University of AL-Tahdi

Faculty of Science

Department of Mathematics

Oscillation of First Order Neutral Delay Differential Equations

A dissertation submitted to the department of mathematics in partial fulfillment of the requirements for the degree of master of science in mathematics

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Faculty of Science
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Title of Thesis

Oscillation of First Order Neutral Delay Differential Equations

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Contents

In the name of Allah, the Beneficent, the Merciful

Introduction

Recently there has been a great deal of work on the Oscillation Behavior of Solutions of Neutral delay differential Equations Or Simply Neutral differential Equations In short NDE.

A Neutral delay differential Equations is a differential Equation in which The Highest Order Derivative Of the unknown function is evaluated Both at the Present state t and at the past state, that is, both with and without delay. for Example,

$$\frac{d}{dt}[x(t)+Px(t-\tau)]+q(t)x(t-\sigma)=0 \qquad \text{for } t \geq t_0$$

where $q(t)>0,\ P\in R-\{0\},\ \tau\in(0,\infty),\ \sigma\in[0,\infty)$.

The Problem of Oscillatory, non – Oscillatory and asymptotic behavior of solutions of **NDE** is both theoretical as well as of practical interest.

The Neutral delay differential Equations of this type appear in networks containing loss less transmission lines. Such networks arise, for Example, in high speed computers where the lass less transmission lines are used to interconnect switching circuits.

It is known that there are drastic differences in the behaviour of solutions between Neutral delay differential Equations and non-neutral delay differential Equations.

This work is divided into 4 chapters.

In the 1st chapter, we give definitions, lemmas and remarks, which are needed in the subsequent chapters. In the 2nd chapter, New Technique to Analyze the Generalized Characteristic Equations to Obtain Some Infinite Integral Conditions for Oscillation of the Delay Differential Equations.

In the 3 rd chapter, integral conditons for the Oscillation of all Solutions of Linear First Order Neutral Delay Differential Equations.

In the 4th chapter, We obtain some new sharp sufficient conditions for the oscillation of all solutions of the first order neutral differential equation with positive and negative coefficients .

CHAPTER (I)

In this chapter, we give definitions, lemmas and theorems which are to be used in this research.

Definition (1.1):

A differential equation in which the highest order derivative of the unknown function appears both with and without delays is called Neutral Delay Differential Equation "in short NDDE".

A general form of Neutral Delay Differential Equation of first order is of the form:

$$\frac{d}{dt}\left[x(t)+g\left(t,x\left(t-\tau_1(t)\right)\right),.....,x\left(t-\tau_m(t)\right)\right]$$
$$+f\left(t,x(t),x\left(t-\sigma_1(t)\right)\right),.....,x\left(t-\sigma_n(t)\right)=0$$

where

$$\lim_{t\to\infty}\left(t-\tau_i(t)\right)=\infty=\lim_{t\to\infty}\left(t-\sigma_j(t)\right),\ i=1,2,.....,m$$
$$j=1,2,.....,n$$

In this work, we shall be discussing the following first order:

$$\frac{d}{dt}\left[x(t)+px(t-\tau)\right]+q(t)x(t-\sigma)=0 \qquad (1.1)$$

for $t \geq t_0$

where $\tau \in (0,\infty), \sigma [0,\infty),\ p \in R,$

and $q:[t_0,\infty) \to R$ is continuous with $q(t) > 0$

in this search we shall be discussing the following delay and neutral delay differential equations for $t \geq t_0$.

Linear first order Neutral Delay Differential Equation:

$$\frac{d}{dt}[x(t) + Px(t-\tau)] + q(t)x(t-\sigma) = 0 \qquad (1.2)$$

where $q(t) > 0,\ P \in R - \{0\},\ \tau \in (0,\infty),\ \sigma \in [0,\infty).$

Remark (1.1):

If $P = 0,\ \sigma \in (0,\infty)$ *then the equation* (1.1).

becomes linear delay differential equation of first order.

$$\frac{d}{dt}[x(t)] + q(t)x(t-\sigma) = 0 \qquad (1.3)$$

Example (1.1):

$$\frac{d}{dt}\left[x(t)+P\,x(t-\tau)\right]+\sum_{j=1}^{n}q_j\,(t)x(t-\sigma_j)=0$$

where $P\in R,\quad q(t)>0,\quad \tau\in(0,\infty),\quad \sigma\in[0,\infty).$

Example (1.2):

$$\frac{d}{dt}\left[x(t)+P\,x(t-\tau)\right]=A\,x(t)-q(t)x(t-\sigma)$$

Example (1.3):

n-th order Neutral Delay Differential Equation

$$\frac{d^n}{dt^n}\left[x(t)+P\,x(t-\tau)\right]+q(t)x(t-\sigma)=0$$

where $\tau\in(0,\infty),\sigma\in[0,\infty),P\in R,P\neq 0$ and $q(t)>0$

Definition (1.2):

The solution of equation (1.1) is a function $x(t)\in C([t_0-r,\infty),R)$

Where $r=\max\{\tau,\sigma\}$ such that

(i) $x(t)+P\,x(t-\tau)$ is continuously differentiable for $t\in[t_0,\infty)$.

(ii) $x\,(t)$ satisfies the equation (1.1) for $t\geq t_0$ for some $t_0\in R$

Remark (1.2):

(i) Let that ϕ be initial function and continuous on the interval $([t_0 - r, t_0], R)$ where $r = \max\{\tau, \sigma\}$ then $x(t) = \phi(t)$ on the interval $[t_0 - r, t_0]$.

(ii) By using the method of steps it follows that if $\phi(t) \in C([t_0 - r, t_0], R)$ then equation (1.1) has unique solution $x(t)$ such that $x(t) = \phi(t)$ on the interval $[t_0 - r, t_0]$.

Definition (1.3):

Oscillatory and non oscillatory, eventually positive and eventually negative functions.

Assume that $x(t)$ on the interval $[t_0, \infty)$ be a solution o f equation (1.2) then $x(t)$ is called oscillatory.

If $x(t)$ has arbitrarily large zero otherwise $x(t)$ is called non oscillatory.

Definition (1.4):

A solution $x(t)$ on the interval $[t_0, \infty)$ of equation (1.2) is called oscillatory if for every $t_1 > t$ there exist $t_2 > t_1$ such that $x(t_2) = 0$, otherwise non oscillatory if there exists $t_1 > t_0$ such that $x(t) \neq 0$ for some $t > t_1$.

Remark (1.3):

If the solution *x(t)* on the interval $[t_0, \infty]$ of equation (1.2) is non oscillatory it must be either eventually positive that is there exist $T \geq t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq T$ or eventually negative, that is $x(t) < 0$ for $t \geq T$.

Definition (1.5):

Equation (1.2) is called oscillatory.

If for every initial point $t_1 \geq t_0$ and $\phi(t) \in C([t_1 - r \ \ t_1], R)$ the solution of equation (1.2) with initial function ϕ at t_1 is oscillatory.

Lemma (1.1):

Assume that $x(t)$ be a solution of equation (1.2) and so then the following function:

(i) $Z(t) = x(t) + Px(t - \tau)$

(ii) $W(t) = Z(t) + PZ(t - \tau)$

are solution of equation (1.2) where *Z (t)* are continuously differentiable and *w(t)* is a twice continuously differentiable solution.

For reference see [11].

Proof: Lemma (1.1):

(i) since equation (1.2) is autonomous $x(t - \tau)$ is also a solution of equation (1.2) since *Z(t)* is the linear combination of solution of a linear differential equation then it is also a solution:

$$\frac{d}{dt}[x(t) + Px(t - \tau)] = Z'(t) = -q(t)x(t - \sigma).$$

Z(t) is continuously differentiable.

(ii) since equation (1.1) is autonomous and $Z(t)$ is a solution of equation (1.2) and then $Z(t-\tau)$ is also solution of equation (1.2) since $W'(t)=-q(t)Z(t-\sigma)$ and $Z(t)$ is continuously differentiable $W(t)$ is twice differentiable solution.

Lemma (1.2):

let $x(t)$ be an eventually positive solution of equation (1.2) and $Z(t)=x(t)+Px(t-\tau)$.

then $Z(t)$ is continuously differentiable solution of equation (1.2).

Proof:

Follows from lemma (1.1). For reference see [9].

Lemma (1.3):

Assume that $x(t)$ be an eventually positive solution of equation (1.2) and $Z(t)=x(t)+Px(t-\tau)$ then $Z(t)$ strictly decreasing function of t and either

$$\lim_{t\to\infty} Z(t)=0 \qquad \text{or} \qquad \lim_{t\to\infty} Z(t)=-\infty$$

in particular, eventually either $Z(t)>0$ or $Z(t)<0$.

Proof:

Since $\frac{d}{dt}\left[x(t)+Px(t-\tau)\right]=Z(t)=-q(t)x(t-\sigma)<0$

Since $q(t)>0$ and $x(t)$ is eventually positive

Hence $Z(t)$ is strictly decreasing function

So either $\lim_{t\to\infty} Z(t)=-\infty$, or $\lim_{t\to\infty} Z(t)=L\in R$

In the latter case, we show $L=0$.

Suppose $L\neq 0$, then since

$\frac{d}{dt}\left[Z(t)+PZ(t-\tau)\right]=-q(t)Z(t-\sigma)$ it follows

$$\lim_{t\to\infty}\left[\frac{d}{dt}[Z(t)+PZ(t-\tau)\right]=-q(t)\,L\neq 0$$

hence $\lim_{t\to\infty}[Z(t)+PZ(t-\tau)]$ is either ∞ or $-\infty$.

But this is a contradiction because:

$$\lim_{t\to\infty}[Z(t)+PZ(t-\tau)]=\lim_{t\to\infty}Z(t)+P\lim_{t\to\infty}Z(t-\tau)=L+PL, L\in R$$

This implies $\lim_{t\to\infty}Z(t)=0$

Thus either $\lim_{t\to\infty}Z(t)=-\infty$ or $\lim_{t\to\infty}Z(t)=0$

Either, because $\lim_{t\to\infty}Z(t)=0$ and $Z(t)$ is decreasing

This implies $Z(t)>0$ eventually or becomes $\lim_{t\to\infty}Z(t)=-\infty$ and $Z(t)$ is decreasing this implies $Z(t)<0$ eventually.

Lemma (1.3):

Assume that $x(t)$ is eventually positive solution of equation (1.2) and set $Z(t)=x(t)+Px(t-\tau)$

If $\lim_{t\to\infty}Z(t)=0$ then $P>-1$.

Proof:

Suppose that $P\leq -1$

Since $Z(t)=x(t)+Px(t-\tau)>0$ eventually

Then $x(t)>Px(t-\tau)\geq x(t-\tau)$

This implies $x(t)$ is bounded below by a positive constant m, that is $x(t) \geq m$ this implies

$$Z'(t) < -mq(t), \quad \because Z'(t) = -q(t)\,(t-\sigma)$$

this implies $\lim_{t \to \infty} Z(t) = -\infty$

This is a contradiction

$$\therefore P > -1 .$$

Definition (1.6):

A function $f(x)$ is said to have a period P or to be periodic with period P if $f(x+P) = f(x)$ for every x in the domain of f.

CHAPTER (II)

In this chapter, we introduce a new technique to analyze the generalized characteristic equations to obtain some infinite integral conditions for oscillation of the delay differential equations

Consider the first order delay differential equation

$$x'(t)+q(t)\,x(t-\sigma)=0 \qquad (2.1)$$

where $q(t)\geq 0$ is a continuous function and σ is a positive constant, or the more general

$$x'(t)+\sum_{i=1}^{n} q_i(t)\,x(t-\sigma_i)=0 \qquad (2.2)$$

where $q_i(t)\geq 0$ are continuous and σ_i are positive constants.

By a solution of equation (2.1) or (2.2)

We mean a function $x\in C[(t_0-\rho,\infty),R]$

For some t_0, where $\rho=\sigma$ or $\left(\rho=\max_{1\leq i\leq n}\{\sigma_i\}\right)$

Satisfies equation (2.1) or (2.2) for all $t\geq t_0$.

In this section we introduce a new technique to analyze the generalized characteristic equations.

Lemma (2.1):

Assume that:

$$\limsup_{t\to\infty} \int_t^{t+\tau_i} q_i(s)\,ds > 0 .$$

For some i, and $x(t)$ is an eventually positive solution of the equation:

$$x'(t) + \sum_{i=1}^{n} q_i(t)\,x(t-\tau_i) = 0 \qquad (2.3)$$

then for same i, $\liminf_{t\to\infty} \dfrac{x(t-\tau_i)}{x(t)} < \infty$.

Proof: see reference [2]

Suppose that there exist a constant $d > 0$ and a sequence $\{t_k\}$ such that

$t_k \to \infty$ as $k \to \infty$ and $\int_{t_k}^{t_k+\tau_i} q_i(s)\,ds \ge d,\ k = 1,2,\ldots.$

then there exists $\zeta_k \in (t_k, t_k + \tau_i)$ for every k

such that

$$\int_{t_k}^{\zeta_k} q_i(s)\,ds \ge \frac{d}{2} \text{ and } \int_{\zeta_k}^{t_k+\tau_i} q_i(s)\,ds \ge \frac{d}{2} \qquad (2.4)$$

on the other hand, equation (2.1) implies

$x'(t) + q_i(t)\,x(t-\tau_i) \le 0$ .... (2.5) eventually

for same i

By integrating (2.5) over the intervals $[t_k,\ \zeta_k]$ and $[\zeta_k,\ t_k + \tau_i]$ we find

$$x(\zeta_k)-x(t_k)+\int_{t_k}^{\zeta_k} q_i(s)x(s-\tau_i)ds \le 0 \qquad (2.6)$$

and

$$x(t_k+\tau_i)-x(\zeta_k)+\int_{\zeta_k}^{t_k+\tau_i} q_i(s)x(s-\tau_i)ds \le 0 \qquad (2.7)$$

By omitting the first terms (2.6) and (2.7) and by using the decreasing nature of $x(t)$ and (2.4) we find:

$$\int_{t_k}^{\zeta_k} q_i(s)x(s-\tau_i)ds = x(s-\tau_i)\Bigg|_{t_k}^{\zeta_k}\left[\int_{t_k}^{\zeta_k} q_i(s)\,ds\right]-\int_{t_k}^{\zeta_k}\left(\int_{t_k}^{\zeta_k} q_i(s)\,ds\right)x'(s-\tau_i)dx$$

$$\int_{t_k}^{\zeta_k} q_i(s)x(s-\tau_i)ds = \left[x(\zeta_k-\tau_i)-x(t_k-\tau_i)\right]\int_{t_k}^{\zeta_k} q_i(s)\,ds - \int_{t_k}^{\zeta_k}\left[\int_{t_k}^{\zeta_k} q_i(s)\,ds\right]x'(s-\tau_i)dx$$

$$\int_{t_k}^{\zeta_k} q_i(s)x(s-\tau_i)ds = x(\zeta_k-\tau_i)\int_{t_k}^{\zeta_k} q_i(s)\,ds - x(t_k-\tau_i)\int_{t_k}^{\zeta_k} q_i(s)\,ds$$

$$-\int_{t_k}^{\zeta_k}\left[\int_{t_k}^{\zeta_k} q_i(s)\,ds\right]x'(s-\tau_i)dx$$

since $x'(t_k-\tau_i)<0$

$$\therefore \int_{t_k}^{\zeta_k} q_i(s)x(s-\tau_i)ds \ge x(\zeta_k-\tau_i)\int_{t_k}^{\zeta_k} q_i(s)\,ds$$

since $\int_{t_k}^{\zeta_k} q_i(s)ds \ge \frac{d}{2}$

$$\therefore \int_{t_k}^{\zeta_k} q_i(s)x(s-\tau_i)ds \geq x(\zeta_k-\tau_i)\frac{d}{2} \tag{2.8}$$

from (2.6) and (2.8) we have

$$\therefore x(\zeta_k)-x(t_k)+x(\zeta_k-\tau_i)\ \frac{d}{2} \leq 0$$

or

$$-x(t_k)+x(\zeta_k-\tau_i)\frac{d}{2} \leq 0 \tag{2.9}$$

$$-x(t_k)\frac{d}{2}+x(\zeta_k-\tau_i)\left(\frac{d}{2}\right)^2 \leq 0 \tag{2.10}$$

also

$$\int_{\zeta_k}^{t_k+\tau_i} q_i(s)x(s-\tau_i)ds = x(s-\tau_i)\Bigg|_{\zeta_k}^{t_k+\tau_i} \int_{\zeta_k}^{t_k+\tau_i} q_i(s)ds - \int_{\zeta_k}^{t_k+\tau_i}\left(\int_{\zeta_k}^{t_k+\tau_i} q_i(s)\ ds\right)x'(s-\tau_i)\,dx$$

$$\int_{\zeta_k}^{t_k+\tau_i} q_i(s)x(s-\tau_i)ds =$$

$$x(t_k)\int_{\zeta_k}^{t_k+\tau_i} q_i(s)ds - x(\zeta_k-\tau_i)\int_{\zeta_k}^{t_k+\tau_i} q_i(s)ds - \int_{\zeta_k}^{t_k-\tau_i}\left(\int_{\zeta_k}^{t_k+\tau_i} q_i(s)\ ds\right)x'(s-\tau_i)\,dx$$

since $x'(s-\tau_i)<0$

$$\therefore \int_{\zeta_k}^{t_k+\tau_i} q_i(s)x(s-\tau_i)ds \geq x(t_k)\int_{\zeta_k}^{t_k+\tau_i} q_i(s)ds$$

since $\int_{\zeta_k}^{t_k+\tau_i} q_i(s)ds \geq \frac{d}{2}$

$$\therefore \int_{\zeta_k}^{t_k+\tau_i} q_i(s)x(s-\tau_i)ds \geq x(t_k)\frac{d}{2}$$

$$\therefore x(t_k+\tau_i)-x(\zeta_k)+x(t_k)\frac{d}{2} \leq 0$$

$$-x(\zeta_k)+x(t_k)\ \frac{d}{2} \leq 0 \qquad (2.11)$$

from (2.10) and (2.11) we have

$$-x(\zeta_k)+x(\zeta_k-\tau_i)\left(\frac{d}{2}\right)^2 \leq 0$$

$$\therefore x(\zeta_k-\tau_i)\left(\frac{d}{2}\right)^2 \leq x(\zeta_k)$$

$$\frac{x(\zeta_k-\tau_i)}{x(\zeta_k)} \leq \left(\frac{2}{d}\right)^2$$

this complete the proof.

Theorem (2.1): for reference see [2]

Assume that $\int_{t}^{t+\sigma} q(s)ds > 0$ for $t \geq t_0$ for some $t_0 > 0$

and

$$\int_{t_0}^{\infty} q(t)\ln\left(e\int_{t}^{t+\sigma} q(s)ds\right)dt = \infty \qquad (2.12)$$

then every solution of equation (2.1) oscialltes.

Proof:

Assume the contrary.

then we may have an eventually positive solution $x(t)$ of equation (2.1).

obviously $x(t)$ is eventually monotonically decreasing

suppose that $\lambda(t) = \frac{-x'(t)}{x(t)}$

clearly for large t, the function $\lambda(t)$ is non negative and continuous, and

$\lambda(t) = \frac{-x'(t)}{x(t)}$ since $x(t)$ is positive and decreasing so there exists $t_1 \geq t_0$ with

$x(t_1) > 0$

Now we integrate the last equation from t_1 to t we have

$$\int_{t_1}^{t} \lambda(s)\, ds = -\int_{t_1}^{t} \frac{x'(s)}{x(s)} ds$$

$$\int_{t_1}^{t} \lambda(s)\, ds = -\left[\ln x(t) - \ln x(t_1)\right]$$

$$\int_{t_1}^{t} \lambda(s)\, ds = \ln x(t_1) - \ln x(t)$$

$$\int_{t_1}^{t} \lambda(s)\, ds = \ln \frac{x(t_1)}{x(t)}$$

$$\exp \int_{t_1}^{t} \lambda(s)\, ds = \exp\left[\ln \frac{x(t_1)}{x(t)}\right], \; x(t)\left[\exp \int_{t_1}^{t} \lambda(s)\, ds\right] = x(t_1)$$

$$x(t) = x(t_1)\exp\left(-\int_{t_1}^{t}\lambda(s)ds\right) \qquad (2.13)$$

Furthermore $\lambda(t)$ satisfies the generalized characteristic equation

$$x'(t) + q(s)x(t-\sigma) = 0$$

or

$$-\lambda(t)x(t) + q(t)x(t-\sigma) = 0$$

or

$$-\frac{\lambda(t)x(t)}{x(t)} + \frac{q(t)x(t-\sigma)}{x(t)} = 0$$

we get

$$-\lambda(t) + q(t)\frac{x(t-\sigma)}{x(t)} = 0 \qquad (2.14)$$

Now

Take $t_1 = t-\sigma$ of (2.13) we have

$$x(t) = x(t-\sigma)\exp\left(-\int_{t-\sigma}^{t}\lambda(s)ds\right)$$

$$\therefore \frac{x(t)}{x(t-\sigma)} = \exp\left(-\int_{t-\sigma}^{t}\lambda(s)ds\right)$$

$$\therefore \frac{x(t-\sigma)}{x(t)} = \exp\left(\int_{t-\sigma}^{t}\lambda(s)ds\right) \qquad (2.15)$$

From (2.14) and (2.15) we get

$$-\lambda(t)+q(t)\exp\left(\int_{t-\sigma}^{t}\lambda(s)\,ds\right)=0$$

$$\therefore \lambda(t)=q(t)\exp\left(\int_{t-\sigma}^{t}\lambda(s)\,ds\right)$$

by using the inequality

$e^{rx}\geq x+\dfrac{\ln(er)}{r}$ for $r>0$ see reference [2]

and thus

$$\lambda(t)=q(t)\exp\left(A(t).\ \frac{1}{A(t)}\int_{t-\sigma}^{t}\lambda(s)\,ds\right)\geq q(t)\left[\frac{1}{A(t)}\int_{t-\sigma}^{t}\lambda(s)\,ds+\frac{\ln(eA(t))}{A(t)}\right]$$

where $A(t)=\int_{t}^{t+\sigma}q(s)\,ds$, it follows that

$$\lambda(t)\ A(t)\geq q(t)\int_{t-\sigma}^{t}\lambda(s)\,ds+q(t)\left[\ln(e\,A(t))\right]$$

$$\lambda(t)\ A(t)-q(t)\int_{t-\sigma}^{t}\lambda(s)\,ds\geq q(t)\left[\ln(e\,A(t))\right]$$

$$\lambda(t)\int_{t}^{t+\sigma} q(s)ds - q(t)\int_{t-\sigma}^{t} \lambda(s)ds \geq q(t)\ln\left[e\int_{t}^{t+\sigma} q(t)ds\right]$$

then for $N > T$ therefore

$$\int_{T}^{N} \lambda(t)\int_{t}^{t+\sigma} q(s)ds\,dt - \int_{T}^{N} q(t)\int_{t-\sigma}^{t} \lambda(s)ds\,dt \geq$$

$$\int_{T}^{N} q(t)\ln\left[e\int_{t}^{t+\sigma} q(s)ds\right]dt \qquad (2.16)$$

By interchanging the order of integration, we find

$$\int_{T}^{N} q(t)\int_{t-\sigma}^{t} \lambda(s)ds\,dt = \int_{T-\sigma}^{N}\int_{s}^{s+\sigma} q(t)\lambda(s)\,dt\,ds$$

$$\geq \int_{N-\sigma}^{T}\int_{s}^{s-\sigma} q(t)\lambda(s)dt\,ds$$

$$\geq \int_{T}^{N-\sigma}\int_{s}^{s+\sigma} q(t)\lambda(s)dt\,ds$$

Therefore

$$\therefore \int_{T}^{N} q(t)\int_{t-\sigma}^{t} \lambda(s)ds\,dt \geq \int_{T}^{N-\sigma}\int_{s}^{s+\sigma} q(t)\lambda(s)\,dt\,ds$$

$$\int_T^N q(t)\int_{t-\sigma}^{t}\lambda(s)\,ds\,dt \ge \int_T^{N-\sigma}\left(\int_s^{s+\sigma} q(t)\,\lambda(s)\,dt\right)ds$$

$$= \int_T^{N-\sigma}\lambda(s)\int_s^{s+\sigma} q(t)\,dt\,ds$$

$$= \int_T^{N-\sigma}\lambda(t)\int_t^{t+\sigma} q(s)\,ds\,dt \qquad (2.17)$$

From (2.16) and (2.17), it follows that

$$\int_T^N \lambda(t)\int_t^{t+\sigma} q(s)\,ds\,dt - \int_T^{N-\sigma}\lambda(t)\int_t^{t+\sigma} q(t)\,dt\,ds \ge$$

$$\int_T^N q(t)\ln\left(e\int_t^{t+\sigma} q(s)\,ds\right)dt$$

thus

$$\int_{N-\sigma}^{T}\lambda(t)\int_t^{t+\sigma} q(s)\,ds\,dt + \int_T^N \lambda(t)\int_t^{t+\sigma} q(s)\,ds\,dt \ge$$

$$\int_T^N q(t)\ln\left(e\int_t^{t+\sigma} q(s)\,ds\right)dt$$

or

$$\int_{N-\sigma}^{N}\lambda(t)\int_{t}^{t+\sigma}q(s)ds\,dt \geq \int_{T}^{N}q(t)\ln\left(e\int_{t}^{t+\sigma}q(s)ds\right)dt \qquad (2.18)$$

then

$$\int_{N-\sigma}^{N}\lambda(t)dt \geq \int_{T}^{N}q(t)\ln\left(e\int_{t}^{t+\sigma}q(s)ds\right)dt$$

or

$$\ln\frac{x(N-\sigma)}{x(N)} \geq \int_{T}^{N}q(t)\ln\left(e\int_{t}^{t+\sigma}q(s)ds\right)dt$$

$$\lim_{N\to\infty}\ln\frac{x(N-\sigma)}{x(N)} = \int_{T}^{n}q(t)\ln\left(e\int_{t}^{t+\sigma}q(s)ds\right)dt$$

from (2.12)

$$\lim_{t\to\infty}\frac{x(t-\sigma)}{x(t)} = \infty \qquad (2.19)$$

On the other hand (2.12) implies that there exists sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$\int_{t_n}^{t_n+\sigma}q(s)ds > \frac{1}{e} \text{ for all } n$$

Hence by lemma (2.1) we obtain

$$\liminf_{t\to\infty}\frac{x(t-\sigma)}{x(t)} < \infty$$

This contradicts (2.19)

Completes the proof theorem.

Theorem (2.2):

Let $\sigma_n = \max\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$

Suppose that

$$\sum_{i=1}^{n} \int_{t}^{t+\sigma_i} q_i(s)\,ds > 0$$

for $t \ge t_0$ for some $t_0 > 0$ and that

$$\limsup_{t \to \infty} \int_{t}^{t+\sigma_n} q_n(s)\,ds > 0$$

if, in addition

$$\int_{t_0}^{\infty} \left(\sum_{i=1}^{n} q_i(t) \right) \ln \left[e \sum_{i=1}^{n} \int_{t}^{t+\sigma_i} q_i(s)\,ds \right] dt = \infty \qquad (2.20)$$

Then every solution of equation (2.2) oscillates

Proof: see reference [2]

Assume the contrary then equation (2.2) may have an eventually positive and decreasing solution $x(t)$.

Let $\lambda(t) = -x'(t)/x(t)$ therefore

$\lambda(t)$ is non negative and continuous and there exists $t_1 \geq t_0$ with $x(t_1) > 0$

such that $x(t) = x(t_1)\exp\left(-\int_{t_1}^{t}\lambda(s)ds\right)$ (2.21)

$\lambda(t)$ satisfies the generalized characteristic equation

$$x'(t) + \sum_{i=1}^{n} q_i(t)x(t-\sigma_i) = 0$$

or

$$-\lambda(t)x(t) + \sum_{i=1}^{n} q_i(t)x(t-\sigma_i) = 0$$

$$-\lambda(t) + \sum_{i=1}^{n} q_i(t)x\frac{(t-\sigma_i)}{x(t)} = 0 \qquad (2.22)$$

Now

Put $t_1 = t - \sigma_i$, $1 \leq i \leq n$ in equation (2.21)

We get

$$x(t) = x(t-\sigma_i)\exp\left(-\int_{t-\sigma_i}^{t}\lambda(s)\,ds\right)$$

$$\frac{x(t)}{x(t-\sigma_i)} = \exp\left(-\int_{t-\sigma_i}^{t}\lambda(s)\,ds\right) \qquad (2.23)$$

from (2.22) and (2.23) we have

$$-\lambda(t)+\sum_{i=1}^{n} q_i(t)\exp\left(\int_{t-\sigma_i}^{t}\lambda(s)\,ds\right)=0$$

therefore

$$\lambda(t)=\sum_{i=1}^{n} q_i(t)\exp\left(\int_{t-\sigma_i}^{t}\lambda(s)\,ds\right)$$

Assume that

$$B(t)=\sum_{i=1}^{n}\int_{t}^{t+\sigma_i} q_i(s)\,ds\,,$$

By using the inequality

$$e^{rx}\geq x+\frac{\ln(er)}{r}\quad \text{for } r>0$$

we have

$$\lambda(t)=\sum_{i=1}^{n} q_i(t)\exp\left(B(t)\cdot\frac{1}{B(t)}\int_{t-\sigma_i}^{t}\lambda(s)\,ds\right)\geq$$

$$\sum_{i=1}^{n} q_i(t)\left[\frac{1}{B(t)}\int_{t-\sigma_i}^{t}\lambda(s)\,ds+\frac{\ln(eB(t))}{B(t)}\right]$$

$$\lambda(t)B(t)\geq\sum_{i=1}^{n} q_i(t)\int_{t-\sigma_i}^{t}\lambda(s)\,ds+\sum_{i=1}^{n} q_i(t)\,\ln(eB(t))$$

$$\lambda(t)\left(\sum_{i=1}^{n}\int_{t}^{t+\sigma_i} q_i(s)ds\right) \ge \sum_{i=1}^{n} q_i(t) \int_{t-\sigma_i}^{t}\lambda(s)ds$$

$$+\sum_{i=1}^{n} q_i(t)\ \ln\left(e\sum_{i=1}^{n}\int_{t}^{t+\sigma_i} q_i(s)ds\right)$$

then

$$\lambda(t)\left(\sum_{i=1}^{n}\int_{t}^{t+\sigma_i} q_i(s)ds\right) - \sum_{i=1}^{n} q_i(t) \int_{t-\sigma_i}^{t}\lambda(s)ds \ge$$

$$\sum_{i=1}^{n} q_i(t)\ \ln\left(e\sum_{i=1}^{n}\int_{t}^{t+\sigma_i} q_i(s)ds\right)$$

then for $N > T$

$$\int_{T}^{N}\left(\sum_{i=1}^{n}\int_{t}^{t+\sigma_i} q_i(s)ds\right)\lambda(t)\,dt - \int_{T}^{N}\sum_{i=1}^{n} q_i(t)\int_{t-\sigma_i}^{t}\lambda(s)ds\,dt \ge$$

$$\int_{T}^{N}\left(\sum_{i=1}^{n} q_i(s)\right)\ln\left(e\sum_{i=1}^{n}\int_{t}^{t+\sigma_i} q_i(s)ds\right)dt \qquad (2.24)$$

By interchanging the order of integration we find

$$\int_{T}^{N}\sum_{i=1}^{n} q_i(t)\int_{t-\sigma}^{t}\lambda(s)ds\,dt = \sum_{i=1}^{n}\int_{T-\sigma_i}^{N}\int_{s}^{s+\sigma_i} q_i(t)\lambda(s)dt\,ds$$

$$\sum_{i=1}^{n}\int_{T-\sigma_i}^{N}\int_{s}^{s+\sigma_i} q_i(t)\lambda(s)\,dt\,ds \ge \sum_{i=1}^{n}\int_{T}^{N-\sigma_i}\int_{s}^{s+\sigma_i} q_i(t)\lambda(s)\,dt\,ds$$

$$\int_{T}^{N}\sum_{i=1}^{n} q_i(t)\int_{t-\sigma_i}^{t}\lambda(s)\,ds\,dt \ge \sum_{i=1}^{n}\int_{T}^{N-\sigma_i}\int_{s}^{s+\sigma_i} q_i(t)\lambda(s)\,dt\,ds =$$

$$\sum_{i=1}^{n}\int_{T}^{N-\sigma_i}\lambda(s)\int_{s}^{s+\sigma_i} q_i(t)\,dt\,ds =$$

$$\sum_{i=1}^{n}\int_{T}^{N-\sigma_i}\lambda(t)\int_{t}^{t+\sigma_i} q_i(s)\,ds\,dt$$

$$\int_{T}^{N}\sum_{i=1}^{n} q_i(t)\int_{t-\sigma_i}^{t}\lambda(s)\,ds\,dt \ge \sum_{i=1}^{n}\int_{T}^{N-\sigma_i}\lambda(t)\int_{t}^{t+\sigma_i} q_i(s)\,ds\,dt \tag{2.25}$$

From (2.24) and (2.25) we get

$$\sum_{i=1}^{n}\int_{N-\sigma_i}^{N}\lambda(t)\int_{t}^{t+\sigma_i} q_i(s)\,ds\,dt \ge$$

$$\int_{T}^{N}\left(\sum_{i=1}^{n} q_i(t)\right)\ln\left[e\sum_{i=1}^{n}\int_{t}^{t-\sigma_i} q_i(s)\,ds\right]dt \tag{2.26}$$

since $\int_{t}^{t+\sigma_i} q_i(s)\,ds \le 1,\ i = 1, 2 \ldots\ldots n$

then

$$\sum_{i=1}^{n} \int_{N-\sigma_i}^{N} \lambda(t)dt \geq \int_{T}^{N}\left(\sum_{i=1}^{n} q_i(t)\right)\ln\left[e\sum_{i=1}^{n} \int_{t}^{t+\sigma_i} q_i(s)ds\right]dt$$

or

$$\sum_{i=1}^{n} \ln\frac{x(N-\sigma_i)}{x(N)} \geq \int_{T}^{N}\left(\sum_{i=1}^{n} q_i(t)\right)\ln\left[e\sum_{i=1}^{n} \int_{t}^{t+\sigma_i} q_i(s)ds\right]dt$$

in view of (2.20)

$$\lim_{t\to\infty} \prod_{i=1}^{n} \frac{x(t-\sigma_i)}{x(t)} = \infty$$

this implies

$$\lim_{t\to\infty} \frac{x(t-\sigma_n)}{x(t)} = \infty \qquad (2.27)$$

Hence by lemma (2.1) we have

$$\liminf_{t\to\infty} \frac{x(t-\sigma_n)}{x(t)} < \infty$$

This contradicts (2.27)

Completes the proof of theorem.

Example (2.1):

Consider the delay differential equation

$$x'(t) + \exp(k\sin t - 1)x(t-1) = 0 \qquad (2.28)$$

where $q(t) = \exp(k \sin t - 1)$ and

k is a positive constant. Clearly

$$\int_0^\infty q(t) \ln\left(e \int_t^{t+1} q(s) ds \right) dt \geq \int_0^\infty q(t) \int_t^{t+1} k \sin ds \, dt$$

$$= \frac{2k \sin \frac{1}{2}}{e} \int_0^\infty \exp(k \sin t) \sin\left(t + \frac{1}{2}\right) dt$$

on the other hand, it is easy to see that

$\int_0^t \exp(k \sin t) \cos t \, dt$ is bounded and

$$\int_0^{2\pi} \exp(k \sin t) \sin t \, dt > 0$$

it follows that $\int_0^\infty q(t) \ln\left(e \int_t^{t+1} q(s) ds \right) dt = \infty$

Then by theorem (2.1) every solution of equation (2.28) oscillates.

CHAPTER (III)

In this chapter we will study some new integral conditions for the oscillation of all solution of linear first order neutral delay differential equations.

Consider the first order linear neutral delay differential equation

$$\frac{d}{dt}\left[x(t)+px(t-\tau)\right]+q(t)x(t-\sigma)=0 \qquad (3.1)$$

where

(i) $p\in R$, $\tau\in(0,\infty)$, $\sigma\in[0,\infty)$.

(ii) $q:[t_0,\infty)\to R$ is a continuous function with $q(t)>0$.

Lemma (3.1):

Assume that $P\in(1,\infty)$ *and* $\sigma>\tau$

$$\limsup_{t\to\infty}\int_t^{t-\sigma+\tau} q(s)\,ds>0$$

if $x(t)$ is an eventually positive solution of equation (3.1) then

$$\liminf_{t\to\infty}\frac{Z(t-\sigma+\tau)}{Z(t)}<\infty$$

where $Z(t)=x(t)+Px(t-\tau)$

Proof: see reference [2] and [10]

From our hypothesis we say that

$Z(t) > 0$ eventually

and from equation (3.1) we have that

$Z(t)$ is decreasing

since

$$Z'(t) + q(t)x(t-\sigma) = 0$$

$$Z'(t) = -q(t)x(t-\sigma) < 0$$

Since

$$Z(t) = x(t) + Px(t-\tau)$$

$$Px(t-\tau) = Z(t) - x(t) \tag{3.2}$$

take $t = t + \tau$ we obtain

$$Px(t+\tau-\tau) = Z(t+\tau) - x(t+\tau)$$

$$\therefore Px(t) = Z(t+\tau) - x(t+\tau)$$

$$Z(t+\tau) = x(t+\tau) + Px(t)$$

Since $z(t)$ is decreasing, we have

$$Z(t) > Z(t+\tau) \geq Px(t) \tag{3.3}$$

Since $Px(t-\tau)=Z(t)-x(t)$

$$P^2 x(t-\tau)=PZ(t)-Px(t)$$

From (3.2) and (3.3) we have

$$P^2 x(t-\tau)\geq PZ(t)-Z(t)$$

$$x(t-\tau)\geq \frac{PZ(t)-Z(t)}{P^2}$$

$$x(t-\tau)\geq \frac{Z(t)[P-1]}{P^2} \qquad (3.4)$$

take $t=t+\tau-\sigma$ into (3.4) we have

$$x(t-\sigma)\geq \frac{P-1}{P^2}Z(t+\tau-\sigma) \qquad (3.5)$$

from equation (3.1) and (3.5) we have

$$Z'(t)+q(t)x(t-\sigma)=0$$

$$Z'(t)+q(t)\frac{P-1}{P^2}Z(t+\tau-\sigma)\leq 0 \qquad (3.6)$$

by lemma (2.1)

then $\liminf_{t\to\infty} \frac{Z(t-\sigma+\tau)}{Z(t)}<\infty$.

We obtain the desired result.

Lemma (3.2):

Assume that $P \in (1,\infty)$ and $\sigma > \tau$.

If equation (3.1) has eventually positive

Solution then:

$$\int_t^{t+\sigma-\tau} q(s)\, ds \le \frac{p^2}{P-1}, \text{ for sufficiently large } t.$$

Proof:

Proceeding as in the proof of lemma (3.1).

We again obtain (3. 6) integrating (3. 6) from t to $t+\sigma-\tau$ and using the decreasing behaviour of $Z(t)$ we have

from t to $t+\sigma-\tau$ and using the decreasing behaviour of $Z(t)$ we have

$$\int_t^{t+\sigma-\tau} Z'(s)\, ds + \int_t^{t+\sigma-\tau} \left(\frac{P-1}{P^2}\right) q(s)\, Z(s+\tau-\sigma)\, ds \le 0$$

$$Z(t+\sigma-\tau) - Z(t) + \frac{P-1}{P^2} \int_t^{t+\sigma-\tau} q(s)\, Z(s+\tau-\sigma)\, ds \le 0$$

since $Z(t)$ is decreasing then we have

$$Z(t+\sigma-\tau) - Z(t) + \frac{P-1}{P^2} \int_t^{t+\sigma-\tau} q(s)\, Z(s)\, ds. \le 0$$

$u = Z(s)$ $\qquad\qquad$ $du = Z'(s)\, ds$

$$dv = q(s)\,ds \qquad\qquad v = \int_{t}^{t+\sigma-\tau} q(s)\,ds$$

$$\therefore \int_{t}^{t-\sigma-\tau} q(s)Z(s)\,ds = Z(s)\int_{t}^{t+\sigma-\tau} q(s)\,ds -$$

$$\int_{t}^{t+\sigma-\tau}\left(\int_{t}^{t+\sigma-\tau} q(s)\,ds\right) Z'(s)\,ds$$

$$\therefore \int_{t}^{t+\sigma-\tau} Z(s)q(s)\,ds > Z(t)\int_{t}^{t+\sigma-\tau} q(s)\,ds$$

$$\therefore Z(t+\sigma-\tau) - Z(t) + \frac{p-1}{p^2}\left[Z(t)\int_{t}^{t+\sigma-\tau} q(s)\,ds\right] \leq 0$$

$$\therefore Z(t+\sigma-\tau) + \left[\frac{p-1}{p^2}\int_{t}^{t+\sigma-\tau} q(s)\,ds - 1\right] Z(t) \leq 0$$

Since $Z(t) > 0$

$$\frac{p-1}{p^2}\int_{t}^{t+\sigma-\tau} q(s)\,ds \leq 1$$

$$\therefore \int_{t}^{t+\sigma-\tau} q(s)\,ds \leq \frac{p^2}{p-1}, \text{ for large } t.$$

Lemma (3.3):

Assume that d is a positive constant

Let $P \in ([t_0, \infty), R^+)$ and suppose that

$$\liminf_{t \to \infty} \int_{t-d}^{t} q(s)\,ds > \frac{1}{e}$$

then

(i) the inequality:

$$x'(t) - P(t)x(t+d) \le 0, \qquad t \ge t_0$$

has no eventually negative solutions;

(ii) the inequality

$$x'(t) - P(t)x(t+d) \ge 0, \qquad t \ge t_0$$

has no eventually positive solution;

(iii) the inequality

$$x'(t) + P(t)x(t-d) \le 0, \qquad t \ge t_0$$

has no eventually positive solutions;

(iv) the inequality

$$x'(t) + P(t)x(t-d) \le 0, \qquad t \ge t_0$$

has no eventually negative solutions

Proof: see reference [3] and [7]

Theorem (3.1):

Assume that $\sigma > \tau$ *and* $P \in (1, \infty)$, *such that*

$$\limsup_{t \to \infty} \int_{t}^{t+\sigma-\tau} q(s)\, ds > 0 \text{ holds}$$

$$\text{if } \int_{t_0}^{\infty} q(t) \ln\left(\frac{e\,(P-1)}{P^2} \int_{t}^{t+\sigma-\tau} q(s)\, ds \right) dt = \infty \tag{3.7}$$

Then every solution of equation (3.1) oscillates

Proof:

For the sake of contradiction we assume that there is an eventually positive solution $x(t)$ of equation (3.1)

then $Z(t)$ is eventually positive and decreasing and satisfies the inequality (3.6).

$$Z'(t) + \frac{P-1}{P^2} q(t) Z(t+\tau-\sigma) \leq 0$$

let $\lambda(t) = -Z'(t)/Z(t)$ then $\lambda(t)$ is continuous and non negative, since $Z(t)$ is positive and decreasing so there exists $t_1 \geq t_0$ with

$Z(t_1) > 0$ such that

$$\int_{t_1}^{t} \lambda(s)\,ds = -\int_{t_1}^{t} \frac{Z'(s)}{Z(s)}\,ds$$

$$\int_{t_1}^{t} \lambda(s)\,ds = -\ln Z(s)\Big|_{t_1}^{t} = -\big(\ln Z(t) - \ln Z(t_1)\big)$$

then

$$-\int_{t_1}^{t} \lambda(s)\,ds = \ln \frac{Z(t)}{Z(t_1)}$$

or

$$\exp\left(-\int_{t_1}^{t} \lambda(s)\,ds\right) = \frac{Z(t)}{Z(t_1)}$$

thus

$$Z(t) = Z(t_1)\exp\left(-\int_{t_1}^{t} \lambda(s)\,ds\right) \qquad (3.8)$$

Now,

$$-\lambda(t)\,Z(t) + \frac{P-1}{p^2} q(t) Z(t+\tau-\sigma) \le 0$$

$$\frac{-\lambda(t)\,Z(t)}{Z(t)} + \frac{P-1}{p^2} q(t) \frac{Z(t+\tau-\sigma)}{Z(t)} \le 0$$

$$-\lambda(t) + \frac{P-1}{p^2} q(t) \frac{Z(t+\tau-\sigma)}{Z(t)} \le 0 \qquad (3.9)$$

Put $t_1 = t + \tau - \sigma$ in equation (3.8) we get

$$Z(t) = Z(t+\tau-\sigma)\exp\left(-\int_{t+\tau-\sigma}^{t}\lambda(s)\,ds\right)$$

$$\frac{Z(t)}{Z(t+\tau-\sigma)} = \exp\left(-\int_{t+\tau-\sigma}^{t}\lambda(s)\,ds\right) \qquad (3.10)$$

from (3.9) and (3.10) we have

$$-\lambda(t) + \frac{P-1}{p^2}q(t)\exp\left(-\int_{t+\tau-\sigma}^{t}\lambda(s)\,ds\right) \le 0$$

Moreover, $\lambda(t)$ satisfies

$$\lambda(t) \ge \frac{P-1}{p^2}q(t)\exp\int_{t+\tau-\sigma}^{t}\lambda(s)\,ds \qquad (3.11)$$

using the inequality

$e^{rx} \ge x + \ln{(er)}/r$ for $x > 0$ and $r > 0$ to (3.11) we have

$$\lambda(t) \ge \frac{P-1}{p^2}q(t)\exp\frac{A(t)}{A(t)}\int_{t+\tau-\sigma}^{t}\lambda(s)\,ds \ge$$

$$\frac{P-1}{p^2}q(t)\left[\frac{1}{A(t)}\int_{t+\tau-\sigma}^{t}\lambda(s)\,ds + \frac{\ln(eA(t))}{A(t)}\right]$$

where $r = A(t)$,

$$x = \frac{1}{A(t)} \int_{t+\tau-\sigma}^{t} \lambda(s)\,ds$$

we take

$$A(t) = \frac{P-1}{P^2} \int_{t}^{t+\tau-\sigma} q(s)\,ds$$

$$\lambda(t) \geq \frac{P-1}{p^2} q(t) \frac{1}{A(t)} \left[\int_{t+\tau-\sigma}^{t} \lambda(s)\,ds + \ln(e\,A(t)) \right] \tag{3.12}$$

$$\lambda(t)\,A(t) \geq \frac{P-1}{P^2} q(t) \left[\int_{t+\tau-\sigma}^{t} \lambda(s)\,ds + \ln(e\,A(t)) \right]$$

$$\lambda(t) \frac{P-1}{P^2} \int_{t}^{t+\sigma-\tau} q(s)\,ds \geq \frac{P-1}{P^2} q(t) \left[\int_{t+\tau-\sigma}^{t} \lambda(s)\,ds + \ln(e\,A(t)) \right]$$

$$\lambda(t) \int_{t}^{t+\sigma-\tau} q(s)\,ds \geq q(t) \left[\int_{t+\tau-\sigma}^{t} \lambda(s)\,ds + \ln(e\,A(t)) \right]$$

$$\lambda(t) \int_{t}^{t+\sigma-\tau} q(s)\,ds - q(t) \int_{t+\tau-\sigma}^{t} \lambda(s)\,ds \geq q(t) \ln\left(e \left(\frac{P-1}{P^2} \int_{t}^{t+\sigma-\tau} q(s)\,ds \right) \right)$$

Then for $u > T + \sigma - \tau$ we have

$$\int_{T}^{u} \lambda(t) \left(\int_{t}^{t+\sigma-\tau} q(s)\,ds \right) dt - \int_{T}^{u} q(t) \left(\int_{t+\sigma-\tau}^{t} \lambda(s)\,ds \right) dt \geq$$

$$\int_T^u q(t)\ln\left(\frac{e(P-1)}{P^2}\int_t^{t+\sigma-\tau} q(s)\,ds\right)dt \tag{3.13}$$

inter changing the order of integration we obtain

$$\int_T^u q(t)\int_{t+\sigma-\tau}^{t}\lambda(s)\,ds\,dt \geq \int_t^{u+\tau-\sigma}\lambda(t)\left(\int_t^{t+\sigma-\tau} q(s)\,ds\right)dt \tag{3.14}$$

from (3.13) and (3.14) we have

$$\therefore \int_T^u \lambda(t)\left(\int_t^{t+\sigma-\tau} q(s)\,ds\right)dt - \int_T^{u+\tau-\sigma}\lambda(t)\left(\int_t^{t+\sigma-\tau} q(s)\,ds\right)dt \geq$$

$$\int_T^u q(t)\ln\left(\frac{e(P-1)}{P^2}\int_t^{t+\sigma-\tau} q(s)\,ds\right)dt$$

$$\int_T^u \lambda(t)\left[\int_t^{t+\sigma-\tau} q(s)\,ds\right]dt + \int_{u+\tau-\sigma}^{T}\lambda(t)\int_t^{t+\sigma-\tau} q(s)\,ds\ \ dt \geq$$

$$\int_T^u q(t)\ln\left(\frac{e(P-1)}{P^2}\int_t^{t+\sigma-\tau} q(s)\,ds\right)dt$$

$$\therefore \int_{u+\tau-\sigma}^{u}\lambda(t)\left[\int_t^{t+\sigma-\tau} q(s)\,ds\right]dt \geq \int_T^u q(t)\ln\left(\frac{e(P-1)}{P^2}\int_t^{t-\sigma-\tau} q(s)\,ds\right)dt \tag{3.15}$$

From (3.15) and lemma (3.2) we have

$$\int_{u-\tau-\sigma}^{u}\lambda(t)\,dt \geq \frac{P-1}{P^2}\int_{T}^{u} q(t)\ln\left(\frac{e(P-1)}{P^2}\int_{t}^{t+\sigma-\tau} q(s)\,ds\right)dt$$

$$\ln\frac{Z(u+\tau-\sigma)}{Z(u)} \geq \frac{P-1}{P^2}\int_{T}^{u} q(t)\ln\left(\frac{e(P-1)}{P^2}\int_{t}^{t+\sigma-\tau} q(s)\,ds\right)dt$$

But in view of (3. 7) we must have

$$\lim_{t\to\infty}\frac{Z(t+\tau-\sigma)}{Z(t)} = \infty .$$

Which contradicts since by lemma (3.1)

$$\lim_{t\to\infty}\inf\frac{Z(t-\sigma+\tau)}{Z(t)} < \infty$$

Completes the proof of the theorem.

Example (1.1):

Consider the neutral delay differential equations

$$\frac{d}{dt}\left[x(t)+2x(t-1)\right]+\frac{4}{e}\left(1+\frac{1}{t}\right)x(t-2)=0,\quad t\geq 2 \qquad (3.16)$$

Here $P=2,\quad q(t)=\left(\frac{4}{e}\right)\left(1+\frac{1}{t}\right)\quad \sigma=2,\quad \tau=1$

$$\int_{t_0}^{\infty} q(t)\ln\left(\frac{e(P-1)}{P^2}\int_{t}^{t+\sigma-\tau} q(s)\,ds\right)dt = \int_{2}^{\infty}\frac{4}{e}\left(1+\frac{1}{t}\right)\ln\left(\frac{e}{4}\int_{t}^{t+1}\frac{4}{e}\left(1+\frac{1}{s}\right)ds\right)dt$$

$$= \frac{4}{e}\int_{2}^{\infty}\left(1+\frac{1}{t}\right)\left[\ln(s+\ln s)\Big|_{t}^{t+1}\right]dt = \frac{4}{e}\int_{2}^{\infty}\left(1+\frac{1}{t}\right)\ln(1+\ln(t+1)-\ln t)\,dt$$

$$= \frac{4}{e}\int_{2}^{\infty}\left(1+\frac{1}{t}\right)\ln\left(1+\ln\left(1+\frac{1}{t}\right)\right)dt$$

$$= \frac{4}{e}\int_{2}^{\infty}\ln\left(1+\ln\left(1+\frac{1}{t}\right)\right)dt + \int_{2}^{\infty}\frac{1}{t}\ln\left(1+\ln\left(1+\frac{1}{t}\right)\right)dt \geq$$

$$\frac{4}{e}\int_{2}^{\infty}\ln\left(1+\ln\left(1+\frac{1}{t}\right)\right)dt$$

$$\therefore \int_{2}^{\infty} q9t)\ln\left(\frac{P-1}{P^{2}}\int_{t}^{t+1} q(s)\,ds\right) \geq \frac{4}{e}\int_{2}^{\infty}\ln\left(1+\ln\left(1+\frac{1}{t}\right)\right)dt = \infty$$

By theorem (3.1) every solution of equation (3.16) oscillates.

Theorem (3.2):

Assume that $\sigma > \tau$ *and* $P \in (1, \infty)$ *and there exist a constant* $k > 0$ *such that*

$$\frac{1}{e} \leq \int_{t-\sigma+\tau}^{t} q(s)\,ds < k \qquad (3.17)$$

Then every solution of equation (3.1) is oscillation.

Proof: see reference [10]

Assume that $Z(t)$ is eventually positive and decreasing and satisfies

$$Z'(t)+\frac{P-1}{p^2}\,q(t)Z(t+\tau-\sigma)\ \le 0 \text{ and also}$$

$$\lambda(t)\ge\frac{P-1}{p^2}q(t)\exp\left[\int_{t+\tau-\sigma}^{t}\lambda(s)\,ds\right]$$

$$B(t)\lambda(t)\ge\frac{P-1}{p^2}\,B(t)q(t)\exp\left[\frac{B(t)}{B(t)}\int_{t+\tau-\sigma}^{t}\lambda(s)\,ds\right]$$

using the inequality

$$e^{x/r}\ \ge 1+\frac{x}{r^2}\ \text{ for } x>0 \text{ and } r>1$$

we obtain

$$B(t)\ \lambda(t)-q(t)\frac{P-1}{P^2}\int_{t+\tau-\sigma}^{t}\lambda(s)\ ds\ge q(t)\,A(t)$$

where

$$A(t)=\frac{P-1}{p^2}\,B(t)$$

Since

$$x=B(t)\int_{t+\tau-\sigma}^{t}\lambda(s)ds \text{ and } r=\frac{1}{B(t)}$$

we let

$$\exp\left(e \int_{t+\tau-\sigma}^{t} q(s)\,ds \right) = B(t)$$

$$B(t)\,\lambda(t) \geq \frac{P-1}{P^2} B(t) q(t) \left[1 + \frac{B(t)}{B(t)^2} \int_{t-\tau-\sigma}^{t} \lambda(s)\,ds \right]$$

$$B(t)\,\lambda(t) \geq \frac{P-1}{P^2} B(t) q(t) + \frac{P-1}{P^2} q(t) \int_{t+\tau-\sigma}^{t} \lambda(s)\,ds$$

$$B(t)\,\lambda(t) - \frac{P-1}{P^2} q(t) \int_{t+\tau-\sigma}^{t} \lambda(s)\,ds \geq \frac{P-1}{P^2} q(t)\, B(t)$$

$$B(t)\,\lambda(t) - \frac{P-1}{P^2} q(t) \int_{t-\tau-\sigma}^{t} \lambda(s)\,ds \geq A(t) q(t)$$

Then for $u > T + \sigma - \tau$

$$\int_{T}^{u} \lambda(t)\, B(t) - \int_{T}^{u} q(t) \frac{P-1}{P^2} \left(\int_{t+\tau-\sigma}^{t} \lambda(s)\,ds \right) dt \geq \int_{T}^{u} q(t)\, A(t)\, dt \qquad (3.18)$$

Interchanging the order of integration and simplifying, we have

$$\int_{T}^{u} q(t) \left(\int_{t+\tau-\sigma}^{t} \lambda(s)\,ds \right) dt \geq \int_{T}^{u+\tau-\sigma} \lambda(t) \left(\int_{t+\tau-\sigma}^{t} q(s)\,ds \right) dt \qquad (3.19)$$

from (3.18) and (3.19) it follows that:

$$\int_T^u \lambda(t)\,B(t)dt - \frac{P-1}{p^2}\int_T^u \lambda(t)\left(\int_{t+\tau-\sigma}^{t} q(s)\,ds\right)dt \geq \int_T^u q(t)\,A(t)\,dt$$

and so

$$\int_T^u \lambda(t)\,B(t)dt - \int_{u+\tau-\sigma}^{T} \lambda(t)\,B(t)\,dt \geq \int_T^u q(t)\,A(t)\,dt \qquad (3.20)$$

since

$$B(t) = \exp\left(e\int_{t+\tau-\sigma}^{t} q(s)\,ds\right) \geq \int_{t+\tau-\sigma}^{t} q(s)\,ds$$

on the other hand, since

$$e \leq B(t) < k_1$$

for sum $k_1 > 0$ and (3.30) we get

$$\int_{u+\tau-\sigma}^{u} \lambda(t)\,k_1\,dt \geq \int_T^u q(t)\,A(t)\,dt$$

$$\int_{u+\tau-\sigma}^{u} \lambda(t)\,dt \geq \frac{1}{k_1}\int_T^u q(t)\,A(t)\,dt$$

Since (3.17) implies that the integral on the right hand side of the above inequality diverges as $u \to \infty$, the remainder of the proof is similar to that of theorem (3.1) and so we omit the details. This completes the proof of the theorem.

Example (3.2):

Consider the neutral delay differential equation:

$$\frac{d}{dt}\left[x(t)+5x(t-1)\right]+\frac{1}{4}x(t-3)\left(1+x^2(t-3)\right)=0, \quad t \geq 3 \qquad (3.21)$$

Here we have

$\tau = 1, \ \sigma = 3, \ q(t) = \frac{1}{4}$ and we have

$$\frac{1}{e} \leq \int_{t-\sigma+\tau}^{t} q(s)\,ds < k$$

$$\therefore \frac{1}{e} \leq \int_{t-2}^{t} \frac{1}{4}\,ds = \frac{1}{2} < k = 1, \text{ also } \int_{t_0}^{\infty} q(t)\,A(t)\,dt \geq$$

$$\int_{t_0}^{\infty} q(t)\exp\left(e\int_{t-\sigma+\tau}^{t} q(s)\,ds\right) = \int_{t_0}^{\infty}\frac{1}{4}e^{e/2}\,dt = \infty$$

Then the hypothesis of theorem (3.2) are satisfied so every solution of (3.21) is oscillatory.

Theorem (3.3):

Assume that the first order of neutral delay differential equation (3.1). If $P<-1$ then every non oscillatory solution of equation (3.1) tends $t_0+\infty$ or $-\infty$ as $t\to\infty$.

Proof: Theorem (3.3) see reference [7]

Since the negative of a solution of equation (3.1) is again a solution of eq (3.1), it suffices to prove the theorem in the case of an eventually positive solution so suppose that $x(t)$ is an eventually positive solution of equation (3.1).

Set $\quad Z(t)=x(t)+Px(t-\tau)$ .... (3.22)

then $\quad Z'(t)=-q(t)\,x(t-\sigma)$ .... (3.23)

and eventually $Z'(t)<0$.

We wish to show that $Z(t)$ is eventually negative,

Suppose for the sake of contradiction that eventually $Z(t)\geq 0$ then (3.22) it follows that eventually

$$Z(t)\geq 0 \Rightarrow x(t)+Px(t-\tau)\geq 0$$

$$\therefore -Px(t-\tau)\leq x(t)$$

which implies, by iteration, we have

$$0 < x(t) \le \left(\frac{1}{-P}\right)^n x(t+n\pi), \quad n = 1,2,...$$

eventually so as $-P > 1$ we see that $x(t) \to +\infty$ as $t \to \infty$ which contradicts the assumption that eventually $Z(t) \ge 0$.

Thus, we see that eventually $Z(t) < 0$, since $Z'(t) < 0$ eventually, we have

$$0 > \lim_{t \to \infty} Z(t) = L \ge -\infty$$

we wish to show $L = -\infty$, and so shall assume for the sake of contradiction, that $L > -\infty$ then by integrating (3.23) from t_0 to t we find

$$Z(t) - Z(t_0) + \int_{t_0}^{t} q(s)x(s-\sigma)ds = 0$$

and so

$$\int_{t_0}^{\infty} q(s)x(s-\sigma)ds = Z(t_0) - L$$

so, since $q(t) > 0$ eventually, we say that $x \in L_1(t_0, \infty)$, thus from (3.22).

we conclude that $Z \in L_1(t_0, \infty)$ this implies that $L = 0$ which is impossible.

Hence we have that

$\lim_{t\to\infty} Z(t) = -\infty$ and as eventually

$Z(t) > Px(t-\tau)$ we conclude that

$$\lim_{t\to\infty} x(t) = +\infty$$

The proof is complete.

Theorem (3.4):

Consider the neutral delay differential equation (3.1).

Assume that $P < -1$, $\tau > \sigma$ and that

$$\frac{1}{-P} \liminf_{t\to\infty} \int_{t}^{t+(\tau-\sigma)} q(s)\,ds > \frac{1}{e} \qquad (3.24)$$

then every solution of equation (3.1) oscillates.

Proof: Theorem (3.4)

Otherwise there is an eventually positive solution $x(t)$ of equation (3.1)

Set $Z(t) = x(t) + Px(t-\tau)$

Then by theorem (3.3) it follows that eventually $Z(t) < 0$, $Z'(t) < 0$ and that $Z(t) > Px(t-\tau)$.

From this last inequality, we find that eventually

$$Z(t) > Px(t-\tau)$$

$$Z(t) > Px(t-\tau) \text{ put } t = t+\tau-\sigma \text{ we get}$$

$$Z(t+\tau-\sigma) > Px(t-\tau)$$

$$q(t)\, Z(t+\tau-\sigma) > Px(t-\sigma)q(t)$$

$$\frac{1}{P}q(t)Z(t+\tau-\sigma) < x(t-\sigma)q(t)$$

$$-\frac{1}{P}q(t)Z(t+\tau-\sigma) > -x(t-\sigma)q(t) = Z'(t)$$

and hence

$$-\frac{1}{P}q(t)Z(t+\tau-\sigma) > Z'(t)$$

$$-Z'(t) - \frac{1}{P}q(t)Z(t+\tau-\sigma) > 0$$

$$Z'(t) - \left(\frac{1}{-P}\right)q(t)Z(t+\tau-\sigma) < 0$$

But by (3.24) and lemma (3.3) it is impossible for this inequality to have an eventually negative solution.

The proof is complete.

Theorem (3.5):

Assume that the neutral delay differential equation (3.1) and $P < -1$, $\tau > \sigma$, $q(t)$ *is periodic with period* τ *finally, suppose that*

$$-\frac{1}{1+P}\liminf_{t\to\infty}\int_{t}^{t-(\tau-\sigma)} q(s)\,ds > \frac{1}{e} \qquad (3.25)$$

Then every solution of equation (3.1) oscillates.

Proof: see reference [7]

Suppose for the sake of contradiction that there is an eventually positive solution $x(t)$.

Set $Z(t) = x(t) + Px(t-\tau)$

and $w(t) = Z(t) + PZ(t-\tau)$

then since $q(t)$ is periodic with period τ it is easy to see that Z and ω are also solution to equation (3.1).

As in the proof of theorem (3.3) we eventually have $Z(t) < 0$ and $Z'(t) < 0$ the same argument when applied to $-Z(t)$ implies that eventually $w(t) > 0$ and $\omega'(t) > 0$.

Since

$$Z(t) < 0 \Rightarrow -Z(t) < 0$$

$$-w(t) = -Z(t) - PZ(t-\tau)$$

$$\therefore -w(t) < 0 \Rightarrow w(t) > 0 \text{ and}$$

$$-w'(t) < 0 \Rightarrow \omega'(t) > 0$$

$$w(t) = Z(t) + PZ(t-\tau) < (1+P)Z(t-\tau)$$

Since

$$w(t) = Z(t) + PZ(t-\tau) \text{ and } Z(t) < Z(t-\tau)$$

then we have

$$w(t) < Z(t-\tau) + PZ(t-\tau)$$

$$w(t) < [1+P]Z(t-\tau)$$

eventually and so

$$w(t+\tau-\sigma) \leq [1+P]Z(t-\sigma)$$

$$-\frac{1}{1+P}q(t)w(t+(\tau-\sigma)) \leq -q(t)Z(\tau-\sigma) = w'(t)$$

from which we find that eventually

$$w'(t) - \left(\frac{-1}{1+P}\right)q(t)w(t+(\tau-\sigma)) \geq 0$$

But by (3.25) and the lemma (3.3) this is impossible.

The proof of the theorem is complete.

Theorem (3.6):

Consider the neutral delay differential equation (3.1).

Assume that $P > 0$ then every non oscillatory solution of equation (3.1) tends to zero as $t \to \infty$.

Proof:

It suffices to show that every eventually positive solution $x(t)$ of equation (3.1) tend to zero as $t \to \infty$, so

Let $x(t)$ be an eventually positive solution of eq. (3.1)

Set $\quad Z(t) = x(t) + P\,x(t-\tau)$

Then $\; Z'(t) = -q(t)x(t-\tau)$

and so $Z'(t)$ is eventually negative. sin $P > 0$.

$Z(t)$ is eventually positive.

Thus $\lim_{t \to \infty} Z(t) = L$ exists, is finite, and $L \geq 0$

By integrating (3.23) from t_0 to t we obtain

$$Z(t) - Z(t_0) + \int_{t_0}^{t} q(s)x(s-\sigma)ds = 0$$

$$Z(t_0) - \ell + \int_{t_0}^{\infty} q(s)x(s-\sigma)\,ds = 0$$

and therefore

$$\int_{t_0}^{t} q(s)x(s-\sigma)ds = Z(t_0) - L$$

So, as $q(t) \geq 0$ eventually we say that $x \in L_1(t_0, \infty)$ and so by (3.22) $Z \in L_1(t_0, \infty)$. Hence $L = 0$

Which implies that $\lim_{t \to \infty} x(t) = 0$.

The proof is complete.

Theorem (3.7):

Consider the neutral delay differential equation (3.1).

Assume that $P > 0, \sigma > \tau$ *and* $q(t)$ *is* τ *periodic.*

Finally, suppose that

$$\frac{1}{1+P} \liminf_{t \to \infty} \int_{t-(\sigma-\tau)}^{t} q(s)ds > \frac{1}{e} \qquad (3.26)$$

then every solution of equation (3.1) is oscillates.

Proof: see reference [3] and [7]

Otherwise there is an eventually positive solution $x(t)$ of equation (3.1).

Set $Z(t) = x(t) + Px(t-\tau)$

and $\omega(t) = Z(t) + PZ(t-\tau)$

then eventually

$$Z(t) > 0,\ \ Z'(t) < 0,\ \omega(t) > 0,\ \omega'(t) < 0$$

Hence

$$\omega(t) = Z(t) + P\,Z(t-\tau) < (1+P)\,Z(t-\tau)$$

and so

$$-\frac{1}{1+P}q(t)\omega\big(t-(\sigma-\tau)\big) \geq -q(t)\,Z(t-\sigma)$$

But the fact $q(t)$ is periodic with period Z.

Implies that Z, and also ω, are solutions of eq. (3.1)

and so

$$-q(t)\,Z(t-\sigma) = \omega'(t)$$

that is to say

$$\omega'(t) + \frac{1}{1+P}q(t)\omega\big(t-(\sigma-\tau)\big) \leq 0$$

in view of (3.26) and lemma (3.3) this is impossible.

The proof of the theorem is complete.

CHAPTER FOUR (IV)

In this chapter, we give some new sharp sufficient conditions for the oscillation of all solutions of the following equation.

Consider the first order neutral delay differential equations with positive and negative coefficients of the from

$$\frac{d}{dt}(x(t)-R(t)x(t-r))+P(t)x(t-\tau)-Q(t)x(t-\delta)=0 \qquad (4.1)$$

where

$$P,Q,R\in C([t_0,\infty),R^+),\ r\in(0,\infty)\ \ and\ \ \tau,\delta,\in R^+ , \qquad (4.2)$$

$$\tau>\delta,\ \overline{P}(t)=P(t)-Q(t-\tau+\delta)\geq 0 \qquad (4.3)$$

When Q(t) ≡ 0 , equation (4.1) reduces to

$$\frac{d}{dt}(x(t)-R(t)x(t-r))+P(t)x(t-\tau)=0 \qquad (4.4)$$

Let $T_0=\max\{r,\tau,\delta\}$. By a solution of equation .(4.1) we mean a function $x\in C([t_0-T_0,\infty),R)$ *for some* $t_1\geq t_0$ such that *x(t) –R(t) x(t-r)* is continuously differentiable on $[t_1,\infty)$ and such that equation (4.1) is satisfied for $t\geq t_1$.

Theorem (4.1)

Assume that (4.2) and (4.3) hold and that

$$R(t) + \int_{t-\tau+\delta}^{t} Q(s)\,ds \le 1 \quad , \; t \ge t_1 \ge t_0 \qquad (4.5)$$

let $x(t)$ be an eventually positive solution of equation (4.1) and set

$$y(t) = x(t) - R(t)x(t-\tau) - \int_{t-\tau+\delta}^{t} Q(s)\, x(s-\delta)\,ds \qquad (4.6)$$

then

$$y'(t) \le 0 \;\; and \;\; y(t) > 0 \qquad (4.7)$$

Proof Theorem (4.1)

Since x (t) is eventually positive solution then From (4.6) we have

$$y'(t) = \frac{d}{dt}(x(t) - R(t)\, x(t-\tau)) - \frac{d}{dt}\int_{t-\tau+\delta}^{t} Q(s)x(s-\delta)\,ds$$

$$y'(t) = \frac{d}{dt}[x(t) - R(t)\, x(t-\tau)\,] - \int_{t+\tau+\delta}^{t} \frac{d}{ds}[Q(s)x(s-\delta)\,]ds$$

$$y'(t) = \frac{d}{dt}[x(t) - R(t)\, x(t-\tau)] - [Q(t)x(t-\delta) - Q(t-\tau+\delta)x(t-\tau)]$$

$$y'(t) = \frac{d}{dt}[x(t) - R(t)\, x(t-\tau)] - Q(t)x(t-\delta) + Q(t-\tau+\delta)\, x(t-\tau)$$

$$y'(t) = \frac{d}{dt}[x(t) - R(t)\, x(t-\tau)] + P(t)x(t-\tau) - Q(t)\, x(t-\delta) - P(t)x(t-\tau)$$
$$+ Q(t-\tau+\delta)x(t-\tau)]$$

$$y'(t) = -P(t)x(t-\tau) + Q(t-\tau+\delta)x(t-\tau)$$

$$y'(t) = -\left[P(t) + Q(t-\tau+\delta)\right]x(t-\tau)$$

from (4.3) we get

$$y' = -\overline{P}(t)\, x(t-\tau)$$

since

$$\overline{P}(t) \geq 0 \text{ and } x(t-\tau) > 0$$

therefore $y'(t) \leq 0$

Theorem (4.2)

Assume that $\delta>0$, $R\in C([t_0,\infty),R^+)$, $\lambda\in C([t_0-\delta,\infty),R^+)$ *and.*

That $\lambda(t)\ge R(t)\exp\left(\int_{t-\delta}^{t}\lambda(s)\,ds\right),\qquad t\ge t_0$

Then conduction

$$\liminf_{t\to\infty}\int_{t-\delta}^{t}R(S)\,ds>0$$

implies that

$$\liminf\int_{t-\delta}^{t}\lambda(S)\,ds<\infty$$

Proof Theorem (4.2)

That $\lambda(t)\ge R(t)\exp\left(\int_{t-\delta}^{t}\lambda(s)\,ds\right),\qquad t\ge t_0$

Now define

$$A(t)=\liminf_{t\to\infty}\int_{t-\delta}^{t}R(s)\,ds>0$$

By using inequality

$$e^{rx}\ge x+\frac{\ln(er)}{r}\qquad for\qquad r>0$$

we fined that

$$\lambda(t) \geq R(t) \exp\left[(A(t) \frac{1}{A(t)} \int_{t-\delta}^{t} \lambda(s)\, ds) \right],$$

$$\geq R(t) \exp\left[(\frac{1}{A(t)} \int_{t-\delta}^{t} \lambda(s)\, ds) + \frac{\ln(eA(t))}{A(t)} \right],$$

or

$$\left(\liminf_{t\to\infty} \int_{t-\delta}^{t} R(s)\, ds \right) \lambda(t) - R(t) \int_{t-\delta}^{t} \lambda(s)\, ds$$

$$\geq R(t)\, (\ln e \liminf_{t\to\infty} \int_{t\to\delta}^{t} R(s)\, ds)$$

or

$$-R(t) \int_{t\to\delta}^{t} \lambda(s)\, ds \geq R(t)\, (\ln e \liminf_{t\to\infty} \int_{t\to\delta}^{t} R(s)\, ds) \; - (\liminf_{t\to\infty} \int_{t\to\delta}^{t} R(s)\, ds)\lambda(t)$$

$$\int_{t\to\delta}^{t} \lambda(s)\, ds \leq (\ln e \liminf_{t\to\infty} \int_{t\to\delta}^{t} R(s)\, ds)) + \frac{\lambda(t)}{R(t)} (\liminf_{t\to\infty} \int_{t\to\delta}^{t} R(s)\, ds))$$

there for

$$\liminf_{t\to\infty} \int_{t-\delta}^{t} \lambda(s)\, ds < \infty$$

Theorem (4.3)

(i) Assume that (4.2) and (4.3)and (4.5) held and that

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} \overline{P}(s)\, ds > 0$$

(ii) There sexists a positive continuous function H (t) such that

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} H(s)\, ds > 0$$

(iii) Either

$$\inf_{\lambda>0,\, t\geq T}\left\{\frac{R(t-\tau)P(t)H(t-r)}{\overline{P}(t-r)H(t)}\exp(\lambda\int_{t-r}^{t}H(s)\,ds)+\frac{1}{H(t)\lambda}\overline{P}\exp(\lambda\int_{t-\tau}^{t}H(s)\,ds)+\right.$$

$$\left.\frac{\overline{P}(t)}{H(t)}\int_{t-\tau-\delta}^{t}\frac{Q(s-\tau)H(s-\delta)}{\overline{P}(s-\delta)}\exp(\lambda\int_{s-\delta}^{t}H(u)\,du)ds\right\}>1$$

$for\ \overline{P}(t)>0\ ,\ t\geq T$

or

$$\inf_{\lambda>0,\, t\geq T}\left\{\frac{1}{H(t)\lambda}\exp(\lambda\int_{t-\tau}^{t}H(s)\overline{P}(s)\,ds)+\frac{H(t-r)R(t-\tau)}{H(t)}\exp(\lambda\int_{t-r}^{t}H(s)\overline{P}(s)\,ds)+\right.$$

$$\left.\frac{1}{H(t)}\int_{t-\tau-\delta}^{t}Q(s-\tau)H(s-\delta)\exp(\lambda\int_{s-\delta}^{t}H(u)\overline{P}(u)\,du)ds\right\}>1$$

$for\ \overline{P}(t)\geq 0\ ,\ t\geq T$

Then every solution of equation (4.1) is oscillatory

Proof theorem (4.3)

Without loss of generality assume that the equation (4.1) has and eventually positive solution x (t) .

Let y(t) be defined by (4.6) then by theorem (4.1) we have

$y'(t)\leq 0\ \ and\ \ y(t)>0\ \ for\ \ \ t\geq t_1\geq t_0$

from (4.1) we have

$$y' = -\overline{P}(t)\,x(t-\tau) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(I)}$$

sine $x(t-\tau) = y(t-\tau) + R(t-\tau)\,x(t-\tau-r) + \int_{t-\tau+\delta}^{t} Q(s-\tau)\,x(s-\tau-\delta)\,ds$

Therefore

$$y' = -\overline{P}(t)\,y(t-\tau) + R(t-\tau)\,x(t-\tau-r) + \int_{t-\tau+\delta}^{t} Q(s-\tau)\,x(s-\tau-\delta)\,ds \quad \ldots\ldots\ldots \text{(II)}$$

put t = t - r in equation (I)

$$y'(t-r) = -\overline{P}(t-r)\,x(t-\tau-r)$$

$$x(t-\tau-r) = \frac{y'(t-r)}{\overline{P}(t-r)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(III)}$$

substituting (III) into (II) we get

$$y' = -\overline{P}(t)\left[y(t-\tau) + \frac{R(t-\tau)\,\overline{P}(t)}{\overline{P}(s-\delta)}\,y'(t-r) + \overline{P}(t)\int_{t-\tau+\delta}^{t} \frac{Q(s-\tau)}{\overline{P}(s-\delta)}\,y'(s-\delta)\,ds \right] \qquad (4.12)$$

Assuming condition (4.10) holds

set

$$\lambda(t)\, H(t) = -\frac{y'(t)}{y(t)}$$

Implies That

$$\exp\left(\int_{t-\tau}^{t} \lambda(s)\, H(s)\,ds\right) = \frac{y(t-\tau)}{y(t)}$$

then (4.12) becomes

$$\lambda(t)\, H(t) = \overline{P}(t)\, \frac{y(t-\tau)}{y(t)} - \frac{R(t-\tau)\, \overline{P}(t)}{\overline{P}(t-r)}\, \frac{y'(t-r)}{y(t)}$$

$$-\overline{P}(t) \int_{t-\tau+\delta}^{t} \frac{Q(s-\tau)}{\overline{P}(s-\delta)}\, \frac{y'(s-\delta)}{y(t)}\,ds$$

$$\lambda(t)\, H(t) = \overline{P}(t)\, \frac{y(t-\tau)}{y(t)} + \frac{R(t-\tau)\, \overline{P}(t)\lambda(t-r)H(t-r)}{\overline{P}(t-r)}\, \frac{y(t-r)}{y(t)}$$

$$+\overline{P}(t) \int_{t-\tau+\delta}^{t} \frac{Q(s-\tau)\lambda(s-\delta)H(s-\delta)}{\overline{P}(s-\delta)}\, \frac{y(s-\delta)}{y(t)}\,ds$$

$$\lambda(t)\, H(t) = \overline{P}(t)\exp -\left(\int_{t-\tau}^{t} \lambda(s)\, H(s)\,ds\right)$$

$$+\lambda(t-r)\, H(t-r)\, \frac{R(t-\tau)\, \overline{P}(t)}{\overline{P}(t-r)} \exp\left(\int_{t-r}^{t} \lambda(s)\, H(s)\,ds\right)$$

$$+\overline{P}(t)\int_{t-r+\delta}^{t}\frac{Q(s-\tau)}{\overline{P}(s-\delta)}\lambda(s-\delta)\,H(s-\delta)\exp\left(\int_{s-\delta}^{t}\lambda(u)\,H(u)du\right)ds \tag{4.13}$$

It is obvious that $\lambda(t)\,H(t)\ >0\ for\ \ t\geq t_0$ from (4.13) we have

$$\lambda(t)\,H(t)\geq\overline{P}(t)\exp\left(\int_{t-r}^{t}\lambda(s)\,H(s)ds\right)$$

From (4.8) and theorem (4.2) we get

$$\liminf_{t\to\infty}\int_{t-r}^{t}\lambda(s)\,H(s)ds\ <\ \infty$$

which implies, by using (4.9),that $\liminf_{t\to\infty}\lambda(t)<\infty$.

Now we show that

$$\liminf_{t\to\infty}\lambda(t)>0$$

in fact, if $\liminf_{t\to\infty}\lambda(t)=0$, then there exists a sequence $\{t_n\}$ such that $t_n\geq t_1$, $t_n\to\infty$ as $n\to\infty$, and $\lambda(t_n)\leq\lambda(t)\ \ for\ t\in[t_1\,,t_n]$. From (4.13) we have

$$\lambda(t_n)\,H(t_n)\ \geq\ \bar{P}(t_n)\exp\left(\lambda(t_n)\int_{t_n-r}^{t_n}H(s)\,ds\right)$$

$$+\ \lambda(t_n)\,H(t_n-r)\,\frac{R(t_n-\tau)\,\overline{P}(t_n)}{\overline{P}(t_n-r)}\exp\left(\lambda(t_n)\int_{t_n-r}^{t_n}H(s)ds\right)$$

$$+\overline{P}(t_n)\int_{t_n-\tau+\delta}^{t_n}\frac{Q(s-\tau)}{\overline{P}(s-\delta)}\lambda(t_n)\,H(s-\delta)\exp(\lambda(t_n)\int_{s-\delta}^{t_n}H(u)du)\,ds$$

hence

$$\frac{1}{\lambda(t_n)H(t_n)}P(t_n)\exp(\lambda(t_n)\int_{t_n-\tau}^{t_n}H(s)ds)$$

$$+\frac{H(t_n-r)R(t_n-r)\overline{P}(t_n)}{H(t_n)\overline{P}(t_n-r)}\exp(\lambda(t_n)\int_{t_n-r}^{t_n}H(s)ds)$$

$$+\frac{\overline{P}(t_n)}{H(t_n)}\int_{t_n-\tau+\delta}^{t_n}\frac{Q(s-\tau)H(s-\delta)}{\overline{P}(s-\delta)}\exp(\lambda(t_n)\int_{s-\delta}^{t_n}H(u)du)ds\le 1$$

which contradicts (4.10) and therefore

$$0<\liminf_{t\to\infty}\lambda(t)=h<\infty \tag{4.14}$$

Form (4.10) there exists an $\alpha\in(0,1)$ such than

$$\alpha\inf_{\lambda>0,t\ge T}\left\{\frac{R(t-\tau)\overline{P}(t)H(t-r)}{\overline{P}(t-r)H(t)}(\lambda\int_{t-r}^{t}H(s)ds)\right.$$

$$+\frac{1}{H(t)\lambda}\overline{P}(t)\exp(\lambda\int_{t-r}^{t}H(s)ds$$

$$\left.+\frac{P(t_n)}{H(t_n)}\int_{t-\tau+\delta}^{t}\frac{Q(s-\tau)H(s-\delta)}{\overline{P}(s-\delta)}\exp((\lambda\int_{t-r}^{t}H(s)ds\right\}>1 \tag{4.15}$$

on the other hand , in view of the definition of $\liminf_{t\to\infty} \lambda(t) = h$, there exists a $t_2 > t_1$ such that

$$\lambda(t) > \alpha h \ , \ t \ge t_2 \tag{4.16}$$

substituting (4.16) into (4.13) , we obtain

$$\lambda(t)H(t) \ge \overline{P}(t)\exp(h\alpha \int_{t-\tau}^{t} H(s)\,ds)$$

$$+ \ h\alpha \frac{H(t-r)\ R(t-\tau)\overline{P}(t)}{\overline{P}(t-\tau)} \exp\ (h\alpha \int_{t-\tau}^{t} H(s)\,ds)$$

$$+\overline{P}(t)\alpha h \int_{t-\tau+\delta}^{t} h\alpha \frac{Q(s-\tau)H(s-\delta)}{\overline{P}(s-\delta)} \exp\ (h\alpha \int_{t-\tau}^{t} H(u)\,du)$$

for $t \ge t_2 + T_0$ Hence

$$h \ \ge \ \liminf_{t\to\infty} \left\{ \frac{\overline{P}(t)}{H(t)} \exp\ (h\alpha \int_{t-\tau}^{t} H(s)\,ds) \right.$$

$$+ \ h\alpha \frac{H(t-r)\ R(t-\tau)\overline{P}(t)}{H(t)\overline{P}(t-\tau)} \exp\ (h\alpha \int_{t-\tau}^{t} H(s)\,ds)$$

$$+\frac{\overline{P}(t)}{H(t)}\alpha h \int_{t-\tau+\delta}^{t} h\alpha \frac{Q(s-\tau)H(s-\delta)}{\overline{P}(s-\delta)} \exp\ (h\alpha \int_{t-\tau}^{t} H(u)\,du)$$

which implies that there exists a sequence $\{\bar{t}_n\}$ such that $\bar{t}_n \geq \max\{T, t_2 + T_0\}$, $\bar{t}_n \to \infty$ as $n \to \infty$, and

$$\lim_{t\to\infty}\left\{\frac{\overline{P}(t)}{H(t)}\exp\left(h\alpha\int_{t-r}^{t}H(s)\,ds\right)\right.$$

$$+\ h\alpha\frac{H(\bar{t}_n - r)\,R(\bar{t}_n-\tau)\,\overline{P}(\bar{t}_n)}{H(\bar{t}_n)\overline{P}(\bar{t}_n-\tau)}\exp\left(h\alpha\int_{\bar{t}_n-r}^{\bar{t}_n}H(s)\,ds\right)$$

$$\left.+\frac{\overline{P}(t)}{H(t)}\alpha h\int_{\bar{t}_n-\tau+\delta}^{\bar{t}_n} h\alpha\frac{Q(s-\tau)H(s-\delta)}{\overline{P}(s-\delta)}\exp\left(h\alpha\int_{s-\delta}^{\bar{t}_n}H(u)\,du\right)\right\}=\bar{h}\leq h$$

set $\lambda = h\alpha$, then

$$\alpha\lim_{t\to\infty}\left\{\frac{\overline{P}(t)}{H(t)\lambda}\exp\left(\lambda\int_{t-r}^{t}H(s)\,ds\right)\right.$$

$$+\ \frac{H(\bar{t}_n - r)\,R(\bar{t}_n-\tau)\,\overline{P}(\bar{t}_n)}{H(\bar{t}_n)\overline{P}(\bar{t}_n-\tau)}\exp\left(\lambda\int_{\bar{t}_n-r}^{\bar{t}_n}H(s)\,ds\right)$$

$$\left.+\frac{\overline{P}(t)}{H(t)}\alpha h\int_{\bar{t}_n-\tau+\delta}^{\bar{t}_n} h\alpha\frac{Q(s-\tau)H(s-\delta)}{\overline{P}(s-\delta)}\exp\left(\lambda\int_{s-\delta}^{\bar{t}_n}H(u)\,du\right)\right\}\leq 1$$

which contradicts (4.15) and completes t he proof of this theorem under condition (4.10)

For the condition (4.11) .let

$$\lambda(t)H(t)\overline{P}(t) = \frac{-y'(t)}{y(t)}$$

Then (4.12) becomes

$$\lambda(t)\ H(t) = \frac{y(t-r)}{y(t)} - \frac{R(t-\tau)\ \overline{P}(t)\lambda(t-r)H(t-r)}{\overline{P}(t-r)}\frac{y(t-r)}{y(t)}$$

$$-\int_{t-\tau+\delta}^{t}\frac{Q(s-\tau)\lambda(s-\delta)H(s-\delta)}{\overline{P}(s-\delta)}\frac{y'(s-\delta)}{y(t)}ds$$

$$\lambda(t)\ H(t) = \exp\ (\int_{t-t}^{t}\lambda(s)\ H(s)\ \overline{P}(s)\ ds,)$$

$$+\ \lambda(t-r)\ H(t-r)\ R(t-\tau)\ \exp\ (\int_{t-R}^{t}\lambda(s)\ H(s)\overline{P}(s)ds,)$$

$$+\ \overline{P}(t)\ \int_{t-\tau+\delta}^{t}Q(s-\tau)\lambda(s-\delta)\ H(s-\delta)\ \exp(\int_{s-\delta}^{t}\lambda(u)\ H(u)\overline{P}(u)\ du)ds \qquad (4.17)$$

It is obvious that $\lambda(t)\ H(t) > 0$ *for* $t \geq t_0$

$$\lambda(t)\ H(t) \geq \exp\ (\int_{t-t}^{t}\lambda(s)\ H(s)\ \overline{P}(s)\ ds,)$$

$$\overline{P}(t)\lambda(t)\ H(t) \geq \overline{P}(t)\exp\ (\int_{t-t}^{t}\lambda(s)\ H(s)\ \overline{P}(s)\ ds,)$$

in view OF (4.8) and theorem (4.2) we get

$$\liminf_{t\to\infty} \int \lambda(s)\, H(s)\overline{P}(s)\, ds \quad < \infty \tag{4.18}$$

From (4.8),(4.9) and (4.18) we may conclude that $\liminf_{t\to\infty} \lambda(t) < \infty$ in view of (4.17) and so

$$0 < \liminf_{t\to\infty} \lambda(t) = h < \infty$$

form (4.11) there exists $\alpha \in (0,1)$ such than

$$\alpha \inf_{\lambda>0,\, t\geq T} \left\{ \frac{1}{\lambda H(t)} \exp\left(\int_{t-\tau}^{t} H(s)\, \overline{P}(s) ds \right) \right.$$

$$+ \frac{H(t-\tau) R(t-\tau)}{H(t)} \exp\left(\lambda \int_{t-\tau}^{t} H(s)\, \overline{P}(s) ds \right.$$

$$\left. \frac{1}{H(t)} \int_{t-\tau+\delta}^{t} Q(s-\tau) H(s-\delta) \exp\left(\lambda \int_{s-\delta}^{t} H(u)\, \overline{P}(u) du\right) ds \right\} > 1 .$$

By using a similar method as in the first part of the proof , we can derive a contraction. The proof is complete

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بسم الله الرحمن الرحيم

مقدمة

لقد ظهر في الأونة الأخيرة جهدا لا بأس به من الأبحاث والانجازات التي تتعلق فيما يعرف بالسلوك التذبذبي لمعادلات التأخير المحايدة التفاضلية ومختصرها (المعادلات التفاضلية المحايدة) والمعادلة التفاضلية المحايدة هي معادلة تقيم المشتقة ذات الرتبة الأعلى للدالة المجهولة عند كل من الحالة الجارية t والحالة الماضية.

إن مسألة السلوك التذبذبي وغير التذبذبي و التقاربي لحلول المعادلات التفاضلية المحايدة لها أهميتان نظرية وعملية وهذا النوع من المعادلات التفاضلية المحايدة يظهر في الشبكات المتضمنة لفاقد أقل من خطوط الإرسال و الانتقال, مثل هذه الشبكات تظهر علي سبيل المثال في الحاسبات ذوات السرعة العالية حيث تستخدم خطوط الإرسال والانتقال ذوات الفاقد الأقل لربط دوائر التحويل.

ومن المعلوم أن هنالك فروق كبيرة في سلوك الحلول لكل من المعادلات التفاضلية المحايدة و المعادلات التفاضلية غير المحايدة.

تم وضع هذه الأطروحة في أربعة أبواب حيث نعرض في الباب الأول التعريفات والتمهيدات والملاحظات التي يحتاج إليها البحث في الأبواب اللاحقة ،وفي الباب الثاني يقوم البحث بدراسة أسلوب جديد لتحليل المعادلة المميزة العامة للوصول إلي شروط تكاملية (في صورة تكاملات اللانهائيةِ) لتذبذب المعادلات التفاضلية التأخيرية ،ويقدم الباب الثالث شروط تكاملية (في صورة تكاملات) لتذبذب كل حلول المعادلات التفاضلية الخطية ذات التأخير المحايدة والباب الرابع يقدم بَعْض الشروطِ الكافيةِ الجديدةِ لتذبذبِ كُلّ الحلول المعادلات التفاضلية ذات تأخير من الرتبة الأولي بمعاملات الموجبة والسالبة

الإهداء

إلى سيد الأولين والآخرين حبيبي رسول الله
صلى الله عليه وسلم

إلى والدتي ووالدي بارك الله في عمرهم والذي أكن لهم كل احترام وتقدير..واللذان كانا دائما يحثانني على مجالس العلم وصحبة العلماء.

إلى زوجتي بارك الله في عمرها.

إلى إخوتي وأخواتي.

أهدى عملي هذا إليهم جميعا.

كلية العلوم

قسم الرياضيات

عنوان البحث

تذبذب المعادلات التفاضلية ذات التأخير و المعادلات التفاضلية المحايدة من الرتبة الأولى

مقدمة من الطالب

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تذبذب المعادلات التفاضلية المحايدة ذات التاخير من الرتبة الأولى

أطروحة مقدمة لاستكمال متطلبات التخصص العالي الماجستير في الرياضيات

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