S)=\cap\{S' \subseteq S \mid cl(S')=cl(S)\}$.

Remark

- ***Independence*** is defined in the same way as for matroid . and , if S is a subset of the antimatroid M then S is independent iff $S \subseteq E(S)$.
- We cannot expect the important theorems about basis valid for matroids to be true in this thesis , since they are based on the exchange property .

1-5 :Rings and Modules:

The references of the following material are [4] ,[5], [7], [10] , [25].

This section lists some standard definitions, and a few results , regarding modules and rings ,it is assumed that the reader knows the basis about group theory , vector space , etc, For more information the reader is referred to any books of abstract algebra.

The following definitions are on rings ,modules, etc.

Definition (1-5-1)(Ring)

A ring is a non empty set R with two binary operations called addition + and multiplication . , such that :

1. (R,+) is an abelian group i.e (commutative group).
2. (R,.) is semigroup , i.e. closed and associative .
3. (.) is distributive over + :

   a.(b+c)= a.b +a.c

and (b+c).a=b.a+c.a for all a,b,c ∈R.

Remarks:

1) a.b is written usually as ab.
2) A ring R is called a ring with unity if ∃ e∈R such that ∀ a∈R then a.e=e.a=a .
3) A ring R is called a commutative ring if it's a commutative under (.) i.e. a.b=b.a ∀ a,b∈R.

Example(1-5-1):

Let ℤ be the set of all integers, ℤ is a commutative ring with unity under the addition and multiplication of integers.

Definition (1-5-2)(Zero divisor Z.D)

Let R be a ring and a∈R such that a≠0, a is called a **zero divisor** if ∃ b∈R such that b≠0 and ab=0 or ba=0.

Example(1-5-2):

Let $\mathbb{Z}_6$ be the integers modulo 6, that is $\mathbb{Z}_6=\{0,1,2,3,4,5\}$, $\mathbb{Z}_6$ is a ring under the additive and multiplication of integers and, 2.3=0 but 2≠0 and 3≠0 also 4.3=0 but 4≠0, then {2 ,3 ,4} are zero divisors in $\mathbb{Z}_6$.

Definition (1-5-3)(Integral domain I.D)

An integral domain is a nontrivial commutative ring with no zero divisors.

Example(1-5-3):

The ring of integers $\mathbb{Z}$ under the additive and multiplication of integers is an integral domain.

Definition (1-5-4)(Ideal)

Let I be a subset of the commutative ring R ,Then I is called **an ideal** of R if:

1- I is an additive subgroup of R.

2- For all $a\in I$ and $b\in R$ then $ab\in I$.

Note that:

If I≠R then I is called a proper ideal in R.

Example(1-5-4):

$n\mathbb{Z}$ is an ideal of a ring of integers $\mathbb{Z}$, $n \in \mathbb{N}$

Definition (1-5-5)(Principal ideal)

An ideal I in a ring R is a **principal** if there is an element $r\in R$ such that $I= Rr=\{ rr' \mid r'\in R\}$.

i.e. an ideal I is generated by one element r and denoted by <r>.

Example(1-5-5):

An ideal $2\mathbb{Z}$ in a ring $\mathbb{Z}$, is an ideal generated by one element 2.

$<2>:=\{2.a \mid a\in\mathbb{Z}\} = 2\mathbb{Z}$

Definition (1-5-6)(Principal ideal domain PID)

A principal ideal domain is an integral domain in which every ideal is principal , that is generated by a single element .

Example(1-5-6):

The ring of integers $\mathbb{Z}$ is an integral domain and for all ideals of $\mathbb{Z}$ is a principal ideal (generated by a single element) .

The following theorem shows that all ideals over $\mathbb{Z}$ are principals:

Theorem(1-5-1):

Every ideal of $\mathbb{Z}$ is a principal ideal.

Proof

Let I be an ideal of $\mathbb{Z}$,

If $I=\{0\}$, then $I= <0>$ which is a principal ideal ,

Suppose that $I\neq\{0\}$, then there exist $x\in I$ such that $x \neq 0$,

Let n be the smallest positive integer in I

by *the division algorithm* in integers $x=qn+r$; $0\leq r < n$, $q,r \in\mathbb{Z}$.

$\therefore$ $r = x-qn$, and $x\in I$, $n\in I \Rightarrow qn\in I$, then $r\in I$, but $r< n$,

it is impossible , then $r=0$

thus $x= qn$

$\therefore$ $I=\{qn \mid q\in\mathbb{Z}\}= n\mathbb{Z}= <n>$ is principal ideal □

Definition (1-5-7)(Module)

Let R be a commutative ring , M is an R-module if :

1- M is an additive abelian group

2- There is a map: $R \times M \to M$,

Such that , for all $r,s \in R$ and $m,n \in M$

i. $(r+s)m= rm+sm$

ii. $(rs)m=r(sm)$

iii. $r(m+n)=rm+rn$

iv. $em=m$ if R has unity e.

Definition (1-5-8)(Submodule)

Let M be an R-module and $N \subseteq M$, N is called an **R-submodule** of M , if N is a subgroup of M and $rx \in N$ for all $r \in R$ and $x \in N$.

Definition (1-5-9)(Module homomorphism)

Let R be a ring and M,N be two R-modules, a map $f:M \to N$ is called an **R-module homomorphism** if for any $x,y \in M$ and $r \in R$ we get:

i) $f(x+y)=f(x)+f(y)$.

ii) $f(rx)= rf(x)$ (*scalar multiplication*)

Note:

A subset of module is a submodule iff it's closed under scalr multiplication and sum .

Theorem(1-5-2):

Let $f:M \to N$ be a map between two R-modules , then f is R-module homomorphism iff $f(rx+sy)=rf(x)+sf(y)$ (*linear transformation*) □

Definition (1-5-10)(Isomorphism)

Two R-modules are **isomorphic** if there exist a bijective R-module homomorphism

Definition (1-5-11)(Torsion free)

An R-module M is **torsion free** if for any $r \in R \setminus \{0\}$ and $m \in M \setminus \{0\}$ the product satisfies $rm \neq 0$.

The following three statements are easily seen to be equivalent :

- The R-module M is *torsion free*
- $rm=rn$ implies $m=n$ for any $r \in R \setminus \{0\}$ and $m,n \in M$
- $rm=sm$ implies $r=s$ for any $r,s \in R$ and $m \in M \setminus \{0\}$.

Definition (1-5-12)(Generates)

Let M be an R-module . A subset $G \subseteq M$ **generates** M if

$$M = \left\{ \sum_{i=1}^{n} r_i g_i \mid n \in N, r_i \in R, g_i \in G \right\}.$$

Definition (1-5-13)(Finitely Generated)

If M is generated by a finite subset , then M is **Finitely generated.**

Definition (1-5-14)(linearly independent)

A subset $S \subseteq M$ is **linearly independent** if

$$\sum_{i=1}^{n} r_i s_i = 0, s_i \in S, r_i \in R \Rightarrow r_i = 0, \forall i, 1 \leq i \leq n$$

Definition (1-5-12)(Basis of M)

A linearly independent set that generates M is **a basis of M.**

Definition (1-5-13)(free)

If M has a basis then it is called **free**

1-6: Graph theory :

In this section we define any notations in graph theory are used in this thesis .

The references of the following materials are [14],[21],[30].

What's Graph Theory?

Graph theory has nothing to do with graph paper or x- and y-axes. Graph theory is an area of mathematics that deals with entities (called **nodes**) and the connections (called **links**) between the nodes.

Definition (1-5-13)(directed graph)

If we consider V as a finite nonempty set and E as a subset of the Cartesian product of V x V, then the pair (V, E) is called a Directed Graph or Digraph on V.

Definition (1-5-13)(vertices)

A digraph (directed graph) is a diagram consisting of points, called vertices, joined by directed lines, called arcs

Definition (1-5-13)(graph)

A *graph* G is a pair G = (V,E) of sets V, of vertices (or nodes) and E of edges such that an edge $e \in E$ is associated with an unordered pair of vertices. If there is a unique edge e associated with the vertices v and w,
we write e = (v,w) = vw = (w, v) = wv

Definition (1-5-13)(undirected)

If we need to really emphasize this point we might say *ordinary graph* or *undirected graph.*

Definition (1-5-13)(connected)

A *connected* graph is one in which there is a continuous path through all the branches (any of which may be traversed more than once) which touches all the nodes.

Definition (1-5-14)(Simple graph)

In many applications, one deals with graphs that have neither loops nor multiple copies of the same edge (these are known as parallel edges). Such graphs are called *simple graphs.*

CHAPTER TWO

HUBLER'S DISCRETE GEOMETRY

Chapter Two
"Hűbler's discrete geometry"

In this chapter we study Hűbler's work on discrete geometry is briefly summarized. Since it is hard to get hold of Hűbler's report.
The following martial is taken from [2] ,[8] ,[15],[19] ,[24],[25],[26],[27].

2-1 : Introduction:

Hűbler's report has three main parts, and three approaches to discrete geometry. The first one, totally ignored here, is about so-called digital geometries, and seems to be less abstract than the others. The following two parts, which are summarized here, are about translative neighborhood graphs and axiomatic discrete geometry.

2-2: Discrete geometry on translative neighborhood graphs

First Hűbler's presentation of neighborhood graphs is summarized. These structures are not used in other parts of this thesis, and hence this section can be skipped without much loss. However, neighborhood graphs provide a background setting for Hűbler's axiomatic geometries, and furthermore some results in other parts of this thesis are generalizations of results in this section.

2-2-1: Translative Neighborhood Graphs

Definition (2-2-1-1) (Neighborhood Graphs)

A neighborhood graph is a simple, undirected, connected graph with a nonempty node (point) set and an edge set with the property that each point has a finite number of neighbors. (Two points are each others neighbors if they are connected by an edge.).

Definition (2-2-1-2) (Distance)

The *distance* between two points is the length of the shortest path between them.

Definition (2-2-1-3) (Displacement)

A *displacement* D is a bijection on the point set of a neighborhood graph that preserves neighborhood and has a constant displacement distance (i.e. the distance between p and D(p) equals the distance between q and D(q) for any points p and q).

Note: The identity displacement is denoted by id.

Definition (2-2-1-4) (Translative)

Let $\mathcal{D}$ be the set of all displacements on a neighborhood graph. The graph is said to be *translative* if

1. $\mathcal{D}$ is closed under composition o of displacements,
2. o is commutative,
3. For each of a point's neighbors there always exists a displacement that maps the point to the neighbor,
4. For every displacement D except id, $D^n(p) := (\overbrace{D \circ \ldots\ldots \circ D}^{n-times})(p) \neq p$ for any point p and any $n \in \mathbb{Z}^+$.

Definition (2-2-1-5) (Translation)

The displacements of a translative neighborhood graph are called translations.

Remarks:

- Let us use the term t-graph instead of translative neighborhood graph (these graphs are often denoted by G).
- let P_G be the point set and T_G the displacement (translation) set of G
- The set T_G is easily seen to be an abelian group under the group operation o. This is not a sufficient condition for a neighborhood

graph being a t-graph, though. For t-graphs the power notation D^n can be extended to arbitrary $n \in \mathbb{Z}$ in the standard way (i.e. T_G can be viewed as a $\mathbb{Z}$-module).

Definition (2-2-1-6) (Elementary)

A translation with displacement distance 1 is said to be an *elementary*.

Remarks:

- T-graphs have the property that all points have the same number of neighbors (the neighborhood degree of the graph) .
- For each pair of points p, q there is exactly one translation that maps p to q . Hence the number of elementary translations is finite.

Definition (2-2-1-7) (Dimension)

The dimension of a t-graph is the (well-defined) cardinality of its smallest basis

Definition (2-2-1-8) (Simple)

A translation S is simple if $S \neq T^n$ for all translations T and all $n \in \mathbb{N} \setminus \{1\}$

2-2-2: Lines, Parallelity and convexity

Definition (2-2-2-1) (generator of *l*)

Given a point p and a simple translation S, the associated line *l* is the smallest set that contains p and is closed under S^n for all $n \in \mathbb{Z}$. The translation S is said to be a *generator of l*.

Note:

The only generators of *l* are S and S^{-1}, and $l=\{S^n(q) \mid n \in \mathbb{Z}\}$ for all $q \in l$. Furthermore, to each pair of distinct points there is exactly one line that contains both points.

Definition (2-2-2-2) (parallel)

Two lines *l* and *l'* are said to be parallel ($l \| l'$) if they have the same generators.

Note:

Two parallel lines either have all or no points in common and two lines are parallel iff there is a translation that maps one of the lines bijectively onto the other. Furthermore, for each line *l* and point p there is exactly one line *l'* with $l \| l'$ and $p \in l'$ (compare with the Euclidean parallel axiom).

Definition (2-2-2-3) (Betweens relation)

A *Betweens* relation B is now introduced: B(p, q, r) holds for three points p, q, and r on a line *l* if there are positive integers n_1, n_2 with $n_1 < n_2$ and a generator S of *l* such that $q = S^{n_1}(p)$ and $r = S^{n_2}(p)$. It is easy to check that B(p, q, r) is equivalent to B(r, q, p) and that B(p, q, r) and B(q, r, s) together imply B(p, q, s) and B(p, r, s).

Definition (2-2-2-4) (convex)

A point set P is said to be convex if it is closed under B, i.e. if $p,q \in P$ and B(p, r, q) for a point r, then $r \in P$.

Definition (2-2-2-5) (convexity)

Convexity is also defined just as for neighborhood graphs (a point set is convex if it is closed under B).

Note:

This convexity definition is in some sense weak, though, for there are convex point sets of t-graphs where the induced subgraph associated with such a set is not connected.

2-2-3:Isomorphisms:

Definition (2-2-3-1) (isomorphisms)

Two t-graphs G_1 and G_2 are said to be isomorphic if there exists a bijection φ from $\mathcal{P}G_1$ to $\mathcal{P}G_2$ which is B (betweens) invariant. (i.e.B(p,q, r) iff B(φ(p), φ(q), φ(r)).) .The statement that two t-graphs are isomorphic is equivalent to each of the following four statements:

1. There is a bijection φ between points, mapping lines to lines, which is parallelity invariant ($l \| l'$ iff $\varphi(l) \| \varphi(l')$).
2. There are a bijection Φ between translations and a bijection φ between points for which T(p) = q iff $\Phi(T)(\varphi(p)) = \varphi(q)$ for all points p and q and all translations T.
3. There is a bijection φ between points which is convexity invariant (P is convex iff φ(P) is convex).
4. The two t-graphs have the same dimension.

2-3: Axiomatic Discrete Geometry:

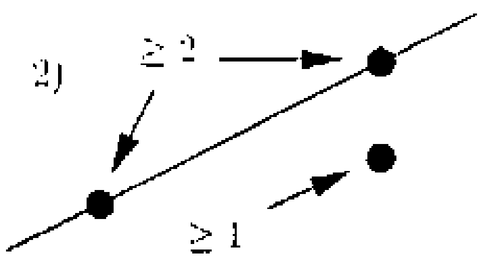
Let us now turn to Hûbler's axiomatisation of discrete geometry.

Basic Axioms and Definitions

The axiom system essences the existence of a point set $\mathcal{P}$ and a nonempty line set $\mathcal{L} \subseteq \wp(\mathcal{P})$

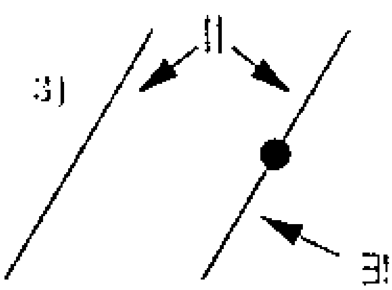
The first axiom: is that for any pair p, q of distinct points there is exactly one line *l* for which the points lie on the line (p,q $\in$ *l*). Let *l* (p,q) denote that unique line.

The second axiom: says that for any line *l* there exist two different points p,q $\in$ *l* and one point r$\notin$ *l*.



The third axiom: states that there is an equivalence relation ||, parallelity, on $\mathcal{L}$,for which, for any line *l* and point p there exists exactly one line *l'* with p $\in$ *l'* and *l* || *l'*.

The corresponding equivalence classes are called *directions*.



Definition(2-3-1) (Translations)

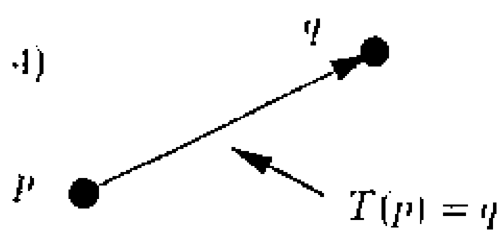
Translations are bijections T on $\mathcal{P}$ satisfying either T = id or has the following properties (referred to simply as the first, second, and third translation properties).

1. $T(l) \parallel l$ for all $l \in L$ (lines are mapped bijectively onto parallel lines),
2. $T(p) \neq p$ for all $p \in \mathcal{P}$,
3. $\{ l(p, T(p)) \mid p \in \mathcal{P} \}$ is an equivalence class of $\parallel$.

Definition(2-3-2) (Simple Translations)

Definition of a simple translation is just as above (S is simple if $S \neq T^n$ for all translations T and all $n \in N \setminus \{1\}$), and a result is that a translation is simple iff it generates a line.

The fourth axiom: now states that for any two points p, q there exists a translation T with T(p) = q.



Remarks :

- This translation can be shown to be unique.
- Another result is that two lines l and l' are parallel iff there exists a translation T such that $T(l) = l'$.
- Furthermore, just as with neighborhood graphs, the set $\mathcal{T}$ of all translations on $\mathcal{P}$ is an abelian group under the group operation

composition (o). Hence $\mathcal{T}$ can be made into a $\mathbb{Z}$-module in the standard way.

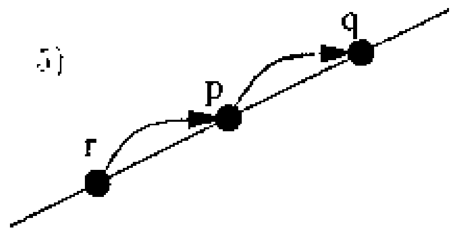
- Hübler goes on to discuss *cyclic translations*, i.e. translations T for which, for some $n \in \mathbb{Z}^+$, $T^n = id$.
- $l \| l' \Leftrightarrow \exists T$, $l = T(l')$.

Definition(2-3-3) (Betweens)

A betweens relation B is yet again defined; for three different points p, q, and r on a line, B(p, q, r) holds if $p < q < r$ or $r < q < p$.

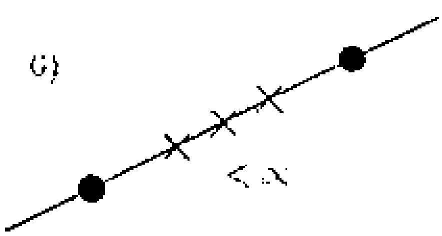
A further assumption is the existence of two opposite total orders $\leq$, $\geq$ defined on the points of each line.

The fifth axiom : For each point p on a line l there are two other, different points $q, r \in l$ with $q < p < r$.



The sixth axiom: introduces discreteness: For any two points p and q on a line l, the set of all points $r \in l$ satisfying $p<r<q$ is finite .

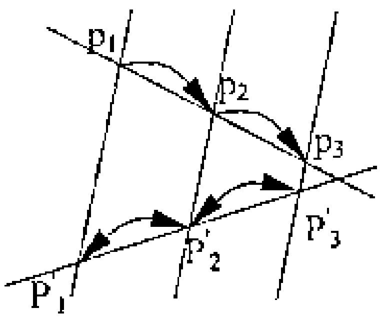
there is at most a finite number of points r such that B(p, r, q).



This means e.g. that every line is a countably infinite set of points.

The seventh axiom: Let l_1, l_2, l_3 be different, parallel lines ,and l ,l' lines that have points p_i, p_i' respectively , in common with all the lines $l_{i, i=\{1,2,3\}}$ then $p_1<p_2<p_3$ holds iff $p'_1<p'_2<p'_3$ or $p'_1>p'_2>p'_3$.

7)



This axiom is the one that rules out cyclic translations.

Remarks:

* For each line l there exists a translation G (a generator) such that $l=\{G^n(p) \mid n \in \mathbb{Z}\}$ for any point $p \in l$. For such a triple (l, G, p) the relation $B(G^i(p),G^j(p),G^k(p))$ holds iff $i<j<k$ or $k<j<i$ ($i, j, k \in \mathbb{Z}$). Furthermore ,each line has exactly two generators (G and G^{-1}), and two lines are parallel iff they have the same generators.

Definition(2-3-4) (line between two lines)

Let l_1, l_2, and l_3 be different, parallel lines. The line l_2 is said to *lie between* l_1 and l_3 ($B(l_1, l_2, l_3)$) if there are a translation T and $i, j \in \mathbb{Z}^+$ such that $T^i(l_1) = l_2$ and $T^j(l_2) = l_3$.

Definition(2-3-5) (A planar set)

A planar set is a point set S, such that

(i) whose points do not all belong to one line, and

(ii) for any four, different points $p_i \in S$, $i \in \{1, 2, 3, 4\}$, one has for the lines l_i, $i \in \{1, 2, 3\}$ with $l_1 = l(p_1, p_2)$, $l_2 \parallel l_1$ with $p_3 \in l_2$, and $l_3 \parallel l_1$ with $p_4 \in l_3$, that if the lines are different, then one of the lines lies between the other two lines.

Definition(2-3-6) (A planar grid)

Let T_1 and T_2 be translations with different directions, and p an arbitrary point. The set PG(p, T_1, T_2) := $\{(T_1^i o T_2^j)(p) \mid i, j \in Z\}$ is the *planar grid spanned* by p, T_1, and T_2.

Definition(2-3-7) (A plane)

A plane is a planar set P for which $P \cup x$ is not planar for any $x \in \mathcal{P} \setminus P$.

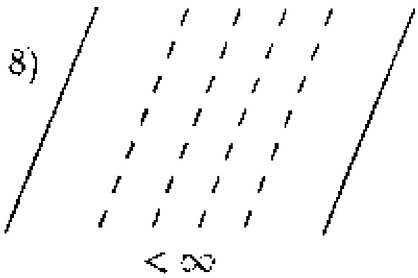
For each planar set S there is exactly one plane P with $S \subseteq P$.

Furthermore l (p, q) $\subseteq$ P holds for any two different points p and q in a plane P, and $P_1 \cap P_2 = l$ (p, q) holds for two different planes P_1 and P_2 whose intersection contains at least two different points p and q.

In his report Hűbler's demonstrates a model of the first seven axioms which, in some sense, is not discrete , but nevertheless motivates why Hűbler introduced an eighth axiom.

Hűbler demonstrates a model of the first seven axioms which is embedded in the real plane; a point set is given and all other concepts are given by the restriction of the corresponding real concept to the point set. Hűbler claims that this model has the property that for each point p in the real plane there are points of the model that are arbitrarily close to p (using the standard Euclidean metric).

The eighth axiom: The set of all lines between two different, parallel lines is finite.



Remarks:

_ Axiom 6 is made redundant by Axiom 8.

_ These two axioms are included to make the geometry discrete.

_ Planes also have generators.

Definition(2-3-8) (Hűbler's geometry)

A *Hűbler's geometry* is a collection of a point set $\mathcal{P}$, a line set $\mathcal{L}$, a parallelity relation $\|$, and a set of total orders $\leq$ for each line $l \in \mathcal{L}$, such that all the eight axioms are satisfied. Such a geometry is uniquely defined by $\mathcal{P}$ and the set $\mathcal{T}$ of all translations on $\mathcal{P}$.

The eighth axiom holds for all planar grids, and given the eighth axiom each plane is a planar grid. Hence all planes are isomorphic to each other. Furthermore the axiom ensures that all bounded subsets of a plane are finite.

Since "discrete image geometry" is not a very descriptive term, at least when it comes to distinguishing between different approaches to discrete geometry, we will use the term Hűbler geometry instead.

2-4: Oriented matroids

The references of the following material are [12] ,[16], [17], [18] , [25] ,[26].

Oriented matroids add extra structure to the ordinary matroids treated in the last section. From Richter-Gebert and Ziegler[17] we get the following description: "Roughly speaking, an oriented matroid is a matroid where in addition every basis is equipped with an orientation."

Definition(2-4-1) (cocircuit.)

Let M be a matroid where the complement M \H of each hyperplane H is partitioned into two possibly empty sets H^- and H^+, the negative and positive side of H. The ordered pair (H^- , H^+) is a *cocircuit*.

If necessary we can change the *orientation* of the cocircuit, i.e. the opposite (H^+ , H^-) is also a cocircuit. There are no other cocircuits.

Definition(2-4-2) (oriented matroid.)

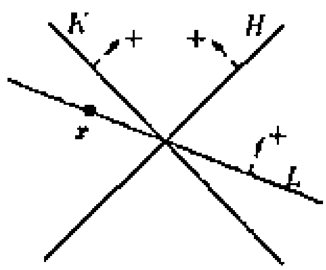
The matroid M together with its cocircuits is an *oriented matroid* if the following requirement is satisfied:

* Let H and K be two hyperplanes intersecting in a subspace of corank 2 and x a point in M \ (H∪K).

If it is possible to choose the orientations of the cocircuits associated with H and K such that $x \in H^+ \cap K^-$ then the hyperplane $L = x \vee (H \wedge K)$ satisfies $L^+ \subseteq H^+ \cup K^+$ and $L^- \subseteq H^- \cup K^-$, given a suitable choice of its orientation.

All the terminology used for ordinary matroids carries over to the oriented case . Some intuition behind the definition is given in Figure (2-4-1).

As with ordinary matroids there are many equivalent definitions oriented matroids, and some of these do not give rise to equivalent definitions when relaxed to the infinite case. The choice to use the definition above is motivated by the connection between convexity and half-spaces, which we turn to now.



Fig(2-4-1)

Fig.(2-4-1). The requirement which the hyperplanes and cocircuits of an oriented matroid have to satisfy. See Definition (2-4-2).

Given a hyperplane H, the sets H^- and H^+ are called *open half-spaces*. The union of an open half-space and the corresponding hyperplane is a *closed half-space*. Let $H_C(M)$ denote all closed half-spaces in M. We can define a *convex closure operator*[.]: $\mathcal{P}$(M)→ $\mathcal{P}$(M) by :

$$[S]=\begin{cases}\cap\{H\in H_C(M)\mid S\subseteq H\}, S..finite \\ \cup\{[S']\mid S'\subseteq S, S'..finite\},..otherwise\end{cases} \qquad (1)$$

Here we use the convention that $\cap\varnothing$= M. The reason for having different cases depending on the cardinality of S is that this definition makes [.] finitary.

CHAPTER THREE

MATROIDS FROM MODULES

Chapter three
"Matroid from modules"

This chapter explores some closure operator's defined on modules over integral domain and , the associated matroids and geometry .

Modules over integral domain are embedded in an associated vector space . The matroids constructed from modules in this chapter turn out to be very similar to the matroids constructed from the corresponding vector spaces

The references of the following material are [2],[5],[7] ,[25], [26] , [27].

3-1: Submodule closure:

The effect of a closure operator is determined by its closed sets (since every set is mapped to the smallest closed set containing it; this set has to be unique). As mentioned above, for a vector space you get a matroid by choosing the vector subspaces as closed sets. This approach does not in general work for modules. The submodules do not always yield a matroid, as we will now prove.

Let us first show that the submodules of a module cannot in general make up the subspaces of a matroid.

Lemma 3-1-1

Let M be an R-module, where R is a ring, and let $<.>_s:\rho(M)\to\rho(M)$ take any subset to the smallest submodule containing it. Then $<.>_s$ is a well-defined closure operator with the explicit characterization

$$<S>_S=\left\{\sum_{i=1}^{n}a_is_i \mid a_i\in R, s_i\in S, n\in N\right\}$$

(The empty sum $\sum_{i=1}^{0}a_is_i$ is interpreted as 0.)

Proof:

To show that $<.>_s$ is well-defined we have to show that every subset is contained in a unique submodule.

Denote the right hand side of the equation by A.

Any submodule containing S has to contain A since all submodules are closed under scalar multiplication and sum and they all contain 0. (The last remark is necessary since we allow n = 0.)

Furthermore A, by exhibiting the properties just listed, is a submodule and hence the operator is well-defined and $<S>_s = A$. By construction the operator satisfies the closure operator axioms. □

Let us now consider the $\mathbb{Z}$-module over $\mathbb{Z}$. Define $n\mathbb{Z} := \{nm | m \in \mathbb{Z}\}$, and for any $a,b \in \mathbb{Z}$, let $<a,b>_s = \{am + bn | m, n \in \mathbb{Z}\}$ is ideal of $\mathbb{Z}$.

Observe that $2 \in \langle 10,3 \rangle_s = \mathbb{Z}$ (since $1 \in \langle 10,3 \rangle_s$, $2 \notin \langle 10 \rangle_s = 10\mathbb{Z}$, and $3 \notin \langle 10,2 \rangle_s = \{10m + 2n \,|m, n \in \mathbb{Z}\} = 2\{5m + n \,|m, n \in \mathbb{Z}\} = 2\mathbb{Z}$.. Hence the exchange property does not hold, and $<.>_s$ is not a matroidal closure operator.

3-2: D-Submodule closure:

Given the previous section we know that we cannot (in general) use submodules as subspaces of a matroid. However, by restricting ourselves to d-submodules and modules over integral domains we get a matroid.

Definition(3-2-1)(D-submodule)

A *d-submodule* of an R-module M is a submodule S with the property that if $rm \in S$ for any $r \in R\backslash 0$ and $m \in M$, then $m \in S$.

We say that a d-submodule is closed under *existing divisors*, i.e. under those divisors which happen to exist. Thus it is easy to see, intuitively, why this approach works, d-submodules emulate vector subspaces .

Theorem(3-2-1)

Let M be an R-module, where R is an integral domain, and let $<.>_d:\rho(M)\to\rho(M)$ take any subset to the smallest d-submodule containing it. Then $<.>_d$ is a well-defined matroidal closure operator with the explicit characterization

$$\langle S\rangle_d=\left\{m\in M\mid bm=\sum_{i=1}^{n}a_is_i,s_i\in S,a_i,b\in R,b\neq 0,n\in N\right\}$$

Proof:

Compare with the proof of Lemma 3-1-1.

Denote the right hand side of the equation by A.

Any d-submodule containing S has to contain A since all d-submodules are closed under scalar multiplication, sum, and existing divisors, and they all contain 0. (The last remark is necessary since we allow n = 0.)

Recall that A is a submodule iff it is *nonempty* and *closed under scalar multiplication* and *sum*. If it is also *closed under existing divisors* then it is a d-submodule.

Nonempty. Because the empty sum is 0 and b0 = 0 for any b $\in$R we have that A is nonempty.

Closed under $\times$. Assume that m $\in$ A. Then $bm=\sum_{i=1}^{n}a_is_i, b\neq 0$. By

multiplying this expression with r $\in$ R, using the commutativity of the integral domain multiplication and the different properties of $\times$, we get

$b(rm)=\sum_{i=1}^{n}(ra_i)s_i$. thus rm$\in$A.

Closed under sum. Let m,m' ∈ A. Then $bm = \sum_{i=1}^{n} a_i s_i$, and $b'm' = \sum_{i=1}^{n'} a_i' s_i', b, b' \neq 0$

By the commutativity of the integral domain and the properties of × we have

$(bb')m = \sum_{i=1}^{n} (b'a_i)s_i, (bb')m' = \sum_{i=1}^{n'} (ba_i')s_i'$, and thus $bb'(m+m') = \sum_{i=1}^{n} (b'a_i)s_i + \sum_{i=1}^{n'} (ba_i')s_i'$.

. Because R is an integral domain and $b, b' \neq 0$ we have bb' ≠ 0, and thus m + m' ∈ A.

Closed under existing divisors. Assume that rm∈ A, r ∈R \{0}, m ∈ M.

Then b(rm) $= \sum_{i=1}^{n} a_i s_i, b \neq 0$. By a property of ×we have that b(rm) = (br)m, and

because b, r ≠ 0 we have that br ≠ 0. Thus m ∈ A.

Hence A is a d-submodule, and thus it is the smallest d-submodule containing S, so $<S>_d = A$.

This means that $<.>_d$ is well-defined, and thus by construction all the closure operator axioms hold.

For the exchange property we use the explicit characterization of $<.>_d$.

Take any $y \in \langle S \cup x \rangle_d$. . Then $by = \sum_{i=1}^{n} a_i s_i + ax$ for some a, b, a_i ∈ R, b ≠ 0,

s_i ∈ S, and n∈ ℕ.

Furthermore a ≠ 0, because otherwise y ∈ $<S>_d$. Thus we have

$ax = \sum_{i=1}^{n} (-a_i)s_i + by$ where a≠ 0, and hence $x \in \langle S \cup y \rangle_d$.

This means that the fourth axiom is satisfied.

To show that $<.>_d$ is finitary, assume that x ∈ $<S>_d$. Then $bx = \sum_{i=1}^{n} a_i s_i$

as usual, and we have that x ∈ $<S'>_d$, where S' := { s_i | i ∈ℕ, 1≤ i ≤ n } is a finite subset of S. □

We note immediately that the matroid obtained from $<.>_d$ is not simple, since $<\emptyset>_d = \{0\}$. Furthermore all subspaces contain 0, which ensures that they cannot be interpreted as affine lines, planes, etc. Because of this we introduce a-submodules in the next section.

3-3: A-Submodule closure:

To get something reminiscent of an affine geometry we define a-submodules. (The term stems from affine submodule, but since the resulting geometry is not in general affine the full name is not used.)

Definition(3-3-1)(A-submodule)

An *a-submodule* A of a module M is a subset of the form $A = D + m$ where $D \subseteq M$ is a d-submodule and $m \in M$ is any element.

We define **Addition** of an element to a set is to be $D + m := \{ d + m | d \in D\}$ and **Subtraction** to be $D - m := \{ d - m | d \in D \}$.

Lemma 3-3-1.

Let D be a d-submodule with $m \in D$. Then $D + m = D$.

Proof.

We have $k \in D \Leftrightarrow k - m \in D \Leftrightarrow k \in D + m$. □

Lemma 3-3-2

Let A be an a-submodule. Then for any $m \in A$ the set $A - m$ is a d-submodule, and all d-submodules obtained from A in this way are equal.

Proof.

By the definition of a-submodule we know that $A = D + n$ for some d-submodule D and element $n \in M$.

Since $D = A-n$ we have $m-n \in D$, thus, since D is a submodule, also $n-m = -(m-n) \in D$.

Hence $D = D+n-m$ or $A-m = D$, so A-m is a d-submodule, and all the obtainable d-submodules are equal to D. □

Corollary 3-3-1

A is an a-submodule with $a \in A$ iff $A - a$ is a d-submodule.

Theorem(3-3-1)

Let M be an R-module, where R is an integral domain, and let $\langle\rangle_a : p(M) \to p(M)$ take any nonempty subset to the smallest a-submodule containing it and $\emptyset$ to $\emptyset$. Then $\langle\rangle_a$ is a well-defined matroidal closure operator with the explicit characterization

$$\langle S\rangle_a = \left\{ m \in M \mid bm = \sum_{i=1}^{n} a_i s_i, s_i \in S, a_i, b \in R, b = \sum_{i=1}^{n} a_i \neq 0, n \in Z^+ \right\}$$

Furthermore, for any $s \in \langle S\rangle_a$, $\langle S\rangle_a = \langle S-s\rangle_d + s$.

Proof.

Compare with the proofs of Lemma 3-1-1 and Theorem 3-2-1. Denote the right hand side of the equation by A. When $S = \emptyset$ we have that $\langle S\rangle_a = \emptyset = A$, so assume that $S \neq \emptyset$.

Take any $s \in A$. Notice that (leaving out all the side conditions)

$$\begin{aligned} A-s &= \left\{ m-s \mid bm = \sum_{i=1}^{n} a_i s_i \right\} \\ &= \left\{ m \mid b(m+s) = \sum_{i=1}^{n} a_i s_i \right\} \\ &= \left\{ m \mid bm = \sum_{i=1}^{n} a_i s_i - bs \right\} \\ &= \left\{ m \mid bm = \sum_{i=1}^{n} a_i (s_i - s) \right\}, \end{aligned}$$

where the last step follows since $b = \sum_{i=1}^{n} a_i$, Except for the conditions n > 0,

and $b = \sum_{i=1}^{n} a_i$ the last expression is equal to $\langle S\text{-}s\rangle_d$.

The first condition does not play any role since $0 \in$ A-s.

The second condition can also be dispensed with: Assume that we have

$m \in \langle S-s\rangle_d$, i.e. $bm = \sum_{i=1}^{n} a_i (s_i - s)$.

Then we also have (assuming that $b's = \sum_{i=1}^{n'} a'_i s'_i, b' = \sum_{i=1}^{n'} a'_i \neq 0$)

$$\begin{aligned} b'b(m+0) &= b'\sum_{i=1}^{n} a_i (s_i - s) + b'\left(b - \sum_{i=1}^{n} a_i\right)(s-s) \\ &= b'\sum_{i=1}^{n} a_i (s_i - s) + \left(b - \sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n'} a'_i (s'_i - s)\right), \end{aligned}$$

and since $b'b = b' \sum_{i=1}^{n} a_i + (b - \sum_{i=1}^{n} a_i) \sum_{i=1}^{n'} a'_i \neq 0$ we have m $\in$ A - s.

In other words, A-s = $\langle S\text{-}s\rangle_d$. Thus A is an a-submodule.

Now take any a-submodule A' containing S.

Since A' - s is a d-submodule we know that $\langle S\text{-}s\rangle_d \subseteq$ A'-s.

Hence A is the smallest a-submodule containing S, and $\langle . \rangle_a$ is well-defined.

We also get that $\langle S\rangle_a = \langle S\text{-}s\rangle_d + s$.

It remains to show that $<.>$ is a matroidal closure operator. Refer to Definition 1-2-1.

All the axioms are easily seen to hold when $A = \varnothing$, so assume that $A \neq \varnothing$, and pick an element $a \in A$. Now it is easy to see that all the matroid axioms are satisfied by using $<A>_a = <A-a>_d + a$, and the fact that $<.>_d$ satisfies all axioms. □

Corollary 3-3-2

For any $s \in M$, $<S>_a - s = <S-s>_a$.

Proof.

Just inspect the explicit representation of $<.>_a$. □

Corollary 3-3-3

The matroid defined in Theorem 3-3-1 is a geometry iff the underlying module is torsion free.

Proof.

Since $<\varnothing>_a = \varnothing$ we have to check when we have $<m>_a = \{m\}$ for arbitrary $m \in M$. Since $<m>_a = \{n \in M | bn = bm,\ b \neq 0\}$ the corollary follows immediately.□

3-4: Rank:

From now on let all modules be modules over integral domains.

Let us distinguish between different kinds of independence and rank. We say that B is d- (a-)independent if it is independent using $<.>_d$ ($<.>_a$) as the closure operator. Furthermore the rank attained using d- (a-)closure is called d- (a-)rank. This terminology is extended in the obvious way to other concepts, sometimes also using the prefix s- which is associated to the submodule closure of Section 3-1.

Note that if we use the term a-geometry we implicitly assume that the underlying module is torsion free; otherwise we do not have a geometry.

Proposition 3-4-1.

Let D be a d-submodule of the R-module M and $B \subseteq D$ with $p \in B$. Then if B is d-independent, then $B \cup 0$ is a-independent, and if B is a-independent then $(B-p) \setminus 0$ is d-independent. Furthermore $\langle B \rangle_d = \langle B \cup 0 \rangle_a$,and if $\langle B \rangle_a = D$, then $\langle (B-p) \setminus 0 \rangle_a = D$.

proof

First assume that B is d-independent. Take any $q \in B \cup 0$. Assume for a contradiction that $q \in \langle (B \cup 0) \setminus q \rangle_a$. If $q \neq 0$, then we have $q \in \langle (B \cup 0) \setminus q \rangle_a = \langle (B \setminus q) \cup 0 \rangle_a = \langle ((B \setminus q) \cup 0) - 0 \rangle_d + 0$.

since $0 \in \langle (B \cup 0) \setminus q \rangle_a$ and 0 does not play any role in d-closure. This is a contradiction since B is d-independent, so assume that $q = 0$ instead. Then $b0 = \sum_{i=1}^{n} a_i b_i$ for some coefficients b, $a_i \in R$, some $b_i \in B$, and one $n \geq 2$ (since $b = \sum_{i=1}^{n} a_i \neq 0$).

We can assume that all the coefficients are nonzero, so we have $a_1 b_1 = \sum_{i=2}^{n} (-a_i) b_i$, and B is not d-independent, which yet again is a contradiction. Hence $B \cup 0$ is a-independent.

Now assume that B is a-independent. Take any $q \in (B - p) \setminus 0$. Assume that $q \in \langle (B-p) \setminus \{0,q\} \rangle_d$. Then $bq = \sum_{i=1}^{n} a_i (b_i - p)$ for some $b \neq 0$, $a_i \in R$, $n \in \mathbb{N}$,and $b_i \in B \setminus \{ p, q+p \}$. By adding bp to both sides we get $b(q+p) = \sum_{i=1}^{n} a_i (b_i - p) + bp$, and since the coefficients add up we have

$q+p \in \langle B \setminus \{q+p\} \rangle_a$ (note that $q \neq 0$, i.e. $p \neq q+p$).

This is a contradiction ,so $(B-p) \setminus 0$ is d-independent.

We immediately have that $\langle B \cup 0 \rangle_a = \langle (B \cup 0) - 0 \rangle_d + 0 = \langle B \rangle_d$ since $0 \in \langle B \cup 0 \rangle_a$.

Now assume that $\langle B \rangle_a = D$, which is a d-submodule as well as an a-submodule. We have (omitting the set generator conditions $b \in R \setminus 0$, $a_i \in R$, $n \in \mathbb{N}$).

$$
\begin{aligned}
&\langle (B-p) \setminus 0 \rangle_d + p \\
&= \left\{ m \in M \mid bm = \sum_{i=1}^{n} a_i b_i, b_i \in (B \setminus p) - p \right\} + p \\
&= \left\{ m \in M \mid b(m-p) = \sum_{i=1}^{n} a_i (b_i - p), b_i \in B \setminus p \right\} \\
&= \left\{ m \in M \mid bm = \sum_{i=1}^{n} a_i b_i + \left(b - \sum_{i=1}^{n} a_i \right) p, b_i \in B \setminus p \right\} \\
&= \left\{ m \in M \mid bm = \sum_{i=1}^{n} a_i b_i + a_{n+1} p, b = \sum_{i=1}^{n+1} a_i, b_i \in B \setminus p \right\} \\
&= \langle B \rangle_a .
\end{aligned}
$$

Since $p \in \langle B \rangle_a$ and $\langle B \rangle_a$ is a d-submodule Lemma 3-3-1 gives that $\langle B \rangle_a = \langle B \rangle_a - p$. and we are done. □

Lemma 3-4-1.

If B is an a-basis for a d-submodule D, then B-p is also an a-basis for D, for any $p \in \langle B \rangle_a$.

(If $p \in B$, then this is an immediate corollary of the preceding proposition.)

Proof.

Since $\langle B \rangle_a$ is a d-submodule we have, according to Lemma 3-3-1, that $\langle B \rangle_a = \langle B \rangle_a - p = \langle B-p \rangle_a$ where the last step follows by Corollary 3-3-2. The independence of B-p follows by the equipotence of all bases. □

3-5:Degree:

Let us now determine the degree of d- and a-matroids. First note that a large class of a-matroids are not projective, and hence not of degree 0. (All matroids of rank 2 or less are of degree 0.)

Proposition 3-5-1.

Let M be an R-module of a-rank at least 3. Then the associated a-matroid does not satisfy the projective law.

Proof

Let B be an a-basis of M. Due to Lemma 3-4-1 we can assume that $0 \in$ B. Choose two other elements in B and form B' = {0,x,y} which is an a-basis for a rank 3 subspace. We know that $x + y \in \langle 0,x,y\rangle_a$, and we will show that $x+y \notin \cup\{\langle u,v\rangle_a \mid u \in \langle 0,x\rangle_a, v \in \langle y\rangle_a\}$, thus showing that the projective law does not hold.

First notice that $\langle 0,x\rangle_a = \left\{\frac{ax}{b} \mid a,b \in R, b \neq 0\right\}$ and $\langle y\rangle_a = \left\{\frac{cy}{c} \mid c \in R\setminus 0\right\}$

Assume for a contradiction that $x+y \in \left\langle \frac{ax}{b}, \frac{cy}{c} \right\rangle_a$ for some a,b,c $\in$R, b,c$\neq$0.

This implies that $d(x+y) = e\frac{ax}{b} + f\frac{cy}{c}$ for some d,e,f $\in$ R with d =e+f$\neq$ 0

Rewritten this reads bcd(x + y) = ceax + bfcy or

$$(ace- bcd)x = (bcd -bcf)y = bcey$$

We know that b, c $\neq$ 0. Furthermore e = 0 implies that

$$x+y \in \left\{m \in M \mid dm = d\frac{cy}{c}, c,d \neq 0\right\}$$
$$= \{m \in M \mid cdm = cdy, cd \neq 0\} \subseteq \langle y,0\rangle_a,$$

i.e. $x \in \langle y,0\rangle_a - y = \langle 0,-y\rangle_a = \langle -y\rangle_d = \langle y\rangle_d = \langle 0,y\rangle_a$. This contradicts the independence of B', and hence we have bce$\neq$ 0 which shows that $y \in \langle x,0\rangle_a$. This is another contradiction and we are done. □

3-6: Representations:

This section lists some results about the representation of an element in a basis.

The following theorem shows that the representation of an element with respect to a certain basis is uniquely defined up to a scalar factor.

Theorem 3-6-1 (The Representation Theorem).

Let B be an a-basis for the R-module M. Assume that $p \in M$ has the representation

$$cp = \sum_{i=1}^{n} a_i b_i, n \in Z^+, c, a_i \in R \setminus 0, c = \sum_{i=1}^{n} a_i, b_i \in B, b_i \neq b_j \text{ if } i \neq j$$

in this basis. Then the only other representations of p in this basis are

$$dp = \sum_{i=1}^{n} \frac{da_i}{c} b_i,$$

where $d \in R \setminus 0$ and all $\frac{da_i}{c}$ are assumed to be well-defined.

Of course, if the module is not torsion free, then any particular representation does not necessarily stand for a unique module element

proof

Suppose that we have another representation $\left(n', c', \left\{a_i'\right\}_{i=1}^{n'}, \left\{b_i'\right\}_{i=1}^{n'}\right)$ of p in B . First assume that $n \neq n'$ or $n = n'$ but $\{ b_i | 1 \leq i \leq n \} \neq \{b_i' | 1 \leq i \leq n'\}$. Then there is one basis element, say b_1, for which

$$c' a_1 b_1 = \sum_{i=2}^{n} (-c' a_i) b_i + \sum_{i=1}^{n'} c a_i' b_i',$$

where b_1 does not occur in the right hand side of the equation. Since

$$\sum_{i=2}^{n} (-c' a_i) + \sum_{i=1}^{n'} c a_i' = -(c'c - c' a_1) + cc' = c' a_1 \neq 0$$

we get that B is not independent, which is a contradiction. Hence n = n' and $\{b_i | 1 \le i \le n\} = \{b_i' | 1 \le i \le n'\}$.

For simplicity let us reorder the basis elements such that $b_i = b_i'$ for all i, $1 \le i \le n$. If $c'a_i \ne ca_i'$ for some i then we get a contradiction as above, so $c'a_i = ca_i'$ for all i, $1 \le i \le n$. Hence $a_i' = \frac{c'a_i}{c}$, and by noticing that d = c' we are almost done. The only thing remaining is to point out that every choice of $d \ne 0$ such that $\frac{da_i}{c}$ is defined for all i gives a correct representation. (Since R is an integral domain we have $\frac{dc}{c} = d$) □

Corollary 3-6-1

The Representation Theorem also holds for d-representations (where $c = \sum_{i=1}^{n} a_i$ does not necessarily hold).

Proof.

Apply Proposition 3-4-1. If B is a d-basis for M, then $B \cup 0$ is an a-basis for M. A consequence of this is that any d-representation in B of a point $p \in M$ is also an a-representation in $B \cup 0$ of p (using 0 to make the coefficients add up) and vice versa (removing 0). The corollary follows. □

Corollary 3-6-2

If R is well-ordered then we get a canonical representation of p by choosing the smallest possible positive d, and if R is a field then we can choose d = 1 (using the notation of the preceding theorem).

3-7: Embedding:

Let M be a module over an integral domain R. Define an equivalence relation on M ×(R\0) by $(m,r)\sim(m',r')$ iff there is an s∈R\0 such that s(r'm-rm') = 0. The equivalence classes are the elements of the *module of fractions* F(M).

Let M be a module over an integral domain R, and define the canonical projection $\pi : M \to F(M)$ by $\pi(m) := \frac{m}{r}$

Denote the preimage of π by π^{-1}, i.e. $\pi^{-1}\left(\frac{m}{r}\right)=\left\{m'\in M \mid \pi(m')=\frac{m}{r}\right\}$

We also need to define another function, $\mu : F(M) \to M$, which maps 0 to 0 and an element m'∈F(M)\ 0 to an arbitrary (but fixed) element in the nonempty set {m∈M\ 0| r ∈R, m = rm' }.

Let us now show in what way the d-matroid structure carries over to the associated vector space. Note first that in a vector space the d-submodule closure $<.>_d$. equals the vector subspace (linear) closure. Furthermore independence, bases, etc. match the corresponding d-matroid concepts exactly.

Theorem 3-7-1

Let M be an R-module, and let F(M) be the module of fractions associated with M. Denote the d-submodule closure in F(M) by $<.>_d$. Then we have the following properties.

1. For any subset S ⊆M the equality $\langle S\rangle_d = \pi^{-1}\left(\langle\pi(S)\rangle_d\right)$ holds.
2. Let D be a d-submodule of M with d-basis B. Then F(R)π(D) = F(D) is a vector subspace with basis π(B), and the d-rank of D equals the dimension of F(D).

3. Let S be a vector subspace of F(M) with basis B. Then $\pi^{-1}(S)$ is a d-submodule of M with d-basis $\mu(B)$, and the dimension of S equals the d-rank of $\pi^{-1}(S)$.

Remember that the expressions involving π and π^{-1} can be simplified whenever M is torsion free.

Proof:

1- We have

$$\langle S\rangle_d=\left\{m\in M \mid bm=\sum_{i=1}^{n} a_i s_i, b, a_i\in R\setminus 0, s_i\in S, n\in N\right\}$$

and

$$\tilde{S}:=\pi^{-1}\left(\langle\pi(S)\rangle_d\right)=$$

$$\left\{m\in M \mid \pi(m)=\sum_{i=1}^{n} a_i\pi(s_i), a_i\in F(R)\setminus 0, s_i\in S, n\in N\right\}$$

It is obvious that $\langle S\rangle_d\subseteq\tilde{S}$. Now assume $m\in\tilde{S}$, i.e. $\pi(m)=\sum_{i=1}^{n} a_i\pi(s_i)$.

Assume that $a_i=\dfrac{b_i}{c_i}$ for all i, $1\leq i\leq n$, We get

$$\left(\prod_{i=1}^{n} c_i\right)\pi(m)=\sum_{i=1}^{n} b_i\left(\prod_{j\neq i} c_j\right)\pi(s_i)$$

Hence, by the definition of F(M) , we get that for some $s\in R\setminus 0$ the equality

$$s\left(\left(\prod_{i=1}^{n} c_i\right)\pi(m)-\sum_{i=1}^{n} b_i\left(\prod_{j\neq i} c_j\right)s_i\right)=0$$

Holds in m . since $s\prod_{i=1}^{n} c_i\neq 0$ it follows that $m\in\langle S\rangle_d$.

2- We already know that F(D) is a vector subspace. Since we have

$$\sum_i a_i\pi(b_i)=\sum_i a_i\frac{b_i}{1}=\frac{\sum_i a_i b_i}{1}=\pi\left(\sum_i a_i b_i\right)$$

($a_i\in R, b_i\in M$)it is easy to check that $\pi(B)$ generates $F(D)=F(R)\pi(D)$.

By showing that $b,b' \in B$, $b \neq b'$ implies $\pi(b) \neq \pi(b')$ we get that $|B|=|\pi(B)|$. To see this, note that if $\pi(b) = \pi(b')$ then $s(b-b') = 0$ for some $s \in R\backslash 0$. We get that $b \in \langle b' \rangle_d$, i.e. B is not independent, a contradiction.

We also need to check that $\pi(B)$ is independent. Assume that $\pi(b) \in \langle \pi(B)\backslash \pi(b) \rangle_d$ for some $b \in B$. Then we have $b \in \pi^{-1}(\langle \pi(B\backslash b)\rangle_d) = \langle B\backslash b \rangle_d$, another contradiction, and we are done.

3- We already know that $\pi^{-1}(S)$ is a submodule. It is easy to check that it is also closed under existing divisors, so it is a d-submodule. We have that $\pi(\mu(B))$ is a basis of S, so $\langle \pi(\mu(B))\rangle_d = S$. Hence

$$\langle \mu(B) \rangle_d = \pi^{-1}(\langle \pi(\mu(B))\rangle_d = \pi^{-1}(S).$$

Furthermore we trivially have $|B| = |\mu(B)|$.

It remains to show that $\mu(B)$ is independent. Assume

$$b \in \langle \mu(B)\backslash b \rangle_d = \pi^{-1}(\langle \pi(\mu(B)\backslash b)\rangle_d \quad \text{,for some } b \in \mu(B).$$

Since $\pi(\mu(B)\backslash b) = \pi(\mu(B))\backslash \pi(b)$ we get

$$\pi(b) \in \left(\pi o \pi^{-1}\right)\left(\langle \pi(\mu(B))\backslash \pi(b)\rangle_d\right)$$

i.e. $\pi(b) \in \langle \pi(\mu(B))\backslash \pi(b)\rangle_d$, and by the independence of $\pi(\mu(B))$ we are done □.

Corollary (3-7-1).

The d-rank and the module rank of a module (over an integral domain) coincides.

Since a-matroids have subspaces which are just translations of d-subspaces some of the results above can be usefully applied in an a-matroid context. For instance, for any subset $S \subseteq M$ and any $s \in \langle S \rangle_a$ we have

$$\langle S \rangle_a = \langle S - s \rangle_d + s = \pi^{-1}\left(\langle \pi(S-s)\rangle_d\right) + s$$

It is easy to verify that this can be rewritten as

$$\langle S \rangle_a = \pi^{-1}\left(\langle \pi(S) - \pi(s) \rangle_d + \pi(s) \right) = \pi^{-1}\left(\langle \pi(S) \rangle_a \right)$$

where $<.>_a$ is a-submodule closure in F(M). Furthermore the lattice of subspaces of any a-matroid M is isomorphic to that of the a-matroid over F(M); in a geometry the atoms of the lattice are exactly the singletons.

CHAPTER FOUR

AFFINE GEOMETRY, GENERATORS AND ISOMORPHISM

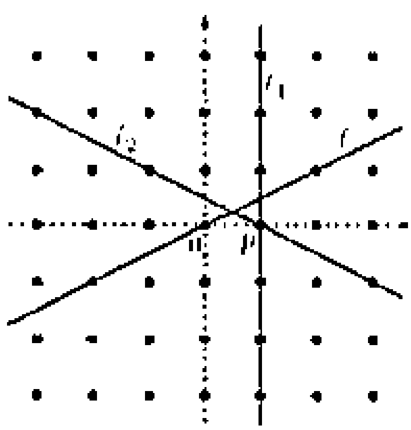
Chapter four
"Affine geometry, generators and isomorphism"

The references of the following material are [2] ,[6], [8], [15], [25], [26], [27].

4-1:Affine geometry:

Given that all a-matroids are of degree 1, and that all a-matroids over modules that are torsion free are geometries, is an a-geometry an affine geometry? Not necessarily, as we will show.

Take the $\mathbb{Z}$-module over $\mathbb{Z}^2$. This module is torsion free and is hence an a-geometry of degree 1. Consider the line $l = <(0,0),(2,1)>_a$ and the point p =(1,0). (All subsets of cardinality two are independent, and hence bases, since this is a geometry.) Both the lines $l_1 = <(1,0),(1,1)>_a$ and $l_2 = <(1,0),(-1,1)>_a$ are parallel to l, so this geometry is not affine. See Figure (4-1-1) for an indication of the situation.



Figure(4-1-1)

Figure (4-1-1): An example demonstrating why the $\mathbb{Z}$-module over $\mathbb{Z}^2$ is not affine. The lines l_1 and l_2 are both parallel to l, and since they intersect the geometry is not affine.

The $\mathbb{Z}$-module over $\mathbb{Z}^2$ clearly has an affine feel to it. The reason why it is not affine is that two lines which are non-parallel in the associated vector space can be disjoint, and hence parallel; the problem lies in the discreteness of the structure. The following proposition shows that it is easy to define a notion of parallelity which at least satisfies some of the usual requirements of a parallelity relation. Of course this definition is influenced by the fact that the same approach gives the proper parallelity relation in an a-geometry over a vector space.

Proposition (4-1-1).

Let M be a torsion free R-module. Define a binary relation ||| on the lines of M by $l|||l'$ iff there is some $p\in M$ such that $l+p=l'$. Then ||| is an equivalence relation, and for any point $p\in M$ and line $l\subseteq M$ there is a unique line l' such that $p\in l'$ and $l|||l'$.

Proof.

The relation is easily seen to be reflexive, symmetric, and transitive. Now consider a point $p\in M$ and a line $l\subseteq M$. Assume $q\in l$.
The set $l'=l+(p-q)$ is then a line with $l|||l'$. Assume that l'' is another such line with $l''=l+r$ and $p\in l''$, $r\in M$.
Notice that l-q is a d-submodule, and hence a subgroup.
(A d-submodule contains 0 and is closed under inverse and addition.) Thus $l'=(l-q)+p$ and $l''=(l-q)+(q+r)$ are non-disjoint cosets of the same subgroup, and hence equal. □

Note:

The proposition does not really require the module to be torsion free, if we replace "line" with "a-rank 2 subspace." This also applies to the following proposition, relating $\|$and $\|\|$, if we weaken the definition of $\|$ to allow matroids that are not geometries.

Proposition (4-1-2)

Let l and l' be two lines of a torsion free R-module M. Then $l \||| l'$ implies $l \| l'$.

Proof.

Assume that $l \||| l'$, i.e. $l + p = l'$ for some $p \in M$. If $l = l'$ then $l \| l'$, so assume $l \neq l'$. This implies, by Proposition(4-1-1), that $l \cap l' = \emptyset$. Assume $l = \langle q,r \rangle_a$, and let $B = \{q, r, q + p\}$. Since $q + p \in l'$ we have $q + p \notin \langle q,r \rangle_a$. Furthermore $\langle q,q+p \rangle_a \neq l$, and hence $\langle q,q+p \rangle_a \cap l = \{q\}$ (two points determine a line uniquely), whereby $r \notin \langle q,q+p \rangle_a$. Analogously $q \notin \langle r, q + p \rangle_a$, so B is a-independent. We obviously have $l \subseteq \langle B \rangle_a$,but also $l' = \langle q,r \rangle_a + p = \langle q+p,r+p \rangle_a \subseteq \langle q+p,q,r \rangle_a = \langle B \rangle_a$ since $br = a_1(q+p)+a_2(r+p)$ implies $br = a_1(q + p) + a_2(r + (q + p) - q)$. Hence $l \vee l' = \langle l \cup l' \rangle_a \subseteq \langle B \rangle_a$. The union of two disjoint lines cannot have a-rank 2, and hence the inclusion is an equality. Thus $r(l \vee l') = 3$ and $l \| l'$ □.

The question about what would make a suitable definition of a discrete affine geometry remains open. However, we can at least motivate why $\|\|$ seems to be a valid parallelity relation (although, in general, it is not).

Note that in an a-geometry over a vector space we have $\|\| = \|$. Let us weaken the definition of $\|\|$ to apply to modules that are not torsion free as well. Then we get the following result.

Proposition(4-1-3)

Let M be an R-module, and let l_1, $l_2 \subseteq M$ be two subspaces a-rank 2. Then $l_1 \parallel\!| \, l_2$ iff $\pi(l_1) \parallel\!| \, \pi(l_2)$, i.e. iff $\pi(l_1) \parallel \pi(l_2)$, where π is the canonical embedding into the associated vector space .

Proof

First note that $\pi(l)$ is a line if l is an a-rank 2 subspace (Theorem 3-7-1). Now assume that $l_1 \parallel\!| \, l_2$. Then $l_1 + p = l_2$ for some $p \in M$. Hence $\pi(l_1 + p) = \pi(l_1) + \pi(p) = \pi(l_2)$, and thus $\pi(l_1) \parallel\!| \, \pi(l_2)$.

Assume instead that $\pi(l_1) \parallel\!| \, \pi(l_2)$. Then $\pi(l_1) + x = \pi(l_2)$ for some $x \in F(M)$. Take any two points $p \in l_1$, $q \in l_2$, and set $y = \pi(q) - \pi(p)$.
We get that y-x is a vector parallel to l_1, and hence

$$\pi(l_2) = \pi(l_1) + x + (y-x) = \pi\,(l_1 + q - p).$$

For any two points r_1, $r_2 \in M$ we have that $\pi(r_1) = \pi(r_2)$ implies $s(r_1 - r_2) = 0$ for some $s \in R \backslash 0$, and hence $r_1 \in \langle r_2 \rangle_a$. It follows that $l_2 = l_1 + (q-p)$, since all sets involved are a-rank 2 subspaces (in either M or F(M)). In other words $l_1 \parallel\!| \, l_2$, and we are done. □

4-2: Generators and Isomorphism:

The generator properties defined below can perhaps serve as an indication of whether a geometry is discrete or not. They are based on a generalization of Hűbler's generators, see section (2-3). Let us say that a set S s-generates a submodule M if $M = \langle S \rangle_a$, where $\langle . \rangle_s$ is the closure operator from Lemma (3-1-1). (The standard terminology in algebra is just to say that S generates M, but the prefix s- reduces the risk of confusion.)

Definition(4-2-1)(Generators)

Let M be a d-matroid over an R-module with d-rank at least n. This matroid has the rank n generator property if all d-submodules of d-rank n are s-generated by a set of points of cardinality n. The elements of this set are called *generators*.

Note that the rank n generator property implies that all a-submodules of rank n + 1 are generated by a set of cardinality n (plus the usual translation).Hence, when n = 1 we use the term *line generator property*, or more often just the generator property. For n = 2 we use the term *plane generator property*.

Let us show that in some cases the properties for different n are not independent. In fact, while we are dealing with the theory of finitely generated, torsion free modules over principal ideal domains we might as well give a result about isomorphism as well. Note that the term finitely generated stands for finitely s-generated. Obviously any a-geometry of d-rank n satisfying the rank n generator property is finitely generated.

Theorem 4-2-1

All finitely generated a-geometries over a principal ideal domain R with d-rank n are isomorphic to the R-module over R^n, and they satisfy the rank m generator property for any $m \leq n$.

Proof.

All finitely generated, torsion free modules over R are free, and all finitely generated, free modules with rank n are isomorphic to the R-module over R^n.

Furthermore all d-submodules of a finitely generated, free module are free and finitely generated with module bases of the same size as the d-rank (by Corollary(3-7-1)), so we are done. □

In the end of this section we will see that the rank 1 generator property is equivalent to the *irreducible element property*.

Definition(4-2-2)(Irreducible element)

Let M be an R-module. An element $m \in M \backslash <\emptyset>_d$ is *irreducible* if $bm = am'$ for some $a,b \in R$, $a \neq 0$, $m' \in M$ implies $a|b$.

The module has the irreducible element property if, given any element $m \in M$, there is an irreducible element $m' \in M$ such that $m = r'm'$ for some $r' \in R$.

Proposition (4-2-1.)

Let M be an R-module of d-rank at least 1. Then the rank 1 generator property is equivalent to the irreducible element property. Furthermore the irreducible elements are exactly those elements which are generators for some d-rank 1 subspace.

Proof.

First assume that M has the rank 1 generator property. Take any element $m \in M$. We know that $<m>_d$ has a generator, say $g \in M$. (Unless $<m>_d = <\emptyset>_d$, in which case we can choose a generator from any d-rank 1 subspace.) It follows that $m = rg$ for some $r \in R$. We will now show that g is irreducible.

Assume that $bg = am'$ for some $a,b \in R$, $a \neq 0$, $m' \in M$. It follows that $m' \in <g>_d$. Hence $m' = cg$ for some $c \in R$. By the Representation

Theorem(3-6-1) we get that $c=\frac{b}{a}$, i.e. a|b

Now let M have the irreducible element property instead. Take any d-rank1 subspace$\langle m\rangle_d$, $m\in M\backslash 0$.

Let $g\in M$ be an irreducible element with $m = rg$, $r\in R$. Since g is irreducible it follows that $\langle m\rangle_d=\langle g\rangle_d=\langle g\rangle_s$, whereby g is a generator.

The procedure above also proves the second statement of the proposition, and we are done. □

Conclusions and future work: Summary

A geometry is a mathematical tool to solve some problems in different sciences, from it's important practicing it uses in engineering.

A new geometry that is using in a technology i.e. computer screen and digital camera and image processing…. Etc, called discrete geometry or digital geometry, Hűbler has developed an axiom system with the intention to capture the essence of discrete geometry as utilized in image processing and computer graphs.

One of the standard examples of a matroid is a vector space with its linear, (vector) subspaces. Modules are hardly ever mentioned. Chapters three and four clearly shows that there is no reason to restrict the attention to vector spaces, modules over integral domains work equally well. In fact, some may say, they work too well. At least those that are torsion free; they are naturally embedded in a unique vector space, and hence they can be treated within the framework of vector space theory. (Of course vector spaces are more well-known than modules, and hence more appropriate for introductory examples.).

Modules over integral and ordered domains have been shown to be useful for characterizing Hűbler's geometry. However, even with some discreteness assumption added (perhaps some kind of generator property) they do not provide a framework for discrete geometry general enough for our purposes. Examples of geometries that are hard or impossible to treat within this framework include many finite geometries and the geometry of a cylinder. Having said this there may be more specialized situations where modules can be used, and some of our results may be useful. As an example there are several proved results in this thesis about Hűbler's system (which has been used by Hűbler himself to prove a practical result regarding lines in the digital plane [25]). Furthermore we have shown quite clearly that torsion free modules over integral (ordered) domains work just as well as vector spaces over (ordered) fields as models of infinite (oriented) matroids. Despite the drawbacks with Hübler's system matroids and oriented matroids may still be of use in a general framework for discrete geometry, since these systems have many models that are not based on modules. Since oriented matroids include the concepts order and convexity they are probably more useful than ordinary matroids.

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بسم الله الرحمن الرحيم

هُوَ الَّذِي جَعَلَ الشَّمْسَ ضِيَاءً وَالْقَمَرَ نُورًا وَقَدَّرَهُ مَنَازِلَ
لِتَعْلَمُوا عَدَدَ السِّنِينَ وَالْحِسَابَ مَا خَلَقَ اللَّهُ ذَلِكَ إِلَّا
بِالْحَقِّ يُفَصِّلُ الْآيَاتِ لِقَوْمٍ يَعْلَمُونَ {5}

من سورة يونس عليه السلام

دراسة فرضيات هندسة هوبلر Hûbler المنفصلة

ملخص

سوف نتناول في هذا البحث فرضيات هندسة هوبلر Hûbler المنفصلة وبعض تطبيقاتها ، وذلك لأن هوبلر Hûbler طور بعض الفرضيات في الأنظمة الهندسية المنفصلة بوضع مفهوم يحصر خلاصة الهندسة المنفصلة أو الرقمية بما هو جديد في تقديم الصور ورسومات الحاسوب .

ما قدمه هوبلر Hûbler من العمل المميز لنظامه وضح جميع القياسات على المناطق الصحيحة Integral Domain.

نوضح عن طريق الاستعانة بالأمثلة بعض الصفات لفرضيات هندسة هوبلر Hûbler وذلك حتى يتأتى لنا فهم الحقائق الأساسية لهذه الأنظمة ......

وقد قسمنا هذا العمل كالآتي : -

الفصل التمهيدي :

"مقدمة حول الهندسات المختلفة "

نتناول في هذا الفصل بعض أنواع الهندسات المختلفة وفرضياتها وبعض مفاهيمها

الفصل الأول :

"أدوات البحث "

سيتناول هذا الفصل المفاهيم الأساسية لهندسة هوبلر Hûbler والتي سنحتاجها خلال الفصول القادمة .

الفصل الثاني :

" هندسة هوبلر Hûbler المنفصلة "

في هذا الفصل نذكر فرضيات هندسة هوبلر وبعض خصائصها

الفصل الثالث :

" مفاهيم رئيسية من القياس Module"

في هذا الفصل نعرض أهم مفاهيم والتي لها علاقة بين القياس الجزئي submodule والماترويدات

الفصل الرابع:

" الارتباطات الهندسية والتشاكل"

ندرس في هذا الفصل تعميم فرضية التوازي في المستقيمات و التشاكل ومولدات هندسة هوبلر...

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الموافق: 22 / 7 / 2006 م

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كليـــة العلـــوم

قســم الرياضيات

عنـــوان البحـــث

((دراســـة فرضيــات هندســـة هـــوبلر المنفصـــلة))

ســرت / ليـبـيـا

مقدمــة مــن الطـــالب

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بحث بعنوان:

دراسة فرضيات هندسة هوبلر Húbler المنفصلة

On Hübler axiomatic discrete geometry

استكمالاً لمتطلبات الإجازة العالية الماجستير في علوم الرياضيات

مقدمة من الطالب:

أنيس إبراهيم فضيل سعد

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