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FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS

PAINLEVE' ANALYSIS OF SOME DIFFUSION EQUATIONS

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ATTIA A. H.MOSTAFA

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REVISED AND SUPERVISED BY

DR. ABULGASSIM ALI ALDERANE

PROFESSOR OF MATHEMATICS

SIRT - LIBYA

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Department of Mathematics

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((Painleve' analysis of some diffusion equations))

By

Attia A. H. Mostafa

Approved by:

Dr. Abulgassim Ali Mohamed ....
(Supervisor)

Dr.Mustafa Sharif Shaieb ....
(External examiner)

Dr. Nabil Zaki Farid ....
(Internal examiner)

Countersigned by:

Dr. Mohamed Ali Salem

(Dean of Faculty of Science)

674 ✉ 054 - 60361 62151 ☎ 054 – 68089

Dedication

To my mother , my father , my brothers my country

Attia

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I am grateful to my teacher professor

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Introduction

Motivation:

Most of phenomena sciences and other fields can be described and classified nonlinear *diffusion equation* which normally result from natural phenomena that appears in one daily life such as the flow of water beneath bridges if the density was high . Also the slow blood in veins increase as a result of the high heart pulse the some application other physic engineering chemical and mathematical phenomena .

In this thesis , we try to find a solution to this type of equations although it normally very difficult to find a clear cut solutions . However through the use of *Painleve' analysis* it is possible to find an analytic solution which may benefit engineers chemists doctors and other to explain the result solution and arrive of this through understanding which could be difficult for mathematicians to explain .

ABSTRACT

In this thesis, we studied Painleve' property and their implementations on some diffusion equations , and by using the truncation technique is used to obtain some analytical solutions.

In chapter one, we gave some important definitions and lemmas which are used in the thesis supported by several examples.

In chapter two, we applied Painleve' property for partial differential equation which is Korteweg-de Vries equation.I.

$$u_t + 12\, uu_x + u_{xxx} = 0,$$

In chapter three, we studied modified Korteweg-de Vries equation.II. And through our study we found that the partial different equation does not satisfy the Painleve' property but we can find an analytical solution .

$$u_t - 6u^2 u_x + u_{xxx} = 0,$$

In chapter four, we are going to illustrate the nature of the Painleve' property on the complex modified Korteweg-de Vries equation.II, instead of a single complex nonlinear (PDE), we preferred to study with a system of real and imaginary parts in Korteweg-de Vries equation.II. finally, we draw some conclusions and review areas of future research .

$$w_t - 6|w|^2 w_x + w_{xxx} = 0,$$

CHAPTER ONE
Preliminaries

CHAPTER ONE

This chapter contains of an important definitions :

Definition 1.1.1 ***Singular point*** [8].

A point z_0 is called a singular point of a function $f(z)$,if $f(z)$ fails to be analytic at z_0 but is holomorphic at some point in every neighborhood of z_0 .

Definition 1.1.2 ***Isolated singular point*** [8].

Let z_0 be a singular point of $f(z)$ if there is some neighborhood of z_0 at which $f(z)$ is analytic except at z_0 then we say that z_0 is isolated a singular point .

Definition 1.1.3 ***Pole of order -K*** [8].

A function whose Laurent expansion about the isolated singular point z_0 contains a finite number of nonzero terms in the Principal part in which the most negative power of $(z-z_0)$ is-K

take $f(z)=\sum_{k=0}^{\infty} a_k (z-z_0)^k + \frac{r_1}{(z-z_0)} + \frac{r_2}{(z-z_0)^2} + \cdots + \frac{r_k}{(z-z_0)^k}$

where $r_k \neq 0$,

then the function is said to have order a pole K of z_0.

Definition 1.1.4 ***Removable singularity*** **[8]**

When a singular point of a function $f(z)$ at z_0 can be removed by suitably defining $f(z)$ at z_0, we say $f(z)$ has removable singular point at z_0,

Theorem 1.1.5 :[10].

$$Let \quad u_t = K(u, u_{(r)}), \text{---------} \quad (1)$$

be a given partial differential equation where ***K*** is a polynomial in ***u*** and in the spatial derivatives up to order ***r***. Furthermore, we know the expansion of ***u*** in the form :

$$u = \sum_{j=0}^{\infty} u_j \varphi^{j-p},$$

Equation (1) passes the Painleve' test .

Definition 1.1.6 ***Painleve' property*** **[5].**

A differential equation has the Painleve' property if all the movable singularities of all its solutions are poles.

A singularity is *movable* if it depends on the constants of integration of the *ordinary differential equation* (ODE).

For instance, the Riccati equation,

$$w'(z) + w^2(z) = 0,$$

has the general solution $w(z) = 1/(z - c)$,where c is constant of integration . Hence, the equation has a movable simple pole at $z = c$ because it depends on the constant of integration.

The solutions of an (ODE) can have various kinds of singularities, including branch points and essential singularities; examples of the various types of singularities are
shown in examples :

Example 1.1.7 ***simple fixed pole*** **[20].**

$$zw' + w = 0 \quad \Rightarrow \quad w(z) = \frac{c}{z}$$

Example 1. 1.8 ***simple movable pole.*** **[20].**

$$w' + w^2 = 0 \quad \Rightarrow \quad w(z) = \frac{1}{(z - c)}$$

Example 1.1.9 ***Movable algebraic branch point*** **[20].** .

$$2ww' - 1 = 0 \quad \Rightarrow \quad w(z) = \sqrt{z - c}$$

Example 1.1.10 ***Movable logarithmic branch point*** **[20].** .

$$w'' + w'^2 = 0 \quad \Rightarrow \quad w(z) = \log(z - c_1) + c_2$$

Example 1.1.11 ***Non-isolated movable essential singularity*** **[20].**

$$\left(1 + w^2\right)w'' + (1 - 2w) = 0 \quad \Rightarrow \quad w(z) = \tan[\ln(c_1 z + c_2)]$$

As a general property, the solutions of linear (ODEs) have only fixed singularities .

Definition 1.1.12 ***Resonance condition*** **[21].**

We say that η is *resonant*, if :

$$\sum_{i=0}^{n} \lambda_i \alpha_i - \eta\lambda_i = 0,$$

for some $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in Z_+^n$, $|\alpha| \geq 2$ *and* i ; $1 \leq i \leq n$.
If η is not resonant , say that η nonresonant.

Theorem 1.1.13 ***Cauchy–Kowalevskaya*** **[23].**

If the number of integral constants is less than n, then :

the series $$w = \sum_{j=0}^{\infty} a_j (z - z_0)^{j+p},$$

is not a general solution of $F(z, w, \frac{dw}{dz}, \ldots, \frac{d^n w}{dz^n}) = 0,$

Therefore it has no Painlevé property.

Property of painleve analysis **[23] :**

An Ordinary differential equation ODE (or Partial differential equation PDE) are said to have the Painleve' property if :

(I) – We get the compatibility in the recursion relation at resonance points

(II) – The number of integral constants in ODE (or arbitrary function in PDE) ,equals to order of (ODE) or (PDE) .

(III) – the number of the integral constant in ODE (or arbitrary function in PDE) ,equals to the number of the resonance of the recursion relation .

and let $\varphi=0$ be the movable non-characteristic (i.e., $\varphi_x \varphi_t \neq 0$) singular manifold of solutions. Assume that ,

$$u = \frac{1}{\varphi^p}\sum_{j=0}^{\infty} u_j \varphi^j$$

where φ and u_j are analytic functions in a neighborhood of the manifold $\varphi=0$,

Putting the above expansion into the equation and analyzing the leading part, we get the value of p and a series of recursive relations for u_j. We say that this equation has ***Painlevé property***, if the following three conditions are satisfied:

(i) p is a positive integer.

(ii) the recursive relations are consistent for all u_j .

(iii) there are enough free functions in the sense of Cauchy–Kowalevskaya Theorem .

Definition 1.1.14 *Schwartzian derivative* [38].

The Schwartzian derivative of panileve' holomorphic in C^n denoted by ,

$$S = \{\varphi, x\} = \frac{\varphi_{xxx}}{\varphi_x} - \frac{3}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2.$$

and ,

$$C = -\frac{\varphi_t}{\varphi_x},$$

S is called ***Schwartzian derivative*** while C has the dimension of velocity . According the compatibility of C and S , we have $S_t+C_{xxx}+2C_xS+CS_x=0$.

besides , let $L = -\dfrac{\varphi_{xx}}{2\varphi_x}$

Lemma 1.1.15 [43].

Let ψ_1 and ψ_2 be two linearly independent solution of the equation,

$$\frac{d^2\psi}{dz^2} + P(z) = 0, \quad \text{-------------------} \ \textbf{(II)}$$

which are defined and holomorphic on some simply connected domain D in complex plane. Then $W(z) = \Psi_1(z)/\Psi_2(z)$ satisfies the equation

$$\{W:Z\} = 2P(z), \quad \text{-------------------} \ \textbf{(III)}$$

Conversel, if $W(z)$ is a solution of (III) at all point of D ,then one can find two linearly holomorphic independent solutions Ψ_1 and Ψ_2 of (II) such that $W(z) = \Psi_1(z)/\Psi_2(z)$.in some neighborhood of $Z_0 \in D$,

Lemma 1.1.16 [43].

The Schwartzian derivative is invariant under fractional linear transformation acting on the first argument, namely,

$$\left\{\frac{aW+b}{cW+d};z\right\} = \{W;z\} \qquad ad - bc \neq 0$$

Where a , b , c and d are constant.

CHAPTER TWO
Korteweg-deVries (or KDVI) equation. I

CHAPTER TWO

In this chapter we study the Korteweg-de Vries equation.I , and through this study we find that the Korteweg-de Vries equation.I satisfies Painlevé property ,and by using truncation technique , we proceed as follow .

The Korteweg-de Vries (or KDV.I) equation.I

Section 2.1
Painlevé property .

$$u_t + 12\,uu_x + u_{xxx} = 0, \tag{2.1.1}$$

Let $u = \dfrac{1}{\varphi^p}\sum_{j=0}^{\infty} u_j\varphi^j$ be the series solution of (2.1.1) ,where φ and u_j are analytic functions in a neighborhood of the manifold $\varphi=0$. First, to find value of p we need to find u_t, u_{xxx} and $12uu_x$ then :

$$u_t = \sum_{j=0}^{\infty}[u_{j,t}\varphi^{j-p} + (j-p)u_j\varphi_t\varphi^{j-p-1}], \tag{2.1.2}$$

$$u_x = \sum_{j=0}^{\infty}[u_{j,x}\varphi^{j-p} + (j-p)u_j\varphi_x\varphi^{j-p-1}],$$

$$u_{xx} = \sum_{j=0}^{\infty}\Big[u_{j,xx}\varphi^{j-p} + 2(j-p)u_{j,x}\varphi_x\varphi^{j-p-1} + (j-p)u_j\varphi_{xx}\varphi^{j-p-1} + (j-p-1)(j-p)u_j\varphi_x^2\varphi^{j-p-2}\Big],$$

$$\begin{aligned}u_{xxx} = \sum_{j=0}^{\infty}\Big[&u_{j,xxx}\varphi^{j-p} + (j-p)u_{j,x}\varphi_x\varphi^{j-p-1} + 2(j-p)u_{j,xx}\varphi_x\varphi^{j-p-1}\\ &+ 2(j-p)u_{j,x}\varphi_{xx}\varphi^{j-p-1} + 2(j-p)(j-p-1)u_{j,x}\varphi_x^2\varphi^{j-p-2}\\ &+ (j-p)u_{j,x}\varphi_{xx}\varphi^{j-p-1} + (j-p)u_j\varphi_{xxx}\varphi^{j-p-1}\\ &+ (j-p)(j-p-1)u_j\varphi_x\varphi_{xx}\varphi^{j-p-2} + (j-p)(j-p-1)u_{j,x}\varphi_x^2\varphi^{j-p-1}\\ &+ 2(j-p)(j-p-1)u_{j,x}\varphi_x\varphi_{xx}\varphi^{j-p-2}\\ &+ (j-p)(j-p-1)(j-p-2)u_j\varphi_x^3\varphi^{j-p-3}\Big]\end{aligned}$$

$$u_{xxx}=\sum_{j=0}^{\infty}[u_{j,xxx}\varphi^{j-p}+3(j-p)u_{j,xx}\varphi_x\varphi^{j-p-1}+3(j-p)u_{j,x}\varphi_{xx}\varphi^{j-p-1}$$
$$+3(j-p)(j-p-1)u_{j,x}\varphi_x^2\varphi^{j-p-2}+(j-p)u_j\varphi_{xxx}\varphi^{j-p-1}$$
$$+3(j-p)(j-p-1)u_j\varphi_x\varphi_{xx}\varphi^{j-p-2}$$
$$+(j-p)(j-p-1)(j-p-2)u_j\varphi_x^3\varphi^{j-p-3}], \qquad (2.1.3)$$

and

$$12uu_x=12\sum_{j=0}^{\infty}\left[\sum_{k=0}^{j-1}u_k u_{j-1-k,x}+\sum_{i=0}^{j}u_{j-i}u_i(i-p)\varphi_x\right]\varphi^{j-2p-1} \qquad (2.1.4)$$

Now substation (2.1.2) , (2.1.3) (2.1.4) into (2.1.1) we get ,

$$\sum_{j=0}^{\infty}\left\{u_{j,t}\varphi^{j-p}+(j-p)u_j\varphi_t\varphi^{j-p-1}\right\}$$
$$+12\sum_{j=0}^{\infty}\left[\sum_{k=0}^{j-1}u_k u_{j-1-k,x}+\sum_{i=0}^{j}u_{j-i}u_i(i-p)\varphi_x\right]\varphi^{j-2p-1}$$
$$+\sum_{j=0}^{\infty}\{u_{j,xxx}\varphi^{j-p}+3(j-p)u_{j,xx}\varphi_x\varphi^{j-p-1}+3(j-p)u_{j,x}\varphi_{xx}\varphi^{j-p-1}$$
$$+3(j-p)(j-p-1)u_{j,x}\varphi_x^2\varphi^{j-p-2}+(j-p)u_j\varphi_{xxx}\varphi^{j-p-1}$$
$$+3(j-p)(j-p-1)u_j\varphi_x\varphi_{xx}\varphi^{j-p-2}$$
$$+(j-p)(j-p-1)(j-p-2)u_j\varphi_x^3\varphi^{j-p-3}\}=0, \qquad (2.1.5)$$

Now by comparing the lowest powers in (2.1.5) to find p ,

$$j-2p-1=j-p-3$$
$$\Rightarrow -2p-1=-p-3$$
$$\Rightarrow p=2$$

Now substituting $p=2$ into (2.1.5) ,we get .

$$\sum_{j=0}^{\infty}\{u_{j,t}\varphi^{j-2}+(j-2)u_j\varphi_t\varphi^{j-3}\}$$
$$+12\sum_{j=0}^{\infty}\left[\sum_{k=0}^{j-1}u_k u_{j-1-k,x}+\sum_{i=0}^{j}u_{j-i}u_i(i-2)\varphi_x\right]\varphi^{j-5}$$
$$+\sum_{j=0}^{\infty}\{u_{j,xxx}\varphi^{j-2}+3(j-2)u_{j,xx}\varphi_x\varphi^{j-3}+3(j-2)u_{j,x}\varphi_{xx}\varphi^{j-3}$$
$$+3(j-2)(j-p-3)u_{j,x}\varphi_x^2\varphi^{j-4}+(j-2)u_j\varphi_{xxx}\varphi^{j-3}$$
$$+3(j-2)(j-3)u_j\varphi_x\varphi_{xx}\varphi^{j-4}+(j-2)(j-3)(j-4)u_j\varphi_x^3\varphi^{j-5}\}=0,$$

Now by associated the summation ,we get .

$$\sum_{j=3}^{\infty}u_{j-3,t}\varphi^{j-5}+\sum_{j=2}^{\infty}(j-4)u_{j-2}\varphi_t\varphi^{j-5}$$
$$+12\sum_{j=0}^{\infty}\left[\sum_{k=0}^{j-1}u_k u_{j-1-k,x}+\sum_{i=0}^{j}u_{j-i}u_i(i-2)\varphi_x\right]\varphi^{j-5}$$
$$+\sum_{j=3}^{\infty}u_{j-3,xxx}\varphi^{j-5}+\sum_{j=2}^{\infty}3(j-4)u_{j-2,xx}\varphi_x\varphi^{j-5}$$
$$+\sum_{j=2}^{\infty}3(j-4)u_{j-2,x}\varphi_{xx}\varphi^{j-5}+\sum_{j=1}^{\infty}3(j-3)(j-4)u_{j-1,x}\varphi_x^2\varphi^{j-5}$$
$$+\sum_{j=2}^{\infty}(j-4)u_{j-2}\varphi_{xxx}\varphi^{j-5}+\sum_{j=1}^{\infty}3(j-3)(j-4)u_{j-1}\varphi_x\varphi_{xx}\varphi^{j-5}$$
$$+\sum_{j=0}^{\infty}(j-2)(j-3)(j-4)u_j\varphi_x^3\varphi^{j-5}=0, \qquad (2.1.6)$$

Now to find u_0 then at $j=0$, we get .

$$-24u_0^2\varphi_x\varphi^{j-5}-24u_0\varphi_x^3\varphi^{j-5}=0,$$

$$\Rightarrow u_0=-\varphi_x^2 \qquad (2.1.7)$$

Then (2.1.6) , becomes .

$$\sum_{j=3}^{\infty}u_{j-3,t}\varphi^{j-5}+\sum_{j=2}^{\infty}(j-4)u_{j-2}\varphi_t\varphi^{j-5}$$
$$+12\sum_{j=1}^{\infty}\left[\sum_{k=0}^{j-1}u_k u_{j-1-k,x}+\sum_{i=0}^{j}u_{j-i}u_i(i-2)\varphi_x\right]\varphi^{j-5}$$
$$+\sum_{j=3}^{\infty}u_{j-3,xxx}\varphi^{j-5}+\sum_{j=2}^{\infty}3(j-4)u_{j-2,xx}\varphi_x\varphi^{j-5}$$
$$+\sum_{j=2}^{\infty}3(j-4)u_{j-2,x}\varphi_{xx}\varphi^{j-5}+\sum_{j=1}^{\infty}3(j-3)(j-4)u_{j-1,x}\varphi_x^2\varphi^{j-5}$$
$$+\sum_{j=2}^{\infty}(j-4)u_{j-2}\varphi_{xxx}\varphi^{j-5}+\sum_{j=1}^{\infty}3(j-3)(j-4)u_{j-1}\varphi_x\varphi_{xx}\varphi^{j-5}$$
$$+\sum_{j=1}^{\infty}(j-2)(j-3)(j-4)u_j\varphi_x^3\varphi^{j-5}=0, \qquad (2.1.8)$$

Now to find u_1 then at $j=1$, we get .

$$12u_0u_{0,x}-36u_0u_1\varphi_x+18u_{0,x}\varphi_x^2+18u_0\varphi_x\varphi_{xx}-6u_1\varphi_x^3=0,$$
$$12(-\varphi_x^2)(-2\varphi_x\varphi_{xx})-36(-\varphi_x^2)u_1\varphi_x+18(-2\varphi_x\varphi_{xx})\varphi_x^2$$
$$+18(-\varphi_x^2)\varphi_x\varphi_{xx}-6u_1\varphi_x^3=0$$

$$\Rightarrow 30u_1 - 30\varphi_{xx} = 0,$$

$$\Rightarrow u_1 = \varphi_{xx} \tag{2.1.9}$$

Then (2.1.8), becomes .

$$\sum_{j=3}^{\infty} u_{j-3,t}\varphi^{j-5} + \sum_{j=2}^{\infty}(j-4)u_{j-2}\varphi_t\varphi^{j-5}$$
$$+12\sum_{j=2}^{\infty}\left[\sum_{k=0}^{j-1} u_k u_{j-1-k,x} + \sum_{i=0}^{j} u_{j-i}u_i(i-2)\varphi_x\right]\varphi^{j-5}$$
$$+\sum_{j=3}^{\infty} u_{j-3,xxx}\varphi^{j-5} + \sum_{j=2}^{\infty} 3(j-4)u_{j-2,xx}\varphi_x\varphi^{j-5}$$
$$+\sum_{j=2}^{\infty} 3(j-4)u_{j-2,x}\varphi_{xx}\varphi^{j-5} + \sum_{j=2}^{\infty} 3(j-3)(j-4)u_{j-1,x}\varphi_x^2\varphi^{j-5}$$
$$+\sum_{j=2}^{\infty}(j-4)u_{j-2}\varphi_{xxx}\varphi^{j-5} + \sum_{j=2}^{\infty} 3(j-3)(j-4)u_{j-1}\varphi_x\varphi_{xx}\varphi^{j-5}$$
$$+\sum_{j=2}^{\infty}(j-2)(j-3)(j-4)u_j\varphi_x^3\varphi^{j-5} = 0, \tag{2.1.10}$$

Now to find u_2 . then at $j=2$, we get .

$$-2u_0\varphi_t\varphi^{-3} + 12\left[\sum_{k=0}^{1} u_k u_{1-k,x} + \sum_{i=0}^{2} u_{2-i}u_i(i-2)\varphi_x\right]\varphi^{-3}$$
$$-6u_{0,xx}\varphi_x\varphi^{-3} - 6u_{0,x}\varphi_{xx}\varphi^{-3} + 6u_{1,x}\varphi_x^2\varphi^{-3}$$
$$-2u_0\varphi_{xxx}\varphi^{-3} + 6u_1\varphi_x\varphi_{xx}\varphi^{-3} = 0,$$

Becomes ,

$$-2u_0\varphi_t + 12u_0u_{1,x} + 12u_1u_{0,x} - 24u_0u_2\varphi_x$$
$$-12u_1^2\varphi_x - 6u_{0,xx}\varphi_x - 6u_{0,x}\varphi_{xx} + 6u_{1,x}\varphi_x^2$$
$$-2u_0\varphi_{xxx} + 6u_1\varphi_x\varphi_{xx} = 0,$$

$$2\varphi_x^2\varphi_t + 12\varphi_x^2\varphi_{xxx} - 24\varphi_x\varphi_{xx}^2 + 24\varphi_x^3u_2 - 12\varphi_x\varphi_{xx}^2$$
$$+12\varphi_x^2\varphi_{xxx} + 12\varphi_x\varphi_{xx}^2 + 12\varphi_x\varphi_{xx}^2 + 6\varphi_x^2\varphi_{xxx} + 2\varphi_x^2\varphi_{xxx}$$
$$+6\varphi_x\varphi_{xx}^2 = 0,$$

Then ,

$$\varphi_x\varphi_t + 12u_2\varphi_x^2 + 4\varphi_x\varphi_{xxx} - 3\varphi_{xx}^2 = 0, \qquad (2.1.11)$$

Divide by $12\varphi_x^2$

$$\Rightarrow u_2 = -\frac{1}{12}\frac{\varphi_t}{\varphi_x} - \frac{1}{3}\frac{\varphi_{xxx}}{\varphi_x} + \frac{1}{4}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2 \qquad (2.1.12)$$

Since $p=2$, by using the technique of truncation , and let $u_j=0$ for all $j>2$.

Then the series solution :

$$u = \sum_{j=0}^{\infty} u_j\varphi^{j-p}$$
$$= \sum_{j=0}^{\infty} u_j\varphi^{j-2}$$
$$u = u_0\varphi^{-2} + u_1\varphi^{-1} + u_2 \qquad (2.1.13)$$

By (2.1.7) and (2.1.9) ,we get .

$$u = u_2 + \frac{d^2}{dx^2}\ln(\varphi)$$

Then ,

$$u = -\left(\frac{\varphi_x}{\varphi}\right)^2 + \frac{\varphi_{xx}}{\varphi} + \frac{1}{4}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2 - \frac{1}{3}\frac{\varphi_{xxx}}{\varphi_x} - \frac{1}{12}\frac{\varphi_t}{\varphi_x}$$

Then (2.1.10) , becomes .

$$\begin{aligned} &u_{j-3,t} + (j-4)u_{j-2}\varphi_t + 12\sum_{k=0}^{j-1} u_k u_{j-1-k,x} \\ &+ 12\sum_{i=0}^{j} u_{j-i}u_i(i-2)\varphi_x + u_{j-3,xxx} + 3(j-4)u_{j-2,xx}\varphi_x \\ &+ 3(j-4)u_{j-2,x}\varphi_{xx} + 3(j-3)(j-4)u_{j-1,x}\varphi_x^2 \\ &+ (j-4)u_{j-2}\varphi_{xxx} + 3(j-3)(j-4)u_{j-1}\varphi_x\varphi_{xx} \\ &+ (j-2)(j-3)(j-4)u_j\varphi_x^3 = 0, \end{aligned} \qquad (2.1.14)$$

Now in (2.1.14) , to find all coefficient of u_j . where $u_j \equiv 0$ for all $j<0$.

$$if \quad i = 0 \Rightarrow 12\sum_{i=o}^{j} u_{j-i}u_i(i-2)\varphi_x = 24\varphi_x^3 u_j$$

$$if \quad i = j \Rightarrow 12\sum_{i=o}^{j} u_{j-i}u_i(i-2)\varphi_x = -12\varphi_x^3(j-2)u_j$$

Then (2.1.14), becomes .

$$u_{j-3,t}+(j-4)u_{j-2}\varphi_t+24\varphi_x^3u_j-12\varphi_x^3(j-2)u_j$$
$$+12\sum_{k=0}^{j-1}u_ku_{j-1-k,x}+12\sum_{i=1}^{j-1}u_{j-i}u_i(i-2)\varphi_x$$
$$+u_{j-3,xxx}+3(j-4)u_{j-2,xx}\varphi_x+3(j-4)u_{j-2,x}\varphi_{xx}$$
$$+3(j-3)(j-4)u_{j-1,x}\varphi_x^2+(j-4)u_{j-2}\varphi_{xxx}$$
$$+3(j-3)(j-4)u_{j-1}\varphi_x\varphi_{xx}$$
$$+(j-2)(j-3)(j-4)u_j\varphi_x^3=0,$$

Thus the recursion relation is :

$$(j+1)(j-4)(j-6)\varphi_x^3u_j=-u_{j-3,t}-(j-4)u_{j-2}\varphi_t$$
$$-12\sum_{k=0}^{j-1}u_ku_{j-1-k,x}-12\sum_{i=1}^{j-1}u_{j-i}u_i(i-2)\varphi_x$$
$$-u_{j-3,xxx}-3(j-4)u_{j-2,xx}\varphi_x-3(j-4)u_{j-2,x}\varphi_{xx}$$
$$-3(j-3)(j-4)u_{j-1,x}\varphi_x^2-(j-4)u_{j-2}\varphi_{xxx}$$
$$-3(j-3)(j-4)u_{j-1}\varphi_x\varphi_{xx} \qquad (2.1.15)$$

Clearly . the resonance point are $j = -1$, 4 , 6 .correspond to the free singularity manifold function φ(t,x) , and arbitrary function u_4 ,u_6 .

Now at $j=3$ in (2.1.15), we have .

$$12u_3\varphi_x^3 + u_{0,t} - u_1\varphi_t + 12\sum_{k=0}^{2} u_k u_{2-k,x}$$

$$+12\sum_{i=1}^{2} u_{3-i}u_i(i-2)\varphi_x + u_{0,xxx} - 3u_{1,xx}\varphi_x -$$

$$-3u_{1,x}\varphi_{xx} - u_1\varphi_{xxx} = 0,$$

By using (2.1.7), (2.1.9) and (2.1.12).

$$u_{0,xxx} = -2\varphi_x\varphi_{xxxx} - 6\varphi_{xx}\varphi_{xxx} \tag{I}$$

By differentiating (2.1.11) with respect to x, we get .

$$\varphi_x\varphi_{xt} + \varphi_t\varphi_{xx} + 12\varphi_x^2 u_{2,x} + 24\varphi_x\varphi_{xx}u_2$$
$$+4\varphi_x\varphi_{xxxx} + 4\varphi_{xx}\varphi_{xxx} - 6\varphi_{xx}\varphi_{xxx} = 0,$$

$$-12\varphi_x^2 u_{2,x} = \varphi_x\varphi_{xt} + \varphi_t\varphi_{xx} + 24\varphi_x\varphi_{xx}u_2$$
$$+4\varphi_x\varphi_{xxxx} - 2\varphi_{xx}\varphi_{xxx}, \tag{II}$$

By (2.1.11), (I) and (II), we get.

$$12u_3\varphi_x^3 = \varphi_x\varphi_{xt} + 12\varphi_x\varphi_{xx}u_2 + \varphi_x\varphi_{xxxx},$$

Division by $12\varphi_x^3$,

$$u_3 = \frac{1}{12}\frac{\varphi_{xt}}{\varphi_x^2} + \frac{\varphi_{xx}u_2}{\varphi_x^2} + \frac{1}{12}\frac{\varphi_{xxxx}}{\varphi_x^2}, \tag{2.1.16}$$

Now , at $j = 4$ in (2.1.15), we get .

$$u_{1,t} + 12\sum_{k=0}^{3} u_k u_{3-k,x} + 12\sum_{i=1}^{3} u_{4-i}u_i(i-2)\varphi_x$$
$$+ u_{1,xxx} = 0,$$

$$u_{1,t} + 12u_0u_{3,x} + 12u_1u_{2,x} + 12u_2u_{1,x} + 12u_3u_{0,x}$$
$$-12u_3u_1\varphi_x + 12u_1u_3\varphi_x + u_{1,xxx} = 0,$$

$$\varphi_{xxt} - 12\varphi_x^2 u_{3,x} + 12\varphi_{xx}u_{2,x} + 12\varphi_{xxx}u_2 - 24\varphi_x\varphi_{xx}u_3$$
$$+ \varphi_{xxxxx} = 0,$$

$$\varphi_{xxt} - 12\varphi_x^2\left[\frac{\varphi_x\left(\varphi_{xxt} + 12\varphi_{xxx}u_2 + 12\varphi_{xx}u_{2,x} + \varphi_{xxxxx}\right)}{12\varphi_x^3}\right]$$
$$+12\varphi_x^2\left[\frac{2\varphi_{xx}\left(\varphi_{xt} + 12\varphi_{xx}u_2 + \varphi_{xxxx}\right)}{12\varphi_x^3}\right] + 12\varphi_{xx}u_{2,x} + 12\varphi_{xxx}u_2$$
$$-24\varphi_x\varphi_{xx}\left[\frac{\varphi_{xt} + 12\varphi_{xx}u_2 + \varphi_{xxxx}}{12\varphi_x^2}\right] + \varphi_{xxxxx} = 0,$$

$$\varphi_{xxt} - \varphi_{xxt} - 12\varphi_{xxx}u_2 - 12\varphi_{xx}u_{2,x} - \varphi_{xxxxx}$$
$$+ \frac{2\varphi_{xx}\varphi_{xt}}{\varphi_x} + \frac{24\varphi_{xx}^2 u_2}{\varphi_x} + \frac{2\varphi_{xx}\varphi_{xxxx}}{\varphi_x} + 12\varphi_{xxx}u_2$$
$$+ 12\varphi_{xx}u_{2,x} - \frac{2\varphi_{xx}\varphi_{xt}}{\varphi_x} - \frac{24\varphi_{xx}^2 u_2}{\varphi_x}$$
$$- \frac{2\varphi_{xx}\varphi_{xxxx}}{\varphi_x} + \varphi_{xxxxx} = 0,$$
$$0 = 0,$$

Now at $j=5$ in (2.1.15) .

$$u_{2,t}+12\sum_{k=0}^{4}u_{k}u_{4-k,x}+12\sum_{i=1}^{4}u_{5-i}u_{i}(i-2)\varphi_{x}$$
$$+u_{2,xxx}+3u_{3,xx}\varphi_{x}+3u_{3,x}\varphi_{xx}+6u_{4,x}\varphi_{x}^{2}$$
$$+u_{3}\varphi_{xxx}+6u_{4}\varphi_{x}\varphi_{xx}+u_{3}\varphi_{t}-6\varphi_{x}^{3}u_{5}=0,$$

$$u_{2,t}+u_{3}\varphi_{t}+12u_{0}u_{4,x}+12u_{1}u_{3,x}+12u_{2}u_{2,x}$$
$$+12u_{3}u_{1,x}+12u_{4}u_{0,x}-12u_{4}u_{1}\varphi_{x}+12u_{2}u_{3}\varphi_{x}$$
$$+24u_{1}u_{4}\varphi_{x}+u_{2,xxx}+3u_{3,xx}\varphi_{x}+3u_{3,x}\varphi_{xx}$$
$$+6u_{4,x}\varphi_{x}^{2}+u_{3}\varphi_{xxx}+6u_{4}\varphi_{x}\varphi_{xx}-6\varphi_{x}^{3}u_{5}=0,$$

Since $u_j=0$ for all $j>2$, then .

$$u_{2,t}+12\,u_{2}u_{2,x}+u_{2,xxx}=0, \qquad (2.1.17)$$

Then u_2 is a solution of the Korteweg-de Vries (KDV.I) equation.I

By, at in (2.1.15) with $j=6$ we have .

$$u_{3,t}+2u_{4}\varphi_{x}+12u_{0}u_{5,x}+12u_{1}u_{4,x}+12u_{2}u_{3,x}$$
$$+12u_{3}u_{2,x}+12u_{4}u_{1,x}+12u_{5}u_{0,x}-12u_{5}u_{1}\varphi_{x}$$
$$+12u_{3}^{2}\varphi_{x}+24u_{2}u_{4}\varphi_{x}+36u_{1}u_{5}\varphi_{x}+u_{3,xxx}$$
$$+6u_{4,xx}\varphi_{x}+6u_{4,x}\varphi_{xx}+18u_{5,x}\varphi_{x}^{2}+2u_{4}\varphi_{xxx}=0,$$

Since $u_j=0$ for all $j>2$, then .

$0=0,$

Then the Korteweg-de Vries (KDV.I) equation.I, have the Painleve' property.

Section 2.2
Analytic solution :

In this section, we follow the idea to derive analytic solution. They are invariant under this transformation ,

$$H:\ \phi \mapsto \frac{\alpha\varphi+\beta}{\gamma\varphi+\delta} \qquad (\alpha\delta-\beta\gamma\neq 0).$$

They are the Schwartzian derivative.

$$S(\varphi)=\frac{\varphi_{xxx}}{\varphi_x}-\frac{3}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2, \qquad (2.2.1)$$

and Dimension of velocity .

$$C(\phi)=-\frac{\varphi_t}{\varphi_x}\ , \qquad (2.2.2)$$

furthermore we define ,

$$L=-\frac{\varphi_{xx}}{2\varphi_x}, \qquad (2.2.3)$$

Since

$$L_x=-L^2-\frac{1}{2}S \qquad (2.2.4)$$

and

$$L_t=-CL_x-LC_x+\frac{1}{2}C_{xx}$$

The compatibility of S and C given by ,

$$S_t + C_{xxx} + 2C_x S + CS_x = 0, \qquad (2.2.5)$$

To prove this .
First we find S_t , C_x .

$$S_t = \frac{\varphi_x \varphi_{xxxt} - \varphi_{xt}\varphi_{xxx}}{\varphi_x^2} - 3\left(\frac{\varphi_{xx}}{\varphi_x}\right)\left(\frac{\varphi_x \varphi_{xxt} - \varphi_{xt}\varphi_{xx}}{\varphi_x^2}\right),$$

$$= \left(1/\varphi_x^4\right)\left(\varphi_{xxxt}\varphi_x^3 - \varphi_{xt}\varphi_{xxx}\varphi_x^2 - 3\varphi_{xxt}\varphi_x^2\varphi_{xx} + 3\varphi_{xt}\varphi_{xx}^2\right) \qquad (2.2.6)$$

$$C_x = -\frac{\varphi_x \varphi_{tx} - \varphi_t \varphi_{xx}}{\varphi_x^2} = \left(1/\varphi_x^2\right)\left(-\varphi_{tx}\varphi_x + \varphi_t \varphi_{xx}\right)$$

and the $2SC_x$.

$$2C_x S = 2\left(-\frac{\varphi_{xt}\varphi_x - \varphi_t \varphi_{xx}}{\varphi_x^2}\right)\left[\frac{\varphi_{xxx}}{\varphi_x} - \frac{3}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2\right].$$

$$= \left(\frac{1}{\varphi_x^2}\right)\left[\varphi_t \varphi_{xx} - \varphi_{xt}\varphi_x\right]\left(\frac{1}{\varphi_x^2}\right)\left[\varphi_x \varphi_{xxx} - \frac{3}{2}\varphi_{xx}^2\right].$$

$$= \left(1/\varphi_x^2\right)\left[2\varphi_t \varphi_x \varphi_{xxx} - 3\varphi_t \varphi_{xx}^3 - 2\varphi_{tx}\varphi_x^2\varphi_{xxx} + 3\varphi_{tx}\varphi_x \varphi_{xx}^2\right], \qquad (2.2.7)$$

Now , to find S_x , CS_x .

$$S_x=\frac{\varphi_x\varphi_{xxxx}-\varphi_{xx}\varphi_{xxx}}{\varphi_x^2}-3\left(\frac{\varphi_{xx}}{\varphi_x}\right)\left(\frac{\varphi_x\varphi_{xxx}-\varphi_{xx}^2}{\varphi_x^2}\right),$$

$$=\left(\frac{1}{\varphi_x^3}\right)\left(\varphi_{xxxx}\varphi_x^2-4\varphi_x\varphi_{xx}\varphi_{xxx}+3\varphi_{xx}^3\right),$$

$$CS_x=\left(-\frac{\varphi_t}{\varphi_x}\right)\left(\varphi_{xxxx}\varphi_x^2-4\varphi_x\varphi_{xx}\varphi_{xxx}+3\varphi_{xx}^3\right),$$

$$=\left(\frac{1}{\varphi_x^4}\right)\left[-\varphi_t\varphi_x^2\varphi_{xxxx}+4\varphi_t\varphi_x\varphi_{xx}\varphi_{xxx}-3\varphi_t\varphi_{xx}^3\right], \qquad (2.2.8)$$

and to find C_{xxx} .

$$C_x=-\frac{\varphi_x\varphi_{tx}-\varphi_t\varphi_{xx}}{\varphi_x^2}=\left(\frac{1}{\varphi_x^2}\right)\left(-\varphi_{tx}\varphi_x+\varphi_t\varphi_{xx}\right)$$

$$C_{xx}=\left(\frac{1}{\varphi_x^4}\right)\left[\varphi_x^2\left(-\varphi_{txx}\varphi_x-\varphi_{tx}\varphi_{xx}+\varphi_t\varphi_{xxx}+\varphi_{tx}\varphi_{xx}\right)-\left(-\varphi_{tx}\varphi_x+\varphi_t\varphi_{xx}\right)2\varphi_x\varphi_{xx}\right],$$

$$=\left(\frac{1}{\varphi_x^3}\right)\left[-\varphi_{txx}\varphi_x^2+\varphi_t\varphi_x\varphi_{xxx}+2\varphi_{tx}\varphi_x\varphi_{xx}-2\varphi_t\varphi_x^2\right],$$

$$C_{xxx}=\left(\frac{1}{\varphi_x^6}\right)\left\{\begin{array}{l}\varphi_x^3\left[\begin{array}{l}-\varphi_{txxx}\varphi_x^2-2\varphi_{txx}\varphi_x\varphi_{xx}+\varphi_{tx}\varphi_x\varphi_{xxx}+\varphi_t\varphi_x\varphi_{xxxx}+\varphi_t\varphi_{xx}\varphi_{xxx}\\+2\varphi_{txx}\varphi_x\varphi_{xx}+2\varphi_{tx}\varphi_x\varphi_{xxx}+2\varphi_{tx}\varphi_{xx}^2-2\varphi_{tx}\varphi_{xx}^2-4\varphi_t\varphi_{xx}\varphi_{xxx}\end{array}\right]\\-\left[-\varphi_{txx}\varphi_x^2+\varphi_t\varphi_x\varphi_{xxx}+2\varphi_{tx}\varphi_x\varphi_{xx}-2\varphi_t\varphi_{xx}^2\right]3\varphi_x^2\varphi_{xx}\end{array}\right\},$$

$$=\left(\frac{1}{\varphi_x^4}\right)\left\{\begin{array}{l}-\varphi_{txxx}\varphi_x^3-2\varphi_{txx}\varphi_x^2\varphi_{xx}+\varphi_{tx}\varphi_x^2\varphi_{xxx}+\varphi_t\varphi_x^2\varphi_{xxxx}+\varphi_t\varphi_x\varphi_{xx}\varphi_{xxx}\\+2\varphi_{txx}\varphi_x^2\varphi_{xx}+2\varphi_{tx}\varphi_x^2\varphi_{xxx}+2\varphi_{tx}\varphi_{xx}^2-2\varphi_{tx}\varphi_x\varphi_{xx}^2-4\varphi_t\varphi_x\varphi_{xx}\varphi_{xxx}\\+3\varphi_{txx}\varphi_x^2\varphi_{xx}-3\varphi_t\varphi_x\varphi_{xx}\varphi_{xxx}-6\varphi_{tx}\varphi_x\varphi_{xx}^2+6\varphi_t\varphi_{xx}^3\end{array}\right\},$$

Then ,

$$C_{xxx} = \left(\frac{1}{\varphi_x^4} \right) \begin{Bmatrix} -\varphi_{txxx}\varphi_x^3 + 3\varphi_{tx}\varphi_x^2\varphi_{xxx} + \varphi_t\varphi_x^2\varphi_{xxxx} - 6\varphi_t\varphi_x\varphi_{xx}\varphi_{xxx} \\ +3\varphi_{txx}\varphi_x^2\varphi_{xx} - 6\varphi_{tx}\varphi_x\varphi_{xx}^2 + 6\varphi_t\varphi_{xx}^3 \end{Bmatrix}, \tag{2.2.9}$$

Now substitute (2.2.6) ,(2.2.7) , (2.2.8) and (2.2.9) into (2.2.5) ,
We get .

$$\left(\frac{1}{\varphi_x^4} \right) \begin{Bmatrix} -\varphi_{txxx}\varphi_x^3 - \varphi_{tx}\varphi_x^2\varphi_{xxx} - 3\varphi_{txx}\varphi_x^2\varphi_{xx} + 3\varphi_{tx}\varphi_x\varphi_{xx}^2 - \varphi_{txxx}\varphi_x^3 \\ +3\varphi_{tx}\varphi_x^2\varphi_{xxx} + \varphi_t\varphi_x^2\varphi_{xxxx} - 6\varphi_t\varphi_x\varphi_{xx}\varphi_{xxx} + 3\varphi_{txx}\varphi_x^2\varphi_{xx} \\ -6\varphi_{tx}\varphi_x\varphi_{xx}^2 + 6\varphi_t\varphi_{xx}^3 + 2\varphi_t\varphi_x\varphi_{xx}\varphi_{xxx} - 3\varphi_t\varphi_{xx}^3 - 2\varphi_{tx}\varphi_x^2\varphi_{xxx} \\ +3\varphi_{tx}\varphi_x\varphi_{xx}^2 - \varphi_t\varphi_x^2\varphi_{xxxx} + 4\varphi_t\varphi_x\varphi_{xx}\varphi_{xxx} - 3\varphi_t\varphi_{xx}^3 \end{Bmatrix} = 0,$$

then.

$0=0,$

Now , by (2.1.12) , (2.1.16) . and $u_j=0$ for all $j \geq 3$,
we obtain.

$$0 = \frac{\varphi_{xt}}{\varphi_x} + \frac{12\varphi_{xx}}{\varphi_x}\left(-\frac{1}{12}\frac{\varphi_t}{\varphi_x} - \frac{1}{3}\frac{\varphi_{xxx}}{\varphi_x} + \frac{1}{4}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2 \right) + \frac{\varphi_{xxxx}}{\varphi_x},$$

$$0 = \frac{\varphi_{xt}}{\varphi_x} - \frac{\varphi_t\varphi_{xx}}{\varphi_x^2} - \frac{4\varphi_{xx}\varphi_{xxx}}{\varphi_x^2} + 3\left(\frac{\varphi_{xx}}{\varphi_x}\right)^3 + \frac{\varphi_{xxxx}}{\varphi_x},$$

$$\frac{\varphi_t\varphi_{xx}}{\varphi_x^2} - \frac{\varphi_{xt}}{\varphi_x} = \frac{\varphi_{xxxx}}{\varphi_x} - \frac{4\varphi_{xx}\varphi_{xxx}}{\varphi_x^2} + 3\left(\frac{\varphi_{xx}}{\varphi_x}\right)^3,$$

Then; by compared with (2.2.1) and (2.2.2) we get.

$$C_x = S_x \tag{2.2.10}$$

and

$$u_2 = -\frac{1}{12}\frac{\varphi_t}{\varphi_x} - \frac{1}{3}\frac{\varphi_{xxx}}{\varphi_x} + \frac{1}{4}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2,$$

$$= -\frac{1}{12}\frac{\varphi_t}{\varphi_x} - \frac{1}{3}\frac{\varphi_{xxx}}{\varphi_x} + \frac{1}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2 - \frac{1}{4}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2,$$

$$= -\frac{1}{12}\frac{\varphi_t}{\varphi_x} - \frac{1}{3}\left[\frac{\varphi_{xxx}}{\varphi x} - \frac{3}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2\right] - \left(\frac{\varphi_{xx}}{2\varphi_x}\right)^2.$$

$$u_2 = \frac{1}{12}C - \frac{1}{3}S - \left(\frac{\varphi_{xx}}{2\varphi_x}\right)^2, \qquad (2.2.11)$$

$$\Rightarrow \quad u_2 = \frac{1}{12}C - \frac{1}{3}S - L^2$$

Now to find $u_{2,t}$, $12u_2u_{2,x}$ and $u_{2,xxx}$

$$u_{2,t} = \frac{1}{12}C_t - \frac{1}{3}S_t - 2LL_t, \qquad (2.2.12)$$

$$u_{2,x} = \frac{1}{12}C_x - \frac{1}{3}S_x - 2LL_x$$

$$u_{2,xx} = \frac{1}{12}C_{xx} - \frac{1}{3}S_{xx} - 2\left(LL_{xx} + L_x^2\right)$$

$$u_{2,xxx} = \frac{1}{12}C_{xxx} - \frac{1}{3}S_{xxx} - 2(LL_{xxx} + L_xL_{xx} + 2L_xL_{xx})$$
$$= \frac{1}{12}C_{xxx} - \frac{1}{3}S_{xxx} - 2LL_{xxx} - 6L_xL_{xx}, \qquad (2.2.13)$$

and

$$u_2 u_{2,x} = \left(\frac{1}{12}C - \frac{1}{3}S - L^2\right)\left(\frac{1}{12}C_x - \frac{1}{3}S_x - 2LL_x\right),$$
$$= \frac{1}{12}CC_x - \frac{1}{3}CS_x - 2CLL_x - \frac{1}{3}SC_x + \frac{4}{3}SS_x + 8SLL_x$$
$$- L^2C_x + 4L^2S_x + 24L^3L_x \qquad (2.2.14)$$

By substituting (2.2.12), (2.2.13) and (2.2.14) in the equation, $u_{2,t}+12u_2u_{2,x}+u_{2,xxx}=0$,

we obtain.

$$\frac{1}{12}C_t - \frac{1}{3}S_t - 2LL_t + \frac{1}{12}CC_x - \frac{1}{3}CS_x - 2CLL_x - \frac{1}{3}SC_x + \frac{4}{3}SS_x$$
$$+ 8SLL_x - L^2C_x + 4L^2S_x + 24L^3L_x + \frac{1}{12}C_{xxx} - \frac{1}{3}S_{xxx} - 2LL_{xxx}$$
$$- 6L_xL_{xx} = 0.$$

Then :

$$C_t - 4S_t - 24CL^3 - 12CLS + 24L^2C_x - 12LC_{xx} + CC_x - 4CS_x$$
$$- 4SC_x + 16SS_x - 12L^2C_x + 48L^2S_x + 24CL^3 - 96SL^3 - 288L^5$$
$$+12CLS-48LS^2-144L^3S+C_{xxx}-4S_{xxx}+144L^5+72L^3S-24L^2S_x$$
$$+24L^3S+12LS^2+12LS_{xx}+144L^5+72L^3S-36L^2S_x+72L^3S$$
$$+ 36LS^2 - 18SS_x = 0,$$

$$C_t - 4S_t + 12L^2C_x - 12LC_{xx} + CC_x - 4CS_x - 4SC_x - 2SS_x$$
$$- 12L^2S_x + C_{xxx} - 4S_{xxx} + 12LS_{xx} = 0, \qquad (2.2.15)$$

By (2.2.10), then :

$$12L^2C_x - 12L^2S_x = 12L^2(C_x - S_x) = 0,$$

also

$$- 12LC_{xx} + 12LS_{xx} = 12L(S_{xx} - C_{xx}) = 0,$$

Then (2.2.15) ,becomes .

$$C_t - 4S_t - 3CC_x - 6SC_x - 3C_{xxx} = 0, \tag{2.2.16}$$

And by substituting S_t in (2.2.5), we get .

$$C_t - 4(-C_{xxx} - 2C_x S - CC_x) - 3CC_x - 6SC_x - 3C_{xxx} = 0.$$

Lead to ,

$$C_t + C_{xxx} + 2C_x S + CC_x = 0.$$

by compared with (2.2.5) .,

Then :

$$C_t = S_t \tag{2.2.17}$$

By (2.2.10) and (2.2.17) then :

$$C = S + K \qquad where\ K\ is\ constant.$$
$$for:\ K = 0$$
$$\Rightarrow C = S \tag{2.2.18}$$

Then (2.216) becomes .

$$-3C_t - 9CC_x - 3C_{xxx} = 0,$$
$$\Rightarrow C_t + 3CC_x + C_{xxx} = 0,$$

$$\Rightarrow S_t + 3SS_x + S_{xxx} = 0, \tag{2.2.19}$$

This Korteweg-de Vries like equation .

Section 2.3
Exact Solution .

Solution for constant S .
The constant functions $S=\pm 2\lambda^2$ where λ is a constant are solutions of the Korteweg-de Vries like equation (2.2.19) .

case I :

For S $=-2\lambda^2$: we have

$$S=\{\varphi,x\}=-2\lambda^2$$

Hence P(x) = $-\lambda^2$ in (III) of Chapter One, and two linearly independent solutions are .

$$\Psi_1=E(t)e^{\lambda x}+F(t)e^{-\lambda x} \qquad , \qquad \Psi_2=G(t)e^{\lambda x}+H(t)e^{-\lambda x}$$

Therefore by Lemma (1.1.15) and Lemma (1.1.16) of Chapter One obtains .

$$\varphi(t,x)=\frac{E(t)e^{\lambda x}+F(t)e^{-\lambda x}}{G(t)e^{\lambda x}+H(t)e^{-\lambda x}}, \qquad EH-FG\neq 0, \qquad (2.3.1)$$

By using (2.2.2) and (2.2.18) , then :

$$C=S=-\frac{\varphi_t}{\varphi_x}=-2\lambda^2 \qquad (2.3.2)$$

Now to find the equation of coefficients E(t) , F(t) ,G(t) and H(t) .

$$\varphi_t=\frac{\begin{bmatrix}\left(G(t)e^{\lambda x}+H(t)e^{-\lambda x}\right)\left(E'(t)e^{\lambda x}+F'(t)e^{-\lambda x}\right)\\ -\left(E(t)e^{\lambda x}+F(t)e^{-\lambda x}\right)\left(G'(t)e^{\lambda x}+H'(t)e^{-\lambda x}\right)\end{bmatrix}}{\left(G(t)e^{\lambda x}+H(t)e^{-\lambda x}\right)^2},$$

Then :

$$\varphi_t = \frac{\begin{bmatrix} (G(t)E'(t) - E(t)G'(t))e^{2\lambda x} + (H(t)F'(t) - F(t)H'(t))e^{-2\lambda x} \\ + (G(t)F'(t) - F(t)G'(t)) + (H(t)E'(t) - E(t)H'(t)) \end{bmatrix}}{\left(G(t)e^{\lambda x} + H(t)e^{-\lambda x}\right)^2}$$

and,

$$\varphi_x = \begin{bmatrix} \lambda\left(G(t)e^{\lambda x} + H(t)e^{-\lambda x}\right)\left(E(t)e^{\lambda x} - F(t)e^{-\lambda x}\right) \\ -\lambda\left(E(t)e^{\lambda x} + F(t)e^{-\lambda x}\right)\left(G(t)e^{\lambda x} - H(t)e^{-\lambda x}\right) \end{bmatrix} \Big/ \left(G(t)e^{\lambda x} + H(t)e^{-\lambda x}\right)^2,$$

$$\Rightarrow \varphi_x = \frac{2\lambda\left(H(t)E(t) - G(t)F(t)\right)}{\left(G(t)e^{\lambda x} + H(t)e^{-\lambda x}\right)^2}$$

By (2.3.2) , we get .

$$C = \frac{-\begin{bmatrix} (G(t)E'(t) - E(t)G'(t))e^{2\lambda x} + (H(t)F'(t) - F(t)H'(t))e^{-2\lambda x} \\ + (G(t)F'(t) - F(t)G'(t)) + (H(t)E'(t) - E(t)H'(t)) \end{bmatrix}}{2\lambda\left(H(t)E(t) - G(t)F(t)\right)} = -2\lambda^2,$$

Then :

$$\begin{aligned} &\left(G(t)E'(t) - E(t)G'(t)\right)e^{2\lambda x} + \left(H(t)F'(t) - F(t)H'(t)\right)e^{-2\lambda x} + G(t)F'(t) \\ &\quad - F(t)G'(t) + \left(H(t)E'(t) - E(t)H'(t)\right) \\ &\quad = 4\lambda^3\left(H(t)E(t) - G(t)F(t)\right), \end{aligned}$$

This leads to a system of nonlinear ordinary differential equation in coefficients E(t) , F(t) , G(t) , and H(t) .

$$GE' - EG' = 0 \qquad \text{———} \qquad (I)$$

$$HF' - FH' = 0 \qquad \text{———} \qquad (II)$$

$$(GF' - FG') + (HE' - EH') = 4\lambda^3(HE - GF) \qquad \text{———} \qquad (III)$$

A particular solution of (I) and (II) is :

$E(t) = BG(t)$ and $F(t) = AH(t)$

where A , B are real arbitrary constant .

Substituting these into (III) , we get .

$$A\big(G(t)H'(t) - H(t)G'(t)\big) + B\big(H(t)G'(t) - G(t)H'(t)\big)$$
$$= 4\lambda^3 H(t)G(t)(B - A),$$

$$-(A - B)\big(G(t)H'(t) - H(t)G'(t)\big) = 4\lambda^3 H(t)G(t)(B - A),$$
$$G(t)H'(t) - H(t)G'(t) = 4\lambda^3 H(t)G(t),$$

$$\frac{G(t)H'(t) - H(t)G'(t)}{H(t)G(t)} = -4\lambda^3,$$

$$\frac{H'(t)}{H(t)} - \frac{G'(t)}{G(t)} = -4\lambda^3,$$

by integrating the above .

$$Ln\left(\frac{H}{G}\right) = -4\lambda^3 t,$$

$$\Rightarrow \frac{H(t)}{G(t)} = Exp\left(-4\lambda^3 t\right)$$

Then (2.3.1) , becomes .

$$\varphi(t,x) = \frac{BG(t)\exp(\lambda x) + AG(t)\exp\left(-4\lambda^3 t - \lambda x\right)}{G(t)\exp(\lambda x) + G(t)\exp\left(-4\lambda^3 t - \lambda x\right)}$$

And multiplying by $\frac{1}{G(t)}\exp\left(2\lambda^3 t\right)$

$$\varphi(t,x) = \frac{B\exp\left(2\lambda^3 t + \lambda x\right) + A\exp\left(-2\lambda^3 t - \lambda x\right)}{\exp\left(2\lambda^3 t + \lambda x\right) + \exp\left(-2\lambda^3 t - \lambda x\right)},$$

$$\varphi(t,x) = \frac{Be^{\lambda\xi} + Ae^{-\lambda\xi}}{e^{\lambda\xi} + e^{-\lambda\xi}}, \qquad where\ \xi = 2\lambda^2 t + x$$

$$= \frac{B(\sinh\lambda\xi + \cosh\lambda\xi) + A(\cosh\lambda\xi - \sinh\lambda\xi)}{2\cosh\lambda\xi},$$

$$= \frac{(B + A)\cosh\lambda\xi + (B - A)\sinh\lambda\xi}{2\cosh\lambda\xi},$$

$$\Rightarrow \varphi(t,x) = K_1 + K_2\tanh\lambda\xi \qquad (2.3.3)$$

$$where\ K_1 = (B + A)/2\ ,\ K_2 = (B - A)/2$$

where K_1 and K_2 are arbitrary constant , When $K_1 = 0$, by substituting (2.3.3) in to (2.1.12) , then :

$$\overset{(1)}{u_2} = -\frac{1}{12}\frac{2K_2\lambda^3 \sec h^2 \lambda\xi}{K_2\lambda \sec h^2 \lambda\xi} - \frac{1}{3}\frac{-2K_2\lambda^3 \sec h^4 \lambda\xi + 4K_2\lambda^3 \sec h^2 \lambda\xi \tanh^2 \lambda\xi}{K_2\lambda \sec h^2 \lambda\xi}$$
$$+\frac{1}{4}\frac{4K_2^2\lambda^4 \sec h^4 \lambda\xi \tanh^2 \lambda\xi}{K_2^2\lambda^2 \sec h^4 \lambda\xi},$$

$$\Rightarrow \overset{(1)}{u_2} = \lambda^2\left(\sec h^2 \lambda\xi - \frac{1}{2}\right), \quad where\ \xi = x + 2\lambda^2 t$$

By (2.1.7) , (2.1.9) ,(2.1.13) and (2.3.2), we obtain .

$$\overset{(1)}{u} = \frac{-\varphi_x^2}{\varphi^2} + \frac{\varphi_{xx}}{\varphi} + u_2$$
$$= \frac{-K_2^2\lambda^2 \sec h^4 \lambda\xi}{K_2^2 \tanh^2 \lambda\xi} - \frac{K_2\lambda^2 \sec h^2 \lambda\xi \tanh \lambda\xi}{\tanh \lambda\xi} + u_2$$
$$= \frac{-\lambda^2 \sec h^4 \lambda\xi}{\tanh^2 \lambda\xi} - 2\lambda^2 \sec h^2 \lambda\xi + u_2$$
$$= -\lambda^2 \sec h^2 \lambda\xi\left(c \sec h^2 \lambda\xi + 2\right) + u_2$$

$$\Rightarrow \overset{(1)}{u} = -\lambda^2\left(c \sec h^2 \lambda\xi + \frac{1}{2}\right), \qquad where\ \xi = x + 2\lambda^2 t$$

Hence $\overset{(1)}{u}(t,x)$ and $\overset{(1)}{u_2}(t,x)$ in the above are exact solutions for Korteweg-de Vries (or KDV-I) equation .

case II :

For $S=2\lambda^2$; we have :

$S=\{\varphi,x\}=2\lambda^2$.

Hence $P(x)=\lambda^2$ in (III) of Chapter One .and two linearly independent solutions are .

$V_1=E(t)e^{\lambda x}+F(t)e^{-\lambda x}$, $V_2=G(t)e^{\lambda x}+H(t)e^{-\lambda x}$

Therefore by Lemma (1.1.15) and Lemma (1.1.16) of Chapter One obtains .

$$\varphi(t,x)=\frac{E(t)e^{\lambda x}+F(t)e^{-\lambda x}}{G(t)e^{\lambda x}+H(t)e^{-\lambda x}}, \qquad EH-FG\neq 0, \qquad (2.3.4)$$

By using (2.2.2) and (2.2.18) , then :

$$C=S=-\frac{\varphi_t}{\varphi_x}=2\lambda^2 \qquad (2.3.5)$$

where :

$$\varphi_t=\frac{\begin{bmatrix}(G(t)E'(t)-E(t)G'(t))e^{2\lambda x}+(H(t)F'(t)-F(t)H'(t))e^{-2\lambda x}\\+(G(t)F'(t)-F(t)G'(t))+(H(t)E'(t)-E(t)H'(t))\end{bmatrix}}{\left(G(t)e^{\lambda x}+H(t)e^{-\lambda x}\right)^2},$$

and,

$$\varphi_x=\frac{2i\lambda(H(t)E(t)-G(t)F(t))}{\left(G(t)e^{\lambda x}+H(t)e^{-\lambda x}\right)^2}$$

Then :

$$\frac{\varphi_t}{\varphi_x}=\frac{\begin{bmatrix}(G(t)E'(t)-E(t)G'(t))e^{2\lambda x}+(H(t)F'(t)-F(t)H'(t))e^{-2\lambda x}\\+(G(t)F'(t)-F(t)G'(t))+(H(t)E'(t)-E(t)H'(t))\end{bmatrix}}{-2i\lambda(H(t)E(t)-G(t)F(t))}=2\lambda^2$$

Then the system of nonlinear ordinary differential equation with coefficients E(t) , F(t) , G(t) , and H(t) , becomes .

$$GE' - EG' = 0 \qquad (I)$$

$$HF' - FH' = 0 \qquad (II)$$

$$(GF' - FG') + (HE' - EH') = -4\lambda^3(HE - GF) \qquad (III)$$

A particular solution of (I) and (II) is :

$$E(t) = MG(t) \quad \text{and} \quad F(t) = NH(t)$$

where M , N are real arbitrary constant .

Substituting these in to (III) , becomes .

$$\frac{H(t)}{G(t)} = Exp(4i\lambda^3 t).$$

Then (2.3.4) , becomes .

$$\varphi(t,x) = \frac{MG(t)\exp(\lambda ix) + NG(t)\exp(4\lambda^3 it - \lambda ix)}{G(t)\exp(\lambda ix) + G(t)\exp(4\lambda^3 it - \lambda ix)},$$

And multiplying by $\frac{1}{G(t)}\exp(-2\lambda^3 it)$

$$\varphi(t,x) = \frac{M\exp(-2\lambda^3 it + \lambda ix) + N\exp(2\lambda^3 it - \lambda ix)}{\exp(-2\lambda^3 it + \lambda ix) + \exp(2\lambda^3 it - \lambda ix)},$$

$$= \frac{Me^{\lambda i\xi} + Ne^{-\lambda i\xi}}{e^{\lambda i\xi} + e^{-\lambda i\xi}}, \qquad where\ \xi = x - 2\lambda^2 t$$

Then :

$$\varphi(t,x) = \frac{M(\sin\lambda\xi + \cos\lambda\xi) + N(\cos\lambda\xi - \sin\lambda\xi)}{2\cos\lambda\xi}.$$

$$= \frac{(M+N)\cos\lambda\xi + (M-N)\sin\lambda\xi}{2\cos\lambda\xi}$$

$$\Rightarrow \varphi(t,x) = K_3 + K_4 \tan\lambda\xi \qquad (2.3.6)$$

$$where\ K_3 = (M+N)/2\ ,\ K_4 = (M-N)/2,$$

where K_3 and K_4 are arbitrary constant ,

For $K_3 = 0$, then by substituting (2.3.6) into (2.1.12) , we get :

$$\overset{(2)}{u_2} = -\frac{1}{12}\frac{-2K_4\lambda^3\sec^2\lambda\xi}{K_4\lambda\sec^2\lambda\xi} - \frac{1}{3}\frac{2K_4\lambda^3\sec^4\lambda\xi + 4K_4\lambda^3\sec^2\lambda\xi\tan^2\lambda\xi}{K_4\lambda\sec^2\lambda\xi} + \frac{1}{4}\frac{4K_4^2\lambda^4\sec^4\lambda\xi\tan^2\lambda\xi}{K_4^2\lambda^2\sec^4\lambda\xi}.$$

$$\Rightarrow \overset{(2)}{u_2} = -\lambda^2\left(\sec^2\lambda\xi - \frac{1}{2}\right),\quad where\ \xi = x - 2\lambda^2 t$$

and by substituting (2.3.6) into (2.1.13) , we get :

$$\overset{(2)}{u} = \frac{-\varphi_x^2}{\varphi^2} + \frac{\varphi_{xx}}{\varphi} + u_2$$

$$\Rightarrow \overset{(2)}{u} = -\lambda^2\left(c\sec^2\lambda\xi - \frac{1}{2}\right),\quad where\ \xi = x - 2\lambda^2 t$$

Hence $\overset{(2)}{u}(t,x)$ and $\overset{(1)}{u_2}(t,x)$ in the above are exact solutions for Korteweg-de Vries (or KDV-I) equation .

CHAPTER THREE
Modified Korteweg-deVries (orMKDVII) equation. II

CHAPTER THREE

In this chapter we study the modified Korteweg-de Vries equation.II , and through this study we find that the modified Korteweg-de Vries equation.II didn't satisfies Painlevé property ,but despite that by using truncation technique , we also find analytic solution ,we proceed as follow .

The modified Korteweg-de Vries (or KDV.II) equation.II

Section 3.1
Painlevé property .

$$u_t - 6u^2u_x + u_{xxx} = 0, \tag{3.1.1}$$

Let $u = \frac{1}{\varphi^p}\sum_{j=0}^{\infty} u_j\varphi^j$ be the series solution of (3.1.1) ,where φ and u_j are analytic functions in a neighborhood of the manifold $\varphi=0$,

First, to find value of p, we need to find u_t, u_{xxx} and $12u^2u_x$ then :

$$u_t = \sum_{j=0}^{\infty}[u_{j,t}\varphi^{j-p} + (j-p)u_j\varphi_t\varphi^{j-p-1}], \tag{3.1.2}$$

$$u_x = \sum_{j=0}^{\infty}[u_{j,x}\varphi^{j-p} + (j-p)u_j\varphi_x\varphi^{j-p-1}],$$

$$u_{xx} = \sum_{j=0}^{\infty} [u_{j,xx}\varphi^{j-p} + 2(j-p)u_{j,x}\varphi_x\varphi^{j-p-1} + (j-p)u_j\varphi_{xx}\varphi^{j-p-1} + (j-p-1)(j-p)u_j\varphi_x^2\varphi^{j-p-2}],$$

$$u_{xxx} = \sum_{j=0}^{\infty}[u_{j,xxx}\varphi^{j-p} + 3(j-p)u_{j,xx}\varphi_x\varphi^{j-p-1} + 3(j-p)u_{j,x}\varphi_{xx}\varphi^{j-p-1} + 3(j-p)(j-p-1)u_{j,x}\varphi_x^2\varphi^{j-p-2} + (j-p)u_j\varphi_{xxx}\varphi^{j-p-1} + 3(j-p)(j-p-1)u_j\varphi_x\varphi_{xx}\varphi^{j-p-2} + (j-p)(j-p-1)(j-p-2)u_j\varphi_x^3\varphi^{j-p-3}], \tag{3.1.3}$$

$$u^2 = \sum_{j=0}^{\infty}\sum_{k=0}^{j} u_{j-k}u_k\varphi^{j-2p}$$

and $-6u^2u_x =$

$$-6\sum_{j=0}^{\infty}\left[\sum_{k=0}^{j}\sum_{i=0}^{k} u_{j-k}u_{k-i}u_i(j-k-p)\varphi_x + \sum_{k=0}^{j-1}\sum_{i=0}^{k} u_{j-k-1,x}u_{k-i}u_i\right]\varphi^{j-3p-1} \tag{3.1.4}$$

Now , substituting (3.1.2) , (3.1.3) and (3.1.4) into (3.1.1) .We get .

$$\sum_{j=0}^{\infty}[u_{j,t}\varphi^{j-p} + (j-p)u_j\varphi_t\varphi^{j-p-1}]$$

$$-6\sum_{j=0}^{\infty}\left[\sum_{k=0}^{j}\sum_{i=0}^{k} u_{j-k}u_{k-i}u_i(j-k-p)\varphi_x + \sum_{k=0}^{j-1}\sum_{i=0}^{k} u_{j-k-1,x}u_{k-i}u_i\right]\varphi^{j-3p-1}$$

$$+\sum_{j=0}^{\infty}[u_{j,xxx}\varphi^{j-p} + 3(j-p)u_{j,xx}\varphi_x\varphi^{j-p-1} + 3(j-p)u_{j,x}\varphi_{xx}\varphi^{j-p-1}$$

$$+3(j-p)(j-p-1)u_{j,x}\varphi_x^2\varphi^{j-p-2} + (j-p)u_j\varphi_{xxx}\varphi^{j-p-1}$$

$$+3(j-p)(j-p-1)u_j\varphi_x\varphi_{xx}\varphi^{j-p-2}$$

$$+(j-p)(j-p-1)(j-p-2)u_j\varphi_x^3\varphi^{j-p-3}] = 0, \tag{3.1.5}$$

Now , to find ***P*** we must camper the lowest power in (3.1.5) .

$$j-3p-1 = j-p-3$$

$$\Rightarrow p = 1$$

Now , by substituting $p=1$ into (3.1.5) , we get .

$$\sum_{j=0}^{\infty}[u_{j,t}\varphi^{j-1}+(j-1)u_j\varphi_t\varphi^{j-2}]$$
$$-6\sum_{j=0}^{\infty}\left[\sum_{k=0}^{j}\sum_{i=0}^{k}u_{j-k}u_{k-i}u_i(j-k-1)\varphi_x+\sum_{k=0}^{j-1}\sum_{i=0}^{k}u_{j-k-1,x}u_{k-i}u_i\right]\varphi^{j-4}$$
$$+\sum_{j=0}^{\infty}[u_{j,xxx}\varphi^{j-1}+3(j-1)u_{j,xx}\varphi_x\varphi^{j-2}+3(j-1)u_{j,x}\varphi_{xx}\varphi^{j-2}$$
$$+3(j-1)(j-2)u_{j,x}\varphi_x^2\varphi^{j-3}+(j-1)u_j\varphi_{xxx}\varphi^{j-2}$$
$$+3(j-1)(j-2)u_j\varphi_x\varphi_{xx}\varphi^{j-3}$$
$$+(j-1)(j-2)(j-3)u_j\varphi_x^3\varphi^{j-4}]=0,$$

Now by associated the summation ,we get .

$$\sum_{j=3}^{\infty}u_{j-3,t}\varphi^{j-4}+\sum_{j=2}^{\infty}(j-3)u_{j-2}\varphi_t\varphi^{j-4}$$
$$-6\sum_{j=0}^{\infty}\left[\sum_{k=0}^{j}\sum_{i=0}^{k}u_{j-k}u_{k-i}u_i(j-k-1)\varphi_x+\sum_{k=0}^{j-1}\sum_{i=0}^{k}u_{j-k-1,x}u_{k-i}u_i\right]\varphi^{j-4}$$
$$+\sum_{j=3}^{\infty}u_{j-3,xxx}\varphi^{j-4}\sum_{j=2}^{\infty}3(j-3)u_{j-2,xx}\varphi_x\varphi^{j-4}$$
$$+\sum_{j=2}^{\infty}3(j-3)u_{j-2,x}\varphi_{xx}\varphi^{j-4}+\sum_{j=1}^{\infty}3(j-2)(j-3)u_{j-1,x}\varphi_x^2\varphi^{j-4}$$
$$+\sum_{j=2}^{\infty}(j-3)u_{j-2}\varphi_{xxx}\varphi^{j-4}+\sum_{j=1}^{\infty}3(j-2)(j-3)u_{j-1}\varphi_x\varphi_{xx}\varphi^{j-4}$$
$$+\sum_{j=0}^{\infty}(j-1)(j-2)(j-3)u_j\varphi_x^3\varphi^{j-4}=0, \qquad (3.1.6)$$

Now , to find u_0 then at $j=0$ in (3.1.6), then .

$$6u_0^3\varphi_x\varphi^{j-4} - 6u_0\varphi_x^3\varphi^{j-4} = 0,$$
$$\Rightarrow u_0^2 = \varphi_x^2$$

$$\Rightarrow u_0 = \varepsilon\varphi_x \qquad where\ \varepsilon = \pm 1 \tag{3.1.7}$$

Then (3.1.6) becomes ,

$$\sum_{j=3}^{\infty} u_{j-3,t}\varphi^{j-4} + \sum_{j=2}^{\infty}(j-3)u_{j-2}\varphi_t\varphi^{j-4}$$
$$-6\sum_{j=1}^{\infty}\left[\sum_{k=0}^{j}\sum_{i=0}^{k}u_{j-k}u_{k-i}u_i(j-k-1)\varphi_x + \sum_{k=0}^{j-1}\sum_{i=0}^{k}u_{j-k-1,x}u_{k-i}u_i\right]\varphi^{j-4}$$
$$+\sum_{j=3}^{\infty}u_{j-3,xxx}\varphi^{j-4}\sum_{j=2}^{\infty}3(j-3)u_{j-2,xx}\varphi_x\varphi^{j-4}$$
$$+\sum_{j=2}^{\infty}3(j-3)u_{j-2,x}\varphi_{xx}\varphi^{j-4} + \sum_{j=1}^{\infty}3(j-2)(j-3)u_{j-1,x}\varphi_x^2\varphi^{j-4}$$
$$+\sum_{j=2}^{\infty}(j-3)u_{j-2}\varphi_{xxx}\varphi^{j-4} + \sum_{j=1}^{\infty}3(j-2)(j-3)u_{j-1}\varphi_x\varphi_{xx}\varphi^{j-4}$$
$$+\sum_{j=1}^{\infty}(j-1)(j-2)(j-3)u_j\varphi_x^3\varphi^{j-4} = 0, \tag{3.1.8}$$

Now , to find u_1 then at $j=1$ we have from (3.1.8) .

$$-6\varphi_x \sum_{k=0}^{1}\left[\sum_{i=0}^{k} u_{k-i}u_i\right](-k)u_{1-k}\varphi^{-3} - 6\sum_{k=0}^{0}\left[\sum_{i=0}^{k} u_{k-i}u_i\right]u_{-k,x}\varphi^{-3}$$

$$+6\varphi_x^2 u_{0,x}\varphi^{-3} + 6\varphi_x\varphi_{xx}u_0\varphi^{-3} = 0,$$

$$-6\varphi_x \sum_{k=0}^{1}\left[u_k u_0 + u_{k-1}u_1\right](-k)u_{1-k} - 6u_{0,x}u_o^2 - 6\varphi_x^2 u_{0,x}$$

$$+6\varphi_x\varphi_{xx}u_0 = 0,$$

$$2\varphi_x u_1 u_0^2 - u_{0,x}u_0^2 + \varphi_x^2 u_{0,x} + \varphi_x\varphi_{xx}u_0 = 0,$$

$$2\varphi_x^3 u_1 + \varepsilon\varphi_x^2\varphi_{xx} = 0,$$

$$\Rightarrow u_1 = -\frac{\varepsilon}{2}\frac{\varphi_{xx}}{\varphi_x}, \qquad where\ \ \varepsilon = \pm 1 \qquad (3.1.9)$$

Since $p=1$, by using the technique of truncation ,
and let $u_j = 0$ for all $j > 1$,
Then the series solution :

$$u = \sum_{j=0}^{\infty} u_j \varphi^{j-p}$$

becomes ,

$$u = \frac{u_0}{\varphi} + u_1 \qquad (3.1.10)$$

$$or \qquad u = \varepsilon\left[(\ln\varphi)_x - \frac{1}{2}(\ln\varphi_x)_x\right],$$

Then (3.1.8) , becomes .

$$\sum_{j=2}^{\infty}(j-3)\left(u_{j-2}\varphi_t+3u_{j-2,xx}\varphi_x+3u_{j-2,x}\varphi_{xx}+u_{j-2}\varphi_{xxx}\right)\varphi^{j-4}$$
$$-6\sum_{j=2}^{\infty}\left[\sum_{k=0}^{j}\sum_{i=0}^{k}u_{j-k}u_{k-i}u_i(j-k-1)\varphi_x+\sum_{k=0}^{j-1}\sum_{i=0}^{k}u_{j-k-1,x}u_{k-i}u_i\right]\varphi^{j-4}$$
$$+\sum_{j=3}^{\infty}u_{j-3,t}\varphi^{j-4}+\sum_{j=2}^{\infty}3(j-2)(j-3)\left(u_{j-1,x}\varphi_x^2+u_{j-1}\varphi_x\varphi_{xx}\right)\varphi^{j-4}$$
$$+\sum_{j=3}^{\infty}u_{j-3,xxx}\varphi^{j-4}+\sum_{j=2}^{\infty}(j-1)(j-2)(j-3)u_j\varphi_x^3\varphi^{j-4}=0, \quad (3.1.11)$$

Now to find u_2 then at $j=2$, we get .

$$-u_0\varphi_t-3\varphi_x u_{0,xx}-3u_{0,x}\varphi_{xx}-u_0\varphi_{xxx}$$
$$-6\varphi_x\sum_{k=0}^{2}\left[u_k u_0+u_{k-1}u_1+u_{k-2}u_2\right](1-k)u_{2-k}$$
$$-6\sum_{k=0}^{1}\left[u_k u_0+u_{k-1}u_1\right]u_{1-k,x}=0,$$

$$-u_0(\varphi_t+\varphi_{xxx})-3\varphi_x u_{0,xx}-3u_{0,x}(\varphi_{xx}+4u_0u_1)$$
$$+6\varphi_x(u_0u_1^2+u_2u_0^2)-6u_0^2u_{1,x}=0,$$

By using (3.1.7) and (3.1.8), we obtain,

$$-\varepsilon\varphi_x(\varphi_t+\varphi_{xxx})-3\varepsilon\varphi_x\varphi_{xxx}-3\varepsilon\varphi_{xx}(\varphi_{xx}-2\varphi_{xx})$$
$$+6\varphi_x\left(\frac{\varepsilon}{4}\frac{\varphi_{xx}^2}{\varphi_x}+\varphi_x^2u_2\right)+3\varphi_x^2\left(\frac{\varphi_x\varphi_{xxx}-\varphi_{xx}^2}{\varphi_x^2}\right)=0,$$
$$-\varepsilon\varphi_x(\varphi_t+\varphi_{xxx})+\frac{3\varepsilon}{2}\varphi_{xx}^2+6\varphi_x^3u_2=0,$$

$$\Rightarrow u_2=\frac{\varepsilon}{6\varphi_x}\left(\frac{1}{\varphi_x}[\varphi_t+\varphi_{xxx}]-\frac{3}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2\right),\quad(3.1.12)$$

Then (3.1.11), becomes.

$$\sum_{j=3}^{\infty}(j-3)\left(u_{j-2}\varphi_t+3u_{j-2,xx}\varphi_x+3u_{j-2,x}\varphi_{xx}+u_{j-2}\varphi_{xxx}\right)\varphi^{j-4}$$
$$-6\sum_{j=3}^{\infty}\left[\sum_{k=0}^{j}\sum_{i=0}^{k}u_{j-k}u_{k-i}u_i(j-k-1)\varphi_x+\sum_{k=0}^{j-1}\sum_{i=0}^{k}u_{j-k-1,x}u_{k-i}u_i\right]\varphi^{j-4}$$
$$+\sum_{j=3}^{\infty}u_{j-3,t}\varphi^{j-4}+\sum_{j=3}^{\infty}3(j-2)(j-3)\left(u_{j-1,x}\varphi_x^2+u_{j-1}\varphi_x\varphi_{xx}\right)\varphi^{j-4}$$
$$+\sum_{j=3}^{\infty}u_{j-3,xxx}\varphi^{j-4}+\sum_{j=3}^{\infty}(j-1)(j-2)(j-3)u_j\varphi_x^3\varphi^{j-4}=0,$$

And by equating both sides , we get ,

$$
\begin{aligned}
&u_{j-3,t}+(j-3)\left(u_{j-2}\varphi_t+3u_{j-2,xx}\varphi_x+3u_{j-2,x}\varphi_{xx}+u_{j-2}\varphi_{xxx}\right)\\
&-6\sum_{k=0}^{j}\sum_{i=0}^{k}u_{j-k}u_{k-i}u_i(j-k-1)\varphi_x-6\sum_{k=0}^{j-1}\sum_{i=0}^{k}u_{j-k-1,x}u_{k-i}u_i\\
&+u_{j-3,xxx}+3(j-2)(j-3)\left(u_{j-1,x}\varphi_x^2+u_{j-1}\varphi_x\varphi_{xx}\right)\\
&+(j-1)(j-2)(j-3)u_j\varphi_x^3=0, \qquad (3.1.13)
\end{aligned}
$$

where $u_j \equiv 0$ for all $j<0$, to find all coefficient of u_j we have .

$$
\begin{aligned}
&\text{if } \; k=0 \Rightarrow -6\varphi_x\sum_{k=0}^{j}\left[\sum_{i=0}^{k}u_{k-i}u_i\right](j-k-1)u_{j-k}\\
&\qquad = -6\varphi_x u_0^2(j-1)u_j\\
&\text{if } \; i=k \Rightarrow -6\varphi_x\sum_{k=0}^{j}\left[\sum_{i=0}^{k}u_{k-i}u_i\right](j-k-1)u_{j-k}\\
&\qquad = 6\varphi_x u_0^2 u_j\\
&\text{if } \; k=j \Rightarrow -6\varphi_x\sum_{k=0}^{j}\left[\sum_{i=0}^{k}u_{k-i}u_i\right](j-k-1)u_{j-k}\\
&\qquad = 6\varphi_x u_0^2 u_j
\end{aligned}
$$

Putting this in (3.1.13) , we get .

$$u_{j-3,t}+(j-3)\left(u_{j-2}\varphi_t+3u_{j-2,xx}\varphi_x+3u_{j-2,x}\varphi_{xx}+u_{j-2}\varphi_{xxx}\right)$$
$$-6\sum_{k=1}^{j-1}\left[\sum_{i=0}^{k}u_{k-i}u_i\right]u_{j-k}(j-k-1)\varphi_x-6\sum_{k=0}^{j-1}\left[\sum_{i=0}^{k}u_{k-i}u_i\right]u_{j-k-1,x}$$
$$+6\varphi_x u_0\sum_{i=1}^{j-1}u_{j-i}u_i+3(j-2)(j-3)\left(u_{j-1,x}\varphi_x^2+u_{j-1}\varphi_x\varphi_{xx}\right)$$
$$+u_{j-3,xxx}+(j-1)(j-2)(j-3)u_j\varphi_x^3-6(j-3)u_j\varphi_x^3=0,$$

Thus the recursion relation is :

$$(j+1)(j-3)(j-4)\varphi_x^3u_j=-u_{j-3,t}-6\varphi_x u_0\sum_{i=1}^{j-1}u_{j-i}u_i$$
$$-(j-3)\left(u_{j-2}\phi_t+3u_{j-2,xx}\varphi_x+3u_{j-2,x}\varphi_{xx}+u_{j-2}\varphi_{xxx}\right)$$
$$+6\sum_{k=1}^{j-1}\left[\sum_{i=0}^{k}u_{k-i}u_i\right]u_{j-k}(j-k-1)\varphi_x+6\sum_{k=0}^{j-1}\left[\sum_{i=0}^{k}u_{k-i}u_i\right]u_{j-k-1,x}$$
$$-u_{j-3,xxx}-3(j-2)(j-3)\left(u_{j-1,x}\varphi_x^2+u_{j-1}\varphi_x\varphi_{xx}\right)=0,\quad(3.1.14)$$

Clearly , the resonance points are ***j = -1 , 3 , 4*** .correspond to the free singularity manifold function φ(t,x) , and arbitrary function u_3 , u_4 .

For $j=3$ in (3.1.14) ,we have ,

$$0=-u_{0,t}-u_{0,xxx}-6\varphi_x^2\left(u_2u_1+u_1u_2\right)$$
$$+6\varphi_x\sum_{k=1}^{2}\left[u_ku_0+u_{k-1}u_1+u_{k-2}u_2\right](2-k)u_{3-k}$$
$$+6\sum_{k=0}^{2}\left[u_ku_0+u_{k-1}u_1+u_{k-2}u_2\right]u_{2-k,x}$$

Then :

$$0=-u_{0,t}-u_{0,xxx}-6\varphi_x^2\left(u_2u_1+u_1u_2\right)+12\varphi_xu_0u_1u_2$$
$$+6u_0^2u_{2,x}+12\varphi_xu_0u_1u_{1,x}+6u_{0,x}u_1^2+12u_0u_2u_{0,x}$$

Hence .

$$0=-u_{0,t}-u_{0,xxx}-12\varphi_x^2u_1u_2+12\varphi_xu_0u_1u_2$$
$$-6u_0^2u_{2,x}+12\varphi_xu_0u_1u_{1,x}+6u_{0,x}u_1^2+12u_0u_2u_{0,x}$$

But $u_j\equiv 0$ for all $j\geq 2$, then :

$$0=-u_{0,t}-u_{0,xxx}+12u_0u_1u_{1,x}+6u_{0,x}u_1^2, \qquad (3.1.15)$$

Then, by (3.1.7) , (3.1.9)and (3.1.12), we get .

$$0=-\varepsilon\varphi_{xt}-\varepsilon\varphi_{xxxx}-6\left(\varepsilon\varphi_x\right)^2\left(\frac{\varepsilon}{6}\frac{\varphi_{xt}+\varphi_{xxx}}{\varphi_x^2}\right)$$
$$-6\left(\varepsilon\varphi_x\right)^2\left(-\frac{\varepsilon}{6}\frac{2\varphi_{xx}\varphi_t+5\varphi_{xx}\varphi_{xxx}}{\varphi_x^3}+\frac{3\varepsilon}{4}\frac{\varphi_{xx}^3}{\varphi_x^4}\right)$$
$$+12\left(\varepsilon\varphi_x\right)\left(-\frac{\varepsilon}{2}\frac{\varphi_{xx}}{\varphi_x}\right)\left(\frac{\varepsilon}{2}\left[\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2-\frac{\varphi_{xxx}}{\varphi_x}\right]\right)+\frac{3\varepsilon}{2}\left(\frac{\varphi_{xx}^3}{\varphi_x^2}\right),$$

$$0=-\varepsilon\varphi_{xt}-2\varepsilon\varphi_{xxxx}-\varepsilon\varphi_{xt}+2\varepsilon\frac{\varphi_{xx}\varphi_t}{\varphi_x}$$
$$+8\varepsilon\frac{\varphi_{xx}\varphi_{xxx}}{\varphi_x}-6\varepsilon\left(\frac{\varphi_{xx}^3}{\varphi_x^2}\right),$$

$$2\varphi_{tx} + 2\varphi_{xxxx} = 2\varphi_{xx}\left[\frac{\varphi_t}{\varphi_x} + 4\frac{\varphi_{xxx}}{\varphi_x} - 3\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2\right],$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\varphi_t + \varphi_{xxx}}{\varphi_x} - \frac{3}{2}\frac{\varphi_{xx}^2}{\varphi_x^2}\right) = 0 \tag{3.1.16}$$

Inconsistent at the resonance point 3 ,this means that the modified Korteweg-de Vries equation.II , does not have the Painleve' property.

Now , for *j=4* we have from (3.1.14) .

$$0 = -u_{1,t} - u_{1,xxx} - \varphi_t\varphi_2 - 3\varphi_x u_{2,xx} - 3\varphi_{xx}u_{2,x}$$
$$-\varphi_{xxx}u_2 - 6\varphi_x^2 u_{3,x} - 6\varphi_x\varphi_{xx}u_3 - 6\varphi_x^3\sum_{i=1}^{3} u_{4-i}u_i$$
$$+6\varphi_x\sum_{k=1}^{3}\left[\sum_{i=0}^{k} u_{k-i}u_i\right](3-k)u_{4-k} + 6\sum_{k=0}^{3}\left[\sum_{i=0}^{k} u_{k-i}u_i\right]u_{3-k,x},$$

$$0 = -u_{1,t} - u_{1,xxx} - \varphi_t\varphi_2 - 3\varphi_x u_{2,xx} - 3\varphi_{xx}u_{2,x}$$
$$-\varphi_{xxx}u_2 - 6\varphi_x^2 u_{3,x} - 6\varphi_x\varphi_{xx}u_3 - 12\varphi_x^3 u_1u_3 - 6\varphi_x^3 u_2^2$$
$$+6\varphi_x\left[2u_0u_1u_3 + u_1^2u_2 + u_0u_2^2\right] + 6\left[u_0^2u_{3,x} + 2u_0u_1u_{2,x}\right]$$
$$+6\left[2u_0u_2u_{1,x} + u_1^2u_{1,x} + 2u_0u_3u_{0,x} + 2u_1u_2u_{0,x}\right],$$

$$0 = -u_{1,t} - u_{1,xxx} - 3\varphi_x u_{2,xx} - 3\varphi_{xx} u_{2,x} - 6\varphi_x^2 u_{3,x}$$
$$+ 6u_0^2 u_{3,x} + 12 u_0 u_1 u_{2,x} + 6u_1^2 u_{1,x} ,$$

Since $u_j = 0$ for all $j \geq 2$, then :

$$0 = -u_{1,t} - u_{1,xxx} + 6u_1^2 u_{1,x} ,$$
$$\Rightarrow u_{1,t} - 6u_1^2 u_{1,x} + u_{1,xxx} = 0, \qquad (3.1.17)$$

Then u_1 is a solution of the modified Korteweg-de Vries equation.ll .

Section (2.3).
Exact solution :

In this section, we follow the idea to derive analytic solution First, we define ;

$$S(\varphi) = \frac{\varphi_{xxx}}{\varphi_x} - \frac{3}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2 \qquad (3.2.1)$$

$$C(\varphi) = -\frac{\varphi_t}{\varphi_x} \qquad (3.2.2)$$

Where S and C are the Schwartzian derivative and dimension of velocity .
They are invariant under homographic transformation :

$$H : \varphi \mapsto \frac{\alpha\varphi + \beta}{\gamma\varphi + \delta} \qquad (\alpha\delta - \beta\gamma \neq 0)$$

where α, β, γ *and* δ *are constant*

The compatability S and C gives :

$$S_t + C_{xxx} + 2C_x S + CS_x = 0, \qquad (3.2.3)$$

On the other hand , (3.1.12) and by $u_j = 0$ for all $j \geq 2$.then :

$$\frac{\varphi_t + \varphi_{xxx}}{\varphi_x} - \frac{3}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2 = 0, . \qquad (3.2.4)$$

In term of the invariant derivatives C and S, the PDE (3.2.4) can be Summarized as :

$$S = C \qquad (3.2.5)$$

And by the compatibility equation (3.2.3) , therefore the Schwarzian derivative S must satisfy the Korteweg-de Vries like equation .

$$S_t + 3SS_x + S_{xxx} = 0, \qquad (3.2.6)$$

Now to find solution of Korteweg-de Vries-like equation (3.2.6) .
The constant function $S = \pm 2\lambda^2$ with λ constant are necessary solutions .

For $S = \pm 2\lambda^2$

We have : $S\{\varphi : x\} = \pm 2\lambda^2$

case I :

For $S = -2\lambda^2$: we have

$S\{\varphi : x\} = -2\lambda^2$

Hence $P(x) = \lambda^2$ in (III) of Chapter One .and two linearly independent solutions are .

$\Psi_1 = E(t)e^{\lambda x} + F(t)e^{-\lambda x}$, $\Psi_2 = G(t)e^{\lambda x} + H(t)e^{-\lambda x}$

Therefore by Lemma (1.1.15) and Lemma (1.1.16) of Chapter One obtains .

$$\varphi(t,x) = \frac{E(t)e^{\lambda x} + F(t)e^{-\lambda x}}{G(t)e^{\lambda x} + H(t)e^{-\lambda x}}, \qquad EH - FG \neq 0, \qquad (3.2.7)$$

By using (3.2.2) and (3.2.5) , then :

$$C = S = -\frac{\varphi_t}{\varphi_x} = -2\lambda^2 \tag{3.2.8}$$

Now to find the equation of coefficients E(t) . F(t) ,G(t) and H(t) . then :

$$\varphi_t = \frac{\begin{bmatrix}(G(t)E'(t) - E(t)G'(t))e^{2\lambda x} + (H(t)F'(t) - F(t)H'(t))e^{-2\lambda x} \\ + (G(t)F'(t) - F(t)G'(t)) + (H(t)E'(t) - E(t)H'(t))\end{bmatrix}}{(G(t)e^{\lambda x} + H(t)e^{-\lambda x})^2}$$

And,

$$\varphi_x = \begin{bmatrix}\lambda(G(t)e^{\lambda x} + H(t)e^{-\lambda x})(E(t)e^{\lambda x} - F(t)e^{-\lambda x}) \\ -\lambda(E(t)e^{\lambda x} + F(t)e^{-\lambda x})(G(t)e^{\lambda x} - H(t)e^{-\lambda x})\end{bmatrix} \Big/ (G(t)e^{\lambda x} + H(t)e^{-\lambda x})^2 .$$

$$\Rightarrow \varphi_x = \frac{2\lambda(H(t)E(t) - G(t)F(t))}{(G(t)e^{\lambda x} + H(t)e^{-\lambda x})^2}$$

By (3.2.8) , we get .

$$C = \frac{\begin{bmatrix}(G(t)E'(t) - E(t)G'(t))e^{2\lambda x} + (H(t)F'(t) - F(t)H'(t))e^{-2\lambda x} \\ + (G(t)F'(t) - F(t)G'(t)) + (H(t)E'(t) - E(t)H'(t))\end{bmatrix}}{-2\lambda\,(H(t)E(t) - G(t)F(t))} = -2\lambda^2 .$$

Then :

$$\left(G(t)E'(t) - E(t)G'(t)\right)e^{2\lambda t} + \left(H(t)F'(t) - F(t)H'(t)\right)e^{-2\lambda t} + G(t)F'(t)$$
$$- F(t)G'(t) + \left(H(t)E'(t) - E(t)H'(t)\right) = 4\lambda^3\left(H(t)E(t) - G(t)F(t)\right),$$

This leads to a system of nonlinear ordinary differential equation in coefficients E(t) , F(t) , G(t) , and H(t) .

$$GE' - EG' = 0 \qquad \text{———————} \qquad (I)$$
$$HF' - FH' = 0 \qquad \text{———————} \qquad (II)$$
$$(GF' - FG') + (HE' - EH') = 4\lambda^3(HE - GF) \qquad \text{———} \qquad (III)$$

A particular solution of (I) and (II) is :

$$E(t) = BG(t) \qquad \text{and} \qquad F(t) = AH(t)$$

where A , B are real arbitrary constant.

Then by substituting these into (III) , we get .

$$A\left(G(t)H'(t) - H(t)G'(t)\right) + B\left(H(t)G'(t) - G(t)H'(t)\right)$$
$$= 4\lambda^3 H(t)G(t)(B - A),$$
$$-(A - B)\left(G(t)H'(t) - H(t)G'(t)\right) = 4\lambda^3 H(t)G(t)(B - A),$$
$$G(t)H'(t) - H(t)G'(t) = 4\lambda^3 H(t)G(t),$$

$$\frac{G(t)H'(t) - H(t)G'(t)}{H(t)G(t)} = -4\lambda^3,$$
$$\frac{H'(t)}{H(t)} - \frac{G'(t)}{G(t)} = -4\lambda^3,$$

Then (3.2.7) . becomes .

$$\varphi(t,x)=\frac{BG(t)\exp(\lambda x)+AG(t)\exp(-4\lambda^3 t-\lambda x)}{G(t)\exp(\lambda x)+G(t)\exp(-4\lambda^3 t-\lambda x)}$$

And multiplying by $\frac{1}{G(t)}\exp(2\lambda^3 t)$

$$\varphi(t,x)=\frac{B\exp(2\lambda^3 t+\lambda x)+A\exp(-2\lambda^3 t-\lambda x)}{\exp(2\lambda^3 t+\lambda x)+\exp(-2\lambda^3 t-\lambda x)},$$

$$\varphi(t,x)=\frac{Be^{\lambda\xi}+Ae^{-\lambda\xi}}{e^{\lambda\xi}+e^{-\lambda\xi}}, \qquad where\ \xi=x+2\lambda^2$$

$$=\frac{B(\sinh\lambda\xi+\cosh\lambda\xi)+A(\cosh\lambda\xi-\sinh\lambda\xi)}{2\cosh\lambda\xi},$$

$$=\frac{(B+A)\cosh\lambda\xi+(B-A)\sinh\lambda\xi}{2\cosh\lambda\xi},$$

$$\Rightarrow \varphi(t,x)=K_1+K_2\tanh\lambda\xi \tag{3.2.9}$$

$$where\ K_1=(B+A)/2\ ,\ K_2=(B-A)/2$$

where K_1 and K_2 are arbitrary constant ,

For $K_1=0$.

By substituting (3.2.9) into (3.1.9) . then :

$$\overset{(1)}{u_1}=-\frac{\varepsilon}{2}\frac{-2K_2\lambda^2\sec h^2\lambda\xi\tanh\lambda\xi}{K_2\lambda\sec h^2\lambda\xi},$$

$$\Rightarrow \overset{(1)}{u_1}=\varepsilon\lambda\tanh\lambda\xi \qquad where\ \xi=x+2\lambda^2 t$$

And by substituting (3.2.9) into (3.1.10) ,then :

$$\overset{(1)}{u} = \frac{K_2 \lambda \varepsilon \sec h^2 \lambda \xi}{K_2 \tan \lambda \xi} + u_1$$

$$= \frac{\lambda \varepsilon \sec h^2 \lambda \xi}{\tanh \lambda \xi} + \varepsilon \lambda \tanh \lambda \xi,$$

$$= \lambda \varepsilon \left[\frac{\sec h^2 \lambda \xi + \tanh^2 \lambda \xi}{\tanh \lambda \xi} \right],$$

$$\Rightarrow \overset{(1)}{u} = \lambda \varepsilon \coth \lambda \xi \qquad \textit{where } \xi = x + 2\lambda^2 t$$

Hence $\overset{(1)}{u}(t,x)$ and $\overset{(1)}{u_1}(t,x)$ in the above are exact solutions for Modified Korteweg-de Vries (or KDV-II) equation .

case II :

for $S = 2\lambda^2$: we have :

$S = \{\varphi, x\} = 2\lambda^2$,

Hence $P(x) = -\lambda^2$ in (III) of Chapter One .and two linearly independent solutions are .

$$V_1 = E(t)e^{\lambda x} + F(t)e^{-\lambda x} \qquad , \qquad V_2 = G(t)e^{\lambda x} + H(t)e^{-\lambda x}$$

Therefore by Lemma (1.1.15) and Lemma (1.1.16) of Chapter One obtains ,

$$\varphi(t,x) = \frac{E(t)e^{\lambda x} + F(t)e^{-\lambda x}}{G(t)e^{\lambda x} + H(t)e^{-\lambda x}} , \qquad EH - FG \neq 0, \qquad (3.2.10)$$

By using (3.2.2) and (3.2.5) , then :

$$C = S = -\frac{\varphi_t}{\varphi_x} = 2\lambda^2 \tag{3.2.11}$$

where :

$$\varphi_t = \frac{\left[\begin{array}{l}\left(G(t)E'(t)-E(t)G'(t)\right)e^{2\lambda x}+\left(H(t)F'(t)-F(t)H'(t)\right)e^{-2\lambda x}\\ +\left(G(t)F'(t)-F(t)G'(t)\right)+\left(H(t)E'(t)-E(t)H'(t)\right)\end{array}\right]}{\left(G(t)e^{\lambda x}+H(t)e^{-\lambda x}\right)^2},$$

and.

$$\varphi_x = \frac{2i\lambda\left(H(t)E(t)-G(t)F(t)\right)}{\left(G(t)e^{\lambda x}+H(t)e^{-\lambda x}\right)^2}$$

then :

$$\frac{\varphi_t}{\varphi_x} = \frac{\left[\begin{array}{l}\left(G(t)E'(t)-E(t)G'(t)\right)e^{2\lambda x}+\left(H(t)F'(t)-F(t)H'(t)\right)e^{-2\lambda x}\\ +\left(G(t)F'(t)-F(t)G'(t)\right)+\left(H(t)E'(t)-E(t)H'(t)\right)\end{array}\right]}{-4i\lambda^3\left(H(t)E(t)-G(t)F(t)\right)},$$

Then the system of nonlinear ordinary differential equation with coefficients E(t) , F(t) , G(t) , and H(t) , becomes .

$$GE' - EG' = 0 \qquad\text{———————————}\qquad (I)$$

$$HF' - FH' = 0 \qquad\text{———————————}\qquad (II)$$

$$\left(GF' - FG'\right) + \left(HE' - EH'\right) = -4i\lambda^3\left(HE - GF\right) \qquad\text{————}\qquad (III)$$

A particular solution of (I) and (II) is :

$E(t) = MG(t)$ *and* $F(t) = NH(t)$

where M , N are real arbitrary constant .

Substituting these into (III) , becomes .

$$\frac{H(t)}{G(t)} = Exp(4i\lambda^3 t),$$

Then (3.2.10) , becomes .

$$\varphi(t,x) = \frac{MG(t)\exp(\lambda ix) + NG(t)\exp(4\lambda^3 it - \lambda ix)}{G(t)\exp(\lambda ix) + G(t)\exp(4\lambda^3 it - \lambda ix)},$$

and multiplying by $\frac{1}{G(t)}\exp(-2\lambda^3 it)$

$$\varphi(t,x) = \frac{M\exp(-2\lambda^3 it + \lambda ix) + N\exp(2\lambda^3 it - \lambda ix)}{\exp(-2\lambda^3 it + \lambda ix) + \exp(2\lambda^3 it - \lambda ix)},$$

$$= \frac{Me^{\lambda i\xi} + Ne^{-\lambda i\xi}}{e^{\lambda i\xi} + e^{-\lambda i\xi}}, \qquad where\ \xi = x - 2\lambda^2 t$$

Then :

$$\varphi(t,x) = \frac{M(\sin\lambda\xi + \cos\lambda\xi) + N(\cos\lambda\xi - \sin\lambda\xi)}{2\cos\lambda\xi},$$

$$= \frac{(M+N)\cos\lambda\xi + (M-N)\sin\lambda\xi}{2\cos\lambda\xi}$$

$$\Rightarrow \varphi(t,x) = K_3 + K_4\tan\lambda\xi \qquad (3.2.12)$$

$$where\ K_3 = (M+N)/2\ ,\ K_4 = (M-N)/2,$$

where K_3 and K_4 are arbitrary constant , For $K_3 = 0$, and by substituting (3.2.12) in to (3.1.9),then .

$$\overset{(2)}{u_1} = -\frac{\varepsilon}{2}\frac{2K_4\lambda^2 \sec^2 \lambda\xi \tan \lambda\xi}{K_4 \lambda \sec^2 \lambda\xi},$$

$$\Rightarrow \overset{(2)}{u_1} = -\lambda\varepsilon \tan \lambda\xi \qquad \textit{where } \xi = x - 2\lambda^2 t$$

And by substituting (3.2.12) into (3.1.10) , then :

$$\overset{(2)}{u} = \frac{K_4 \lambda\varepsilon \sec^2 \lambda\xi}{K_4 \tan \lambda\xi} + u_1$$

$$= \frac{\lambda\varepsilon \sec^2 \lambda\xi}{\tan \lambda\xi} - \lambda \tan \lambda\xi_1$$

$$= \lambda\varepsilon\left[\frac{\sec^2 \lambda\xi - \tan^2 \lambda\xi}{\tan \lambda\xi}\right].$$

$$\Rightarrow \overset{(2)}{u} = \lambda\varepsilon \cot \lambda\xi \qquad \textit{where } \xi = x - 2\lambda^2 t$$

Hence $\overset{(2)}{u}(t,x)$ and $\overset{(2)}{u_2}(t,x)$ in the above are exact solutions for Modified Korteweg-de Vries (or KDV-II) equation .

CHAPTER FOUR
Complex modified Korteweg-de Vries (or CMKDVII) equation. .II

CHAPTER FOUR

In this chapter we study the nonlinear reaction-diffusion the complex modified system Korteweg-de Vries-II equation (or CMKDV-II) .

The Complex modified Korteweg-de Vries (or CMKDV-II) Equation-II .

Section 4.1
Painlevé property .

$$w_t - 6|w|^2 w_x + w_{xxx} = 0, \qquad (4.1.1)$$

Now , we are going to illustrate the nature of the Painleve' test on the complex modified Korteweg-de Vries equation-II (4.1.1) . Since the absolute value $|\cdot|$ in (4.1.1) brings some difficulty in the calculations, let $w = u + iv$ and separate the real and imaginary parts in (4.1.1), and obtain the system ;

$$\left.\begin{aligned} u_t - 6(u^2 + v^2)u_x + u_{xxx} &= 0, \leftarrow (I) \\ v_t - 6(u^2 + v^2)v_x + v_{xxx} &= 0, \leftarrow (II) \end{aligned}\right\} \qquad (4.1.2)$$

Let $u = \dfrac{1}{\varphi^p}\sum_{j=0}^{\infty} u_j \varphi^j \quad , \quad v = \dfrac{1}{\varphi^p}\sum_{j=0}^{\infty} v_j \varphi^j$

be the two series solutions of (4.1.1) .where φ & u_j and v_j are analytic functions in a neighborhood of the manifold at $\varphi=0$,
First to find value of p , we need to find u_t, v_t, $(u^2+v^2)u_x$, $(u^2+v^2)v_x$, u_{xxx} and v_{xxx} .

Then :

$$u^2 = \sum_{j=0}^{\infty}\sum_{k=0}^{j} u_{j-k}u_k\varphi^{j-2p}$$

$$u_t = \sum_{j=0}^{\infty}[u_{j,t}\varphi^{j-p} + (j-p)u_j\varphi_t\varphi^{j-p-1}],$$

$$u_x = \sum_{j=0}^{\infty}[u_{j,x}\varphi^{j-p} + (j-p)u_j\varphi_x\varphi^{j-p-1}],$$

$$\begin{aligned} u_{xxx} = \sum_{j=0}^{\infty}[&u_{j,xxx}\varphi^{j-p} + 3(j-p)u_{j,xx}\varphi_x\varphi^{j-p-1} + 3(j-p)u_{j,x}\varphi_{xx}\varphi^{j-p-1} \\ &+ 3(j-p)(j-p-1)u_{j,x}\varphi_x^2\varphi^{j-p-2} + (j-p)u_j\varphi_{xxx}\varphi^{j-p-1} \\ &+ 3(j-p)(j-p-1)u_j\varphi_x\varphi_{xx}\varphi^{j-p-2} \\ &+ (j-p)(j-p-1)(j-p-2)u_j\varphi_x^3\varphi^{j-p-3}], \end{aligned}$$

and

$$v^2 = \sum_{j=0}^{\infty}\sum_{k=0}^{j} v_{j-k}v_k\varphi^{j-2p}$$

$$v_t = \sum_{j=0}^{\infty}[v_{j,t}\varphi^{j-p} + (j-p)v_j\varphi_t\varphi^{j-p-1}],$$

$$v_x = \sum_{j=0}^{\infty}[v_{j,x}\varphi^{j-p} + (j-p)v_j\varphi_x\varphi^{j-p-1}],$$

$$\begin{aligned} v_{xxx} = \sum_{j=0}^{\infty}[&v_{j,xxx}\varphi^{j-p} + 3(j-p)v_{j,xx}\varphi_x\varphi^{j-p-1} + 3(j-p)v_{j,x}\varphi_{xx}\varphi^{j-p-1} \\ &+ 3(j-p)(j-p-1)v_{j,x}\varphi_x^2\varphi^{j-p-2} + (j-p)v_j\varphi_{xxx}\varphi^{j-p-1} \\ &+ 3(j-p)(j-p-1)u_j\varphi_x\varphi_{xx}\varphi^{j-p-2} \\ &+ (j-p)(j-p-1)(j-p-2)v_j\varphi_x^3\varphi^{j-p-3}], \end{aligned}$$

Hence ,

$$-6(u^2+v^2)u_x = -6\sum_{j=0}^{\infty}\left[\sum_{k=0}^{j-1}\sum_{i=0}^{k}(u_iu_{k-i}+v_iv_{k-i})u_{j-k-1,x}\right.$$
$$\left.+\sum_{k=0}^{j}\sum_{i=0}^{k}(u_iu_{k-i}+v_iv_{k-i})u_{j-k}(j-k-1)\varphi_x\right]\varphi^{j-3p-1}$$

and

$$-6(u^2+v^2)v_x = -6\sum_{j=0}^{\infty}\left[\sum_{k=0}^{j-1}\sum_{i=0}^{k}(u_iu_{k-i}+v_iv_{k-i})v_{j-k-1,x}\right.$$
$$\left.+\sum_{k=0}^{j}\sum_{i=0}^{k}(u_iu_{k-i}+v_iv_{k-i})v_{j-k}(j-k-1)\varphi_x\right]\varphi^{j-3p-1}$$

Then (I) in (4.1.2), becomes :

$$\sum_{j=0}^{\infty}[u_{j,t}\varphi^{j-p}+(j-p)u_j\varphi_t\varphi^{j-p-1}]$$
$$-6\sum_{j=0}^{\infty}\left[\sum_{k=0}^{j-1}\sum_{i=0}^{k}(u_iu_{k-i}+v_iv_{k-i})u_{j-k-1,x}\right.$$
$$\left.+\sum_{k=0}^{j}\sum_{i=0}^{k}(u_iu_{k-i}+v_iv_{k-i})u_{j-k}(j-k-1)\varphi_x\right]\varphi^{j-3p-1}$$
$$+\sum_{j=0}^{\infty}[u_{j,xxx}\varphi^{j-p}+3(j-p)u_{j,xx}\varphi_x\varphi^{j-p-1}+3(j-p)u_{j,x}\varphi_{xx}\varphi^{j-p-1}$$
$$+3(j-p)(j-p-1)u_{j,x}\varphi_x^2\varphi^{j-p-2}+(j-p)u_j\varphi_{xxx}\varphi^{j-p-1}$$
$$+3(j-p)(j-p-1)u_j\varphi_x\varphi_{xx}\varphi^{j-p-2}$$
$$+(j-p)(j-p-1)(j-p-2)u_j\varphi_x^3\varphi^{j-p-3}]=0, \qquad (4.1.3)$$

And (II) in (4.1.2), becomes :

$$\sum_{j=0}^{\infty}[v_{j,t}\varphi^{j-p}+(j-p)v_j\varphi_t\varphi^{j-p-1}]$$
$$-6\sum_{j=0}^{\infty}\left[\sum_{k=0}^{j-1}\sum_{i=0}^{k}(u_iu_{k-i}+v_iv_{k-i})v_{j-k-1,x}+\sum_{k=0}^{j}\sum_{i=0}^{k}(u_iu_{k-i}+v_iv_{k-i})v_{j-k}(j-k-1)\varphi_x\right]\varphi^{j-3p-1}$$
$$+\sum_{j=0}^{\infty}[v_{j,xxx}\varphi^{j-p}+3(j-p)v_{j,xx}\varphi_x\varphi^{j-p-1}+3(j-p)v_{j,x}\varphi_{xx}\varphi^{j-p-1}$$
$$+3(j-p)(j-p-1)v_{j,x}\varphi_x^2\varphi^{j-p-2}+(j-p)v_j\varphi_{xxx}\varphi^{j-p-1}$$
$$+3(j-p)(j-p-1)v_j\varphi_x\varphi_{xx}\varphi^{j-p-2}$$
$$+(j-p)(j-p-1)(j-p-2)v_j\varphi_x^3\varphi^{j-p-3}]=0, \quad (4.1.4)$$

By used the compeer of the low power in (4.1.3) or (4.1.4).
Then , clearly ***p=1***.
Now by associated the summation , and substituting ***p=1*** into (4.1.3) and (4.1.4) . we get .

$$\sum_{j=3}^{\infty}u_{j-3,t}\varphi^{j-4}+\sum_{j=2}^{\infty}(j-3)u_{j-2}\varphi_t\varphi^{j-4}$$
$$-6\sum_{j=0}^{\infty}\left[\sum_{k=0}^{j-1}\sum_{i=0}^{k}(u_iu_{k-i}+v_iv_{k-i})u_{j-k-1,x}+\sum_{k=0}^{j}\sum_{i=0}^{k}(u_iu_{k-i}+v_iv_{k-i})u_{j-k}(j-k-1)\varphi_x\right]\varphi^{j-4}$$

$$+\sum_{j=3}^{\infty}u_{j-3,xxx}\varphi^{j-4}+\sum_{j=2}^{\infty}3(j-3)u_{j-2,xx}\varphi_x\varphi^{j-4}+\sum_{j=2}^{\infty}3(j-3)u_{j-2,x}\varphi_{xx}\varphi^{j-4}$$

$$+\sum_{j=1}^{\infty}3(j-2)(j-3)u_{j-1,x}\varphi_x^2\varphi^{j-4}+\sum_{j=2}^{\infty}(j-3)u_{j-2}\varphi_{xxx}\varphi^{j-4}$$

$$+\sum_{j=1}^{\infty}3(j-2)(j-3)u_{j-1}\varphi_x\varphi_{xx}\varphi^{j-4}$$

$$+\sum_{j=0}^{\infty}(j-1)(j-2)(j-3)u_j\varphi_x^3\varphi^{j-4}=0, \quad (4.1.5)$$

And (4.1.4) ,becomes :

$$\sum_{j=3}^{\infty}v_{j-3,t}\varphi^{j-4}+\sum_{j=2}^{\infty}(j-3)v_{j-2}\varphi_t\varphi^{j-4}$$

$$-6\sum_{j=0}^{\infty}\left[\sum_{k=0}^{j-1}\sum_{i=0}^{k}\left(u_iu_{k-i}+v_iv_{k-i}\right)v_{j-k-1,x}\varphi_x\right.$$

$$\left.+\sum_{k=0}^{j}\sum_{i=0}^{k}\left(u_iu_{k-i}+v_iv_{k-i}\right)v_{j-k}(j-k-1)\varphi_x\right]\varphi^{j-4}$$

$$+\sum_{j=3}^{\infty}v_{j-3,xxx}\varphi^{j-4}+\sum_{j=2}^{\infty}3(j-3)v_{j-2,xx}\varphi_x\varphi^{j-4}+\sum_{j=2}^{\infty}3(j-3)v_{j-2,x}\varphi_{xx}\varphi^{j-4}$$

$$+\sum_{j=1}^{\infty}3(j-2)(j-3)v_{j-1,x}\varphi_x^2\varphi^{j-4}+\sum_{j=2}^{\infty}(j-3)v_{j-2}\varphi_{xxx}\varphi^{j-4}$$

$$+\sum_{j=1}^{\infty}3(j-2)(j-3)v_{j-1}\varphi_x\varphi_{xx}\varphi^{j-4}$$

$$+\sum_{j=0}^{\infty}(j-1)(j-2)(j-3)v_j\varphi_x^3\varphi^{j-4}=0, \quad (4.1.6)$$

For ***j=0*** **:**

Then (4.1.5) becomes ,

$$\left(u_0^2 + v_0^2 - \varphi_x^2\right) = 0,$$

and (4.1.6) becomes ,

$$\left(u_0^2 + v_0^2 - \varphi_x^2\right) = 0,$$

They are linearly dependent and yield only the equation .

For ***j=1*** :

The two equations (4.1.5) and (4.1.6) are the same ,
and we have a relation :

$$u_0 u_1 + v_0 v_1 = -\frac{1}{2}\varphi_{xx} \qquad (4.1.8)$$

For ***j=2*** :

Then (4.1.5) becomes ,

$$-u_0\varphi_t - 3u_{0,xxx}\varphi_x - 6\sum_{k=0}^{1}\sum_{i=0}^{k}\left(u_i u_{k-i} + v_i v_{k-i}\right)u_{1-k,x}$$

$$-6\sum_{k=0}^{2}\sum_{i=0}^{k}\left(u_i u_{k-i} + v_i v_{k-i}\right)u_{2-k}\left(1-k\right)\varphi_x - 3u_{0,x}\varphi_{xx} - u_0\varphi_{xxx} = 0,$$

$$6\varphi_x\left[\left(u_1^2+v_1^2\right)u_0+\left(u_0^2-v_0^2\right)u_2+2u_0v_0v_2\right]-6\left(u_0^2+v_0^2\right)u_{1,x}$$
$$-3u_{0,x}\left(4u_0u_1+4v_0v_1+\varphi_{xx}\right)-3\varphi_x u_{0,xx}-u_0\left(\varphi_t+\varphi_{xxx}\right)=0,$$

and (4.1.6), becomes :

$$-v_0\varphi_t-3v_{0,xxx}\varphi_x-6\sum_{k=0}^{1}\sum_{i=0}^{k}\left(u_iu_{k-i}+v_iv_{k-i}\right)v_{1-k,x} \tag{4.1.9}$$
$$-6\sum_{k=0}^{2}\sum_{i=0}^{k}\left(u_iu_{k-i}+v_iv_{k-i}\right)v_{2-k}\left(1-k\right)\varphi_x-3v_{0,x}\varphi_{xx}-v_0\varphi_{xxx}=0,$$

$$6\varphi_x\left[\left(u_1^2+v_1^2\right)v_0+\left(u_0^2-v_0^2\right)v_2+2u_0u_2v_0\right]-6\left(u_0^2+v_0^2\right)v_{1,x}$$
$$-3v_{0,x}\left(4u_0u_1+4v_0v_1+\varphi_{xx}\right)-3\varphi_x v_{0,xx}-v_0\left(\varphi_t+\varphi_{xxx}\right)=0, \tag{4.1.10}$$

Now, to find all coefficient of u_j and v_j for (4.1.5) and (4.1.6) .

For $j \geq 3$:

$$\left[(j-1)(j-2)(j-3)\varphi_x^2-6(j-1)\left(u_0^2+v_0^2\right)+12u_0^2\right]\varphi_x u_j$$
$$+12u_0v_0\varphi_x v_j=\Phi_1(u_0,u_1,\ldots,u_{j-1},\varphi):\ j=0,1,2,\ldots \tag{4.1.11}$$

and

$$+12u_0v_0\varphi_x u_j$$
$$+\left[(j-1)(j-2)(j-3)\varphi_x^2-6(j-1)\left(u_0^2+v_0^2\right)+12v_0^2\right]\varphi_x v_j$$
$$=\Phi_2(v_0,v_1,\ldots,v_{j-1},\varphi):\ j=0,1,2,\ldots \tag{4.1.12}$$

where

$$\Phi_1 = -(j-3)\left(\varphi_t u_{j-2} + 3\varphi_x u_{j-2,xx} + 3\varphi_{xx} u_{j-2,x} + \varphi_{xxx} u_{j-2}\right)$$
$$- u_{j-3,t} - u_{j-3,xxx} - 3(j-2)(j-3)\left(\varphi_x^2 u_{j-1,x} + \varphi_x \varphi_{xx} u_{j-1}\right)$$
$$+ 6\varphi_x u_0 \sum_{i=1}^{j-1}\left(u_i u_{j-i} + v_i v_{j-i}\right) - 6\sum_{k=0}^{j-1}\left[\sum_{i=0}^{k}\left(u_i u_{k-i} + v_i v_{k-i}\right)\right] u_{j-k-1,x}$$
$$- 6\varphi_x \sum_{k=1}^{j-1}\left[\sum_{i=0}^{k}\left(u_i u_{k-i} + v_i v_{k-i}\right)\right](j-k-1) u_{j-k}, \qquad (4.1.13)$$

and,

$$\Phi_2 = -(j-3)\left(\varphi_t v_{j-2} + 3\varphi_x v_{j-2,xx} + 3\varphi_{xx} v_{j-2,x} + \varphi_{xxx} v_{j-2}\right)$$
$$- v_{j-3,t} - v_{j-3,xxx} - 3(j-2)(j-3)\left(\varphi_x^2 v_{j-1,x} + \varphi_x \varphi_{xx} v_{j-1}\right)$$
$$+ 6\varphi_x v_0 \sum_{i=1}^{j-1}\left(u_i u_{j-i} + v_i v_{j-i}\right) - 6\sum_{k=0}^{j-1}\left[\sum_{i=0}^{k}\left(u_i u_{k-i} + v_i v_{k-i}\right)\right] v_{j-k-1,x}$$
$$- 6\varphi_x \sum_{k=1}^{j-1}\left[\sum_{i=0}^{k}\left(u_i u_{k-i} + v_i v_{k-i}\right)\right](j-k-1) v_{j-k}, \qquad (4.1.14)$$

Resonances .

The determinant of the coefficient matrix of the unknowns u_j and v_j in the system of linear algebraic equation (4.1.11) and (4.1.12),is .

$$\begin{vmatrix} \varphi_x\begin{bmatrix}(j-1)(j-2)(j-3) \\ -6(j-1)(u_0^2+v_0^2)+12u_0^2\end{bmatrix}u_j & 12u_0 v_0 \varphi_x v_j \\ 12u_0 v_0 \varphi_x u_j & \varphi_x\begin{bmatrix}(j-1)(j-2)(j-3) \\ -6(j-1)(u_0^2+v_0^2)+12v_0^2\end{bmatrix}v_j \end{vmatrix} = 0.$$

and by relation (4.1.7) ,we get .

$$\begin{vmatrix} \varphi_x\left[(j-1)(j^2-5j)\varphi_x^2+12u_0^2\right]u_j & 12u_0v_0\varphi_x v_j \\ 12u_0v_0\varphi_x u_j & \varphi_x\left[(j-1)(j^2-5j)\varphi_x^2+12u_0^2\right]v_j \end{vmatrix} = 0.$$

Then ,

$$\varphi_x^2\left[(j-1)^2\left(j^2-5j\right)\varphi_x^4+24(j-1)\left(j^2-5j\right)\left(u_0^2+v_0^2\right)\varphi_x^2+144u_0^2v_0^2\right]u_jv_j$$
$$-144u_0^2v_0^2u_jv_j=0,$$
$$\varphi_x^2\left[(j-1)^2\left(j^2-5j\right)\varphi_x^4+12(j-1)\left(j^2-5j\right)\varphi_x^4\right]u_jv_j=0,$$

$$\Rightarrow\left(j^6-12j^5+46j^4-48j^3-47j^2+60j\right)\varphi_x^6u_jv_j=0,$$

Then the resonances of the system are **-1 , 0 , 1 , 3 , 4** and **5** .

Section 4.2

analytic solution :

Let us truncate the series solution at the second term ,and assume that $u_j=0$ for all $j\geq 2$, then we are going to find a truncated series solution to the system (4.1.2) , of the form .

$$u=\frac{u_0}{\varphi}+u_1 \quad , \quad v=\frac{v_0}{\varphi}+v_1 \qquad (4.2.1)$$

Now :

Let $u_2 = v_2 = 0$ in the system algebraic equation (4.1.9) ,and (4.1.10) , we get .

$$C_1 = 6\varphi_x u_0\left(u_1^2 + v_1^2\right) - 6\varphi_x^2 u_{1,x} + 3\varphi_{xx}u_{0,x} - 3\varphi_x u_{0,xx} - \varphi_{xxx}u_0 - \varphi_t u_0 = 0.$$

and (4.2.2)

$$C_2 = 6\varphi_x v_0\left(u_1^2 + v_1^2\right) - 6\varphi_x^2 v_{1,x} + 3\varphi_{xx}v_{0,x} - 3\varphi_x v_{0,xx} - \varphi_{xxx}v_0 - \varphi_t v_0 = 0.$$

For $j=3$:
in the relations (4.1.13) and (4.1.14) .

and let $u_2 = v_2 = u_3 = v_3 = 0$, then (4.1.13) and (4.1.14) successively becomes :

$$C_3 = -6\varphi_x u_0 u_{0,x}\left(u_1^2 + v_1^2\right) + 6\varphi_x\varphi_{xx}u_0 u_{1,x} + \varphi_x u_0 u_{0,xxx} + \varphi_x u_0 u_{0,t} = 0.$$

and (4.2.3)

$$C_4 = -6\varphi_x v_0 v_{0,x}\left(u_1^2 + v_1^2\right) + 6\varphi_x\varphi_{xx}v_0 v_{1,x} + \varphi_x v_0 v_{0,xxx} + \varphi_x v_0 v_{0,t} = 0.$$

For $j=4$:
in the relation (4.1.13) and (4.1.14) .
and let $u_2 = v_2 = u_3 = v_3 = u_4 = v_4 = 0$, then (4.1.13) and (4.1.14) successively ,becomes :

$$C_5 = u_{1,t} - 6(u_1^2 + v_1^2)u_{1,x} + u_{1,xxx} = 0,$$

and (4.2.4)

$$C_6 = v_{1,t} - 6(u_1^2 + v_1^2)v_{1,x} + v_{1,xxx} = 0,$$

Then $u_1 + iv_1$ and $u + iv$ are the two solutions of the original equation of (4.1.1) .

and the special solution in (4.2.3) , if $u_1 = v_1 = 0$ the condition $C_3 = C_4 = 0$,

$$u_{0,xxx} + u_{0,t} = 0,$$
$$v_{0,xxx} + v_{0,t} = 0, \tag{4.2.5}$$

The functions ,

$$u_0(t,x) = A\cos(\alpha x + \alpha^3 t),$$
and ,
$$v_0(t,x) = A\sin(\alpha x + \alpha^3 t)$$

where A is a constant , satisfy the system (4.2.5) .
Hence in this case and by using (4.1.7) , we obtain .

$$u_0(t,x)^2 + v_0(t,x)^2 = \varphi_x^2 = A^2$$
then ,
$$\varphi(t,x) = Ax + B(t),$$
where $B(t)$ *is an arbitrary function* .

With this $\varphi(t,x)$ the truncated solution (4.2.1) , becomes .

$$u(t,x) = \frac{A}{Ax + B(t)}\cos(\alpha x + \alpha^3 t),$$
and ,
$$v(t,x) = \frac{A}{Ax + B(t)}\sin(\alpha x + \alpha^3 t),$$

Two functions in the above equations satisfy the original system (4.1.2).

then

$$w(t,x) = \frac{A[\cos(\alpha x + \alpha^3 t) + i\sin(\alpha x + \alpha^3 t)]}{Ax + B(t)},$$

where α *is arbitrary consatnt .*

Is a special solution of the original equation (4.1.1) .

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تحليل بـــانليفا لبعض معادلات الانتشار

مقدمة:

بسم الله والحمد لله, والصلاة والسلام على رسول الله خاتم الأنبياء والمرسلين .

الكثير من الظواهر تنشأ في العلوم التطبيقية وفي مجالات أخرى يمكن وصفها أو نمذجتها بمعادلات الانتشار الارتدادية الغير خطية . والتي من العادة تنشأ عن ظواهر طبيعية تظهر في حياتنا اليومية , مثل انسياب المياه وجريانها أو تسربها تحت جسر إذا كانت الكثافة عالية , وكذلك تنشأ عن زيادة ضربات القلب وجريان الدم في الشرايين والأوردة , وعدة ظواهر أخرى منها فيزيائية وهندسية ورياضية وكيميائية . وفي هذا البحث نحاول التوصل إلى حل لهذا النوع من المعادلات والتي في اغلب الأحيان يكون من الصعب إيجاد الحل لها في صورة دالة صريحة , ولكن باستخدام تحليل بانليفا وبواسطة القطع التكنيكي للمتسلسلة عند عدد "p" الذي هو نقطة توازن المتسلسلة ,يمكن إيجاد حل تحليلي قد يفيد المهندسين والفيزيائيين والأطباء وغيرهم في تفسير نتائج هذا الحل وللتوصل إلى فهم صريح قد يصعب على الرياضيين تفسيره .

هذا البحث هو دراسة موضوعيه وتطبيقيه لـ **Painleve' analysis** في إيجاد الحلول التحليلية لبعض من معادلات الانتشار .

في الفصل الأول : قمنا بعرض بعض من التعاريف والمفاهيم الأساسية وبعض النظريات والمبرهنات الضرورية في عملنا , مدعمة بعدة أمثلة .

في الفصل الثاني : قمنا بتحقيق خصائص **Painleve'** على معادلة تفاضلية جزئية وهى معادلة **Korteweg-de Vries equation.I** ومن خلال دراستنا وجدنا إنها تحقق خصائص **Painleve'** وباستخدام اسلوب القطع التكنيكي تحققنا من وجود الحل .

أما في الفصل الثالث وجدنا انه ورغم عدم تحقيق خصائص Painleve' للمعادلة التفاضلية الجزئية Korteweg-de Vries equation.II إلا أننا استطعنا إيجاد حل تحليلي لها .

في الفصل الرابع قمنا بدراسة خاصية Painleve' في شكل نظام (system) من المعادلات التفاضلية الجزئية , وتوصلنا فيه إلى بعض النتائج , وهذا الموضوع هو موضوع ختامي , (وهو موضوع مهم كخطة دراسة مستقبلية)

(والله الموفق)

كليــــة العلــــوم

قســـــم الرياضيات

عنـــوان البحـــث

((تحليل بانـليفـا لبعض معادلات الانـتشار))

مقدمـــة مـــن الطـــالب

عطية عبدالباري حسين

* * لجنـــة المناقشـــة :

الدكتور / أبوالقاسم علي الذرعاني ....
(مشـــرف الرســـالـة)

الدكتور / نبيل زكي فريد ....
(ممتحـــن داخلـــي)

الدكتور / مصطفى الشريف بدرالدين ....
(ممتحـــن خارجـي)

يعتمـــد :-
د. محمد علي سالم الفرجاني
أمين اللجنة الشعبية لكلية العلوم

جامعة التحدي – سرت

كلية العلوم – قسم الرياضيات

بحث بعنوان:

تحليل بانليفا لبعض معادلات الانتشار

استكمالاً لمتطلبات الإجازة العالية الماجستير في علوم الرياضيات

مقدمة من الطالب:

عطية عبدالباري حسين

إشراف الأستاذ:

د. أبوالقاسم علي الذرعاني

العام الجامعي 2007ف