University of AL-Tahdi

Faculty of Science

Department of Mathematics

Stability Behaviour of Some Linear and non-Linear Autonomous Differential Systems

A dissertation submitted to the department of mathematics in partial fulfillment of the requirements for the degree of master of science in mathematics

By

FARAJ MOHAMMAD ABDULA

Supervisor

Dr. ABULGASSIM ALI MOHAMMAD

Sirte – Libya

2005

Faculty of Science

Department of Mathematics

Title of Thesis

((STABILITY BEHAVIOUR OF SOME LINEAR AND NON-LINEAR AUTONOMOUS DIFFERENTIAL SYSTEMS))

By

FARAJ MOHAMMAD ABDULA

Approved by

Dr. ABULGASSIM ALI MOHAMMAD ....

(Supervisor)

Dr. FathaAlla Saleh Al-Mesteeri ....

(External examiner)

Dr .NABIL ZAKI FARID ....

(Internal examiner)

Countersigned by:

Dr . Mohamed Ali Salem

(Dean of Faculty of Science)

....

Dedication

To my family.

Faraj

Acknowledgment

الحمد لله

I would like to express my gratitude and appreciation for all who helped me to complete this thesis.

The guidance, encouragement and inspiration of Dr. Abulgassim Ali Mohammad, my major a supervisor, are kindly acknowledged and deeply appreciated special thanks to the department of mathematics.

PREFACE

This thesis studies some aspects of the stability behavior of linear and non-linear autonomous differential equations which have many applications in various fields. The concept was first introduced in the master degree thesis made by the Russian scientist Liapunov titled "The General Case of Movement Stability" published in 1907.

After that came many studies, researches, and efforts which dealt with a certain problem each time.

The main point of this study is to simplify some concepts and present them in a easily comprehensible manner so as to enable researchers to understand the core basics of such concepts. We have chosen certain subjects from a resource or more which have been inflexible and we presented them in a more appropriate manner.

We have treated these concepts with an analytical view. Avoiding completely the geometrical view of these concepts.

Chapter one was reserved to introduce some theorems and definitions which we will need in the following chapters, and in this chapter we introduce the proof of the existence and uniqueness theorem of system of ordinary differential equations.

Chapter two deals with some concepts of stability and asymptotically stable and the main focus in this chapter was on studying the relation between the critical solution and the other solutions for some differential systems.

Continuing in the analytical view of stability concepts we have introduced in chapter three the direct method of the scientist Liapunov presenting the proof for some of the theorems which deal with this concepts.

At the end of the chapter three we have presented some methods through which we can derive the Liapunov functions for some differential systems.

Contents

Chapter One

Introduction

In this chapter, we give some standard definitions and theorems which we shall need later in this thesis.

Definition (1.1):

Let R be the set of all real numbers, and I be an interval on the real line R. Let F be defined on I, and $x^{(n)}$ is the *n-th* derivative of the unknown function x with respect to t, then any *ordinary differential equation* of *n-th* order can be written as

$$F\,(t, x, x^{\cdot}, x^{\cdot\cdot}, .., x^{(n)}) = 0, \quad \text{where } (\,^{\cdot}\,) = \frac{d}{dt},\ x^{(n)} = \frac{d^n x}{dt^n}. \quad (1.1)$$

Definition (1.2):

We can write (1.1) as

$$x^{(n)} = g\,(t, x, x^{\cdot}, x^{\cdot\cdot}, .., x^{(n-1)}), \qquad (1.2)$$

where g is a function defined from F on I.

The differential equation (1.2) is said to be linear if g is *linear* in $x, x^{\cdot}, ..., x^{(n-1)}$, other wise it is *non linear.*

Definition (1.3):

Let $x = \phi(t)$ be defined and n-times differentiable on I, $x = \phi(t)$ is called *a solution* of differential equation (1.2), such that.

$$\phi^{(n)} = g(t, \phi(t), \phi^{\cdot}(t), \dots, \phi^{(n-1)}(t)), \qquad (1.3)$$

For all $t \in I$

Definition (1.4):

Let

$$x_1^{\cdot} = f_1\,(t, x_1, x_2, \dots, x_n),$$
$$x_2^{\cdot} = f_2\,(t, x_1, x_2, \dots, x_n), \qquad (1.4)$$
$$\vdots$$
$$x_n^{\cdot} = f_n\,(t, x_1, x_2, \dots, x_n),$$

where $f_1, f_2, \dots, f_n$ are n given functions of a space of $(n+1)$ dimensions, and $x_1, x_2, \dots, x_n$ are n unknown vectors, the system (1.4) is called *a system of first order differential equations.*

Definition (1.5):

Let $\{\phi_1(t), \phi_2(t), \dots, \phi_n(t)\}$ be a set of n functions defined on I,

then $\{\phi_1(t), \phi_2(t), \dots, \phi_n(t)\}$ is to be a solution of the system (1.4) on I,

if $\phi_1^{\cdot}(t), \phi_2^{\cdot}(t), \dots, \phi_n^{\cdot}(t)$ exists on I, such that

$$\phi_i^{\cdot}(t) = f_i(t, \phi_1(t), \phi_2(t), \dots, \phi_n(t)) \;\; i = 1, 2, \dots, n.$$

Definition (1.6):

The differential equation (1.2) is equivalent to a system (1.4), by using the relations

$$x = x_1,$$
$$x^{\cdot} = x_2,$$
$$x^{\cdot\cdot} = x_3, \qquad (1.5)$$
$$\vdots$$
$$x^{(n)} = x_{n+1}$$

Definition (1.7):

The n-dimensional column vector is define by

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

1

and the vector valued function $X(t)$ defined by

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix},$$

the vector valued function F is defined by

$$F(t, X) = \begin{bmatrix} f_1(t, x_1, x_2, \cdots, x_n) \\ f_2(t, x_1, x_2, \cdots, x_n) \\ \vdots \\ f_l(t, x_1, x_2, \cdots, x_n) \end{bmatrix}.$$

Definition (1.8):

The system of (1.4) can be written as

$$\frac{dX}{dt} = F(t, x), \qquad (1.6)$$

and the solution of system (1.6) on I, is a vector valued $\phi(t)$, where

$$\phi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_n(t)).$$

Definition (1.9):

A system of first order linear differential equations of (1.6) is called *non autonomous* differential system, but the system of differential equations

$$\frac{dX}{dt} = F(X), \qquad \text{where} \quad X = (x_1, x_2, \ldots, x_n) \quad (1.7)$$

is said to be *autonomous*. (independent of t).

Definition (1.10):

The solution $\phi(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_n(t))$ of the system (1.7) is called *trajectory*, (or path or orbit), and a *phase portirait* is obtained from the trajectory by eliminating t.

Definition (1.11):

Let

$$\begin{array}{l} x_1' = a_{11}(t)\,x_1 + a_{12}(t)\,x_2 + \ldots + a_{1n}(t)x_n + f_1(t) \\ x_2' = a_{21}(t)x_1 + a_{22}(t)x_2 + \ldots + a_{2n}(t)x_n + f_2(t) \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ x_n' = a_{n1}(t)\,x_1 + a_{n2}(t)x_2 + \ldots + a_{nn}(t)\,x_n + f_n(t) \end{array}$$

be *a system of first order linear differential equations*, we can write it as

$$x_i' = \sum_{j=1}^{n} a_{ij}(t)x_j + f_i(t), \qquad (1.8)$$

where $a_{ij}(t)$, $(i, j = 1, 2, \ldots, n)$, $f_i(t)$, $i = 1, 2, \ldots, n$ are real valued functions defined on I.

Definition (1.12):

A system of first order linear differential equations in (1.8) can be expressed as

$$\frac{dX}{dt} = A(t)\, X + F(t) \qquad (1.9)$$

where $A(t) = [\, a_{ij}(t)\,]$ is an nxn matrix, and $F(t)$ is an n vector such that $F(t) = (f_1(t), \ldots, f_n(t))$.

Definition (1.13):

The system of (1.9) is called *homogeneous* if $F(t) = 0$, otherwise is called *non- homogeneous*.

Definition (1.14):

Let $y^{(n)} + a_1(t)\, y^{(n-1)} + \ldots + a_n(t)\, y = 0, \qquad (1.10)$

be *n-th* order *linear homogeneous equation with variable coefficients*, where $a_i(t)\ (i = 1, 2, \ldots, n)$ are continuous on I, the equation (1.10) is equivalent to an n-dimensional first order linear homogeneous system

$$\frac{dX}{dt} = A(t) X \ , \qquad (1.11)$$

where

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_n(t) & a_{n-1}(t) & a_{n-2}(t) & \cdots & \cdots & -a_1(t) \end{bmatrix}.$$

Remark:

The differential equation.

$$y^{n} + a_1 y^{n-1} + a_2 y^{n-2} + \dots + a_n y = 0, \qquad (1.12)$$

is a special case of (1.10), where $a_1, a_2, \cdots, a_n$ are constants coefficients, and the differential equation (1.12) is equivalent to the system

$$\frac{dX}{dt} = AX, \qquad (1.13)$$

where $A = [a_{ij}]$ is a constants matrix.

Definition (1.15):

Let $\phi_i' = f_i(t, \phi_1, \phi_2, \dots, \phi_n), \ (i = 1, 2, \dots, n),$

and $\phi_i(t_0) = \phi_{i0} = c_i,$ (1.14)

The system (1.14) is called an *initial value problem* for the system of differential equations.

Definition (1.16):

An initial value problem in (1.14) can be written as

$$\phi_i' = F(t, x),$$

$$\phi(t_0) = \phi_0. \qquad (1.15)$$

where $\phi = (\phi_1, \phi_2, ..., \phi_n)$,

$F = (f_1, f_2, ..., f_i)$,

$\phi_0 = (\phi_{10}, \phi_{20}, ..., \phi_{n0})$,

are vectors in R^{n+1}.

Theorem (1.1) [1]:

An initial value problem

$$\phi_i' = f_i\,(t, \phi_1, \phi_2, ..., \phi_n),\ (i = 1, 2, ..., n),$$

$$\phi_i(t_0) = c_i$$

is equivalent to integral equation

$$\phi_i(t) = c_i + \int_{t_0}^{t} f_i(s, \phi_1, \phi_2, ..., \phi_n)\ ds,\ (i = 1, 2, ..., n).$$

Proof:

$$\frac{d\phi_i}{dt} = f_i(t, \phi_1, \phi_2, ..., \phi_n).$$

$$d\phi_i = f_i(t, \phi_1, \phi_2, ..., \phi_n)\,dt.$$

Integral both sides from t_0 to t.

$$\int_{t_0}^{t} d\phi_i = \int_{t_0}^{t} f_i(s, \phi_1, \phi_2, ..., \phi_n)\,ds.$$

$$\phi_i(t) - \phi_i(t_0) = \int_{t_0}^{t} f_i(s, \phi_1, \phi_2, ..., \phi_n)\,ds.$$

$$\phi_i(t) = \phi_i(t_0) + \int_{t_0}^{t} f_i(s, \phi_1, \phi_2, \ldots, \phi_n)\, ds.$$

$$\phi_i(t) = c_i + \int_{t_0}^{t} f_i(s, \phi_1, \phi_2, \ldots, \phi_n)\, ds.$$

Conversely $\phi_i(t) = c_i + \int_{t_0}^{t} f_i(s, \phi_1, \phi_2, \ldots, \phi_n)\, ds.$

$$\frac{d}{dt}\phi_i = \frac{d}{dt}\left[F_i(s, \phi_1, \phi_2, \ldots, \phi_n)\Big|_{t_0}^{t}\right].$$

$$\frac{d}{dt}\phi_i = \frac{d}{dt}\left[F_i(t, \phi_1, \phi_2, \ldots, \phi_n) - F_i(t_0, \phi_1, \ldots, \phi_n)\right].$$

$$\frac{d\phi_i}{dt} = f_i(t, \phi_1, \phi_2, \ldots, \phi_n).$$

Since $\phi_i(t) = c_i + \int_{t_0}^{t} f_i(s, \phi_1, \phi_2, \ldots, \phi_n)\, ds$.

Put $\phi_i(t_0) = c_i$ we get .

$$\phi_i(t_0) = c_i + \int_{t_0}^{t_0} f_i(s, \phi_1, \phi_2, \ldots, \phi_n)\, ds = c_i.$$

Theorem (1.2) [1]:

If the functions

$$f_i(t, \phi_1, \phi_2, \ldots, \phi_n) \qquad (i = 1, 2, \ldots, n),$$

Satisfy Lipschitz condition such that

$$\left| f_i(t, \bar{\phi}_1, \ldots, \bar{\phi}_n) - f_i(t, \tilde{\phi}_1, \ldots, \tilde{\phi}_n) \right| \le K \left[\left| \bar{\phi}_1 - \tilde{\phi}_1 \right| + \ldots + \left| \bar{\phi}_n - \tilde{\phi}_n \right| \right],$$

where $(t, \bar{\phi}_1, ..., \bar{\phi}_n), (t, \tilde{\phi}_1, ..., \tilde{\phi}_n) \in R$, and k is a constant such that $K > 0$.

And if the functions $f_i(t, \phi_1, \phi_2, ..., \phi_n)$ be continuous on a region D where

$$D: |t - t_0| \le a, |\phi_i - \phi_0| \le b_i,$$

a and b_i are positive real numbers $(i = 1, 2, ..., n)$.

and $\quad |f_i(t, \phi_1, \phi_2, \phi_n)| \le M \quad$ for all $(t, \phi_1, ..., \phi_n) \in R$.

And if $\quad \alpha = \min (a, \frac{b_1}{M}, \frac{b_2}{M}, ..., \frac{b_n}{M}),$

$$M = \max \{M_1, M_2, ..., M_n\},$$

$$M_j = \max |f_i(t, \phi_1, \phi_2, ..., \phi_n)|,$$

then the initial value problem (1.14) has a unique solution on $|t - t_0| \le \alpha$.

We shall prove this theorem by method of successive approximations (Picard method).

Proof:

Define $\quad \phi_{i,0}(t) = c_i$

$\phi_{i,1}(t) = c_i + \int_{t_0}^{t} f_i(s, \phi_{1,0}, ..., \phi_{n,0})\, ds$ is continuous function on $[t_0, t_0 + \alpha]$.

$\phi_{i,2}(t) = c_i + \int_{t_0}^{t} f_i(s, \phi_{1,1}, ..., \phi_{n,1})\, ds$ is continuous function on $[t_0, t_0 + \alpha]$.

$\therefore \phi_{i,j}(t) = c_i + \int_{t_0}^{t} f_i(s, \phi_{i,j-1}, \ldots, \phi_{n,j-1})\, ds$ is a sequence of continuous functions on $[t_0, t_0 + \alpha]$, $(i = 1, 2, \ldots, n)$, $(j = 1, 2, \ldots)$, such that $\phi_i(t) = \lim_{j \to \infty} \phi_{i,j}(t)$.

Next we show that $\phi_{i,j}(t)$ lies inside the region D.

Since $\phi_{i,j}(t) = c_i + \int_{t_0}^{t} f_i(s, \phi_{1,j-1}, \ldots, \phi_{n,j-1})\, ds$.

$$\therefore \left|\phi_{i,j} - c_i\right| = \left|\int_{t_0}^{t} f_i(s, \phi_{1,j-1}, \ldots, \phi_{n,j-1})\, ds\right|.$$

$$\left|\phi_{i,j} - c_i\right| \le M\,(t - t_0).$$

$$\left|\phi_{i,j} - c_i\right| \le M\,\alpha.$$

but $\alpha \le a,\ \alpha \le \dfrac{b_1}{M}, \ldots, \alpha \le \dfrac{b_i}{M}$.

Therefore $\left|\phi_{i,j} - c_i\right| \le M\ \dfrac{b_i}{M} = b_i$, for all i,

and this mean $\phi_{i,j}(t)$ lies inside region D.

Next we show that $\phi_{i,j}(t)$ *converges uniformly* on $[t_0, t_0 + \alpha]$, we shall need the following corollary (1.3).

Corollary (1.3) [1]:

Let $\sum_{n=1}^{\infty} f_n(x)$ be a series of real functions, where $x \in I = [a,b]$.

If $0 \le |f_n(x)| \le M_n\ \forall n$, with the series $\sum_{n=1}^{\infty} M_n$ *converges* on $I = [a,b]$,

then $\sum_{n=1}^{\infty} f_n(x)$ *converges uniformly* on $[a,b]$.

Let $$\left|\phi_{i,1}-\phi_{i,0}\right| = \left|c_i + \int_{t_0}^{t} f_i(s,\phi_{1,0},\ldots,\phi_{n,0})\,ds - c_i\right|.$$

$$\left|\phi_{i,1}-\phi_{i,0}\right| \le M\,(t-t_0).$$

$$\left|\phi_{i,2}-\phi_{i,1}\right| = \left|\int_{t_0}^{t}\left[f_i(s,\phi_{1,1},\ldots,\phi_{n,1}) - f_i(s,\phi_{1,0},\ldots,\phi_{n,0})\right]ds\right|.$$

$$\left|\phi_{i,2}-\phi_{i,1}\right| \le k\int_{t_0}^{t}\left[\left|\phi_{1,1}-\phi_{1,0}\right| + \ldots + \left|\phi_{n,1}-\phi_{n,0}\right|\right]ds.$$

$$\left|\phi_{i,2}-\phi_{i,1}\right| \le k\int_{t_0}^{t} M(s-t_0)\,ds .$$

$$\left|\phi_{i,2}-\phi_{i,1}\right| \le k\,M\,\frac{(t-t_0)^2}{2}.$$

$$\left|\phi_{i,3}-\phi_{i,2}\right| = \left|\int_{t_0}^{t}\left[f_i(s,\phi_{1,2},\ldots,\phi_{n,2}) - f_i(s,\phi_{1,1},\ldots,\phi_{n,1})\right]ds\right|.$$

$$\left|\phi_{i,3}-\phi_{i,2}\right| \le k\int_{t_0}^{t}\left[\left|\phi_{1,2}-\phi_{1,1}\right| + \ldots + \left|\phi_{n,2}-\phi_{n,2}\right|\right]ds.$$

$$\left|\phi_{i,3}-\phi_{i,2}\right| \le k\int_{t_0}^{t} kM\frac{(s-t_0)^2}{3!}\,ds .$$

$$\left|\phi_{i,3}-\phi_{i,3}\right| \le k^2\,M\,\frac{(t-t_0)^3}{3!}.$$

⋮

⋮

By mathematical induction, we can show that,

$$\left|\phi_{i,j+1}-\phi_{i,j}\right| \le M k^j\,\frac{(t-t_0)^{j+1}}{(j+1)!}.$$

$$\left|\phi_{i,j+1}-\phi_{i,j}\right| \le M k^j\,\frac{(\alpha)^{j+1}}{(j+1)!}.$$

Now Let $u_j(t) = (\phi_{i,j}(t) - \phi_{i,j-1}(t))$.

$$\therefore \left|u_j(t)\right| = \left|\phi_{i,j}-\phi_{i,j-1}\right| \le M K^{j-1}\,\frac{(\alpha)^j}{j!}.$$

And let $\sum_{j=1}^{\infty} M\, k^{(j-1)} \frac{(\alpha)^j}{j!} = \sum_{j=1}^{\infty} v_j(t)$.

$$\therefore \frac{v_{j+1}(t)}{v_j} = \frac{1}{(j+1)}\, k\, \alpha \to 0 \quad \text{as } j \to \infty,$$

and $$\lim_{j \to \infty} \frac{v_{j+1}(t)}{v_j(t)} < 1.$$

$\therefore \sum_{j=1}^{\infty} v_j(t)$ is converges by ratio test.

This implies that by corollary (1.3), the series.

$\sum_{n=1}^{\infty} u_j(t) = \sum_{j=1}^{n} (\phi_{i,j} - \phi_{i,j-1})$ converges uniformly, and the partial

sums of this series given by $S_i(t) = \sum_{p=1}^{j} (\phi_{i,p}(t) - \phi_{i,p-1}(t))$

$$\therefore\ S_i(t) = \phi_{i,j}(t) - \phi_{i,0}(t).$$

The sequence $\{\phi_{i,j}(t)\}$ defined by $\phi_{i,j}(t) = S_j(t) + \phi_{i,0}(t)$, $(i = 1, 2, ..., n)$ $(j = 1, 2, ...)$ converges uniformly to a continuous function $\phi_i(t)$ on $[t_0, t_0 + \alpha]$ for all i.

Next we show that $\phi_i(t)$ satisfies the system

$$\phi_i(t) = c_i + \int_{t_0}^{t} f_i(s, \phi_1, \phi_2, ..., \phi_n)\, ds.$$

Since $$\phi_{i,j}(t) = c_i + \int_{t_0}^{t} f_i(s, \phi_{1,j-1}, ..., \phi_{n,j-1})\, ds.$$

$$\therefore \lim_{j \to \infty} \phi_{i,j}(t) = c_i + \lim_{j \to \infty} \int_{t_0}^{t} f_i(s, \phi_{1,j-1}, ..., \phi_{n,j-1})\, ds.$$

$$\therefore \phi_i(t) = c_i + \int_{t_0}^{t} f_i(s, \phi_1, \phi_2, ..., \phi_n)\, ds.$$

Next we show that an initial value problem has unique solution.

Suppose an initial value problem has two solutions $\phi_i(t)$ and $\psi_i(t)$.

Let $z_i(t) = |\phi_i(t) - \psi_i(t)| \ge 0$.

$$\therefore z_i(t_0) = |\phi_i(t_0) - \psi_i(t_0)| = c_i - c_i = 0.$$

$$z_i(t) = \left| \int_{t_0}^{t} f_i(s, \phi_1, \phi_2, \ldots, \phi_n)\, ds - \int_{t_0}^{t} (s, \psi_1, \psi_2, \ldots, \psi_n)\, ds \right|.$$

$$z_i(t) \le k \int_{t_0}^{t} \left[|\phi_1 - \psi_1| + \ldots + |\phi_n - \psi_n| \right] ds.$$

$$z_i(t) \le k \int_{t_0}^{t} z_i(s)\, ds.$$

$$z_i(t) - k \int_{t_0}^{t} z_i(s)\, ds \le 0.$$

$$z_i(t)\, e^{-k(t-t_0)} - k e^{-k(t-t_0)} \int_{t_0}^{t} z_i(s)\, ds \le 0.$$

$$\frac{d}{dt}\left[e^{-k(t-t_0)} \int_{t_0}^{t} z_i(s)\, ds \right] \le 0.$$

$$\therefore e^{-k(t-t_0)} \int_{t_0}^{t} z_i(s)\, ds \le 0.$$

This implies that $\int_{t_0}^{t} z_i(s)\, ds \le 0 \;\; \forall\, t \in [t_0, t_0 + \alpha]$,

And this implies that $z_i(t) \le 0 \;\; \forall\, t \in [t_0, t_0 + \alpha]$

But $z_i(t) \ge 0 \;\; \forall\, t \in [t_0, t_0 + \alpha]$

Therefore $\phi_i(t) = \psi_i(t)$,

and the initial value problem (1.14) has a unique solution.

Theorem (1.4) [2]:

Any combination of the solutions of the system (1.11) is also a solution of the system (1.11).

Theorem (1.5): [2]

If $\phi(t)$ is a solution of the system (1.9), then $\psi(t)$ is a solution of (1.9) if and only if $\phi(t) - \psi(t)$ is a solution of the system $\frac{dX}{dt} = A(t)\,X$.

Remark:

If $\phi(t)$ is a solution of the system $\frac{dX}{dt} = A(t)X$ such that $\phi(t_0) = 0$, $t_0 \in I$ then $\phi(t) = 0$.

Definition (1.17):

Let $\{v_1(t), v_2(t), ..., v_n(t)\}$ be a set of vector valued functions on I, and $c_1, c_2, ..., c_n$ are constants not all zeros, such that $\sum_{i=1}^{n} c_i\, v_i(t) = 0$, the vector valued functions $v_1(t), v_2(t), ..., v_n(t)$ are called *linearly dependent* on I, and are linearly independent on I if $\sum_{i=1}^{n} c_i\, v_i(t) = 0$, for all t implies that $c_1 = c_2 = ... = c_n = 0$.

Theorem (1.6) [2]:

If $\{\phi_1(t), \phi_2(t), ..., \phi_n(t)\}$ is a set of linearly independent solution of the system (1.11), then $\sum_{j=1}^{n} c_j\,\phi_j(t)$ is equal zero if $c_1 = c_2 = ... = c_n = 0$.

Definition (1.18):

A set of the solutions $\{\phi_1(t), \phi_2(t), ..., \phi_n(t)\}$ of the system (1.11) is called *a fundamental system of solutions* of the system (1.11).

Theorem (1.7) [1]:

A fundamental system of solutions of the system (1.11) exists.

Definition (1.19):

Let Φ be an nxn matrix such that

$$\Phi(t) = \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) & \cdots & \phi_{1n}(t) \\ \phi_{21}(t) & \phi_{22}(t) & \cdots & \phi_{2n}(t) \\ \vdots & \vdots & \cdots & \\ \phi_{n1}(t) & \phi_{n2}(t) & \cdots & \phi_{nn}(t) \end{bmatrix},$$

where

$$\phi_j(t) = (\phi_{1j}(t), \phi_{2j}(t), ..., \phi_{nj}(t)),$$

is a solution of the system (1.11), and $\phi_j(t) = e_j$, e_j is the standard unit vectors, $\Phi(t)$ is called *a fundamental matrix* of the system (1.11) , and $\Phi(t_0) = I$, where I is the identity matrix.

Corollary (1.8) [2]:

A fundamental matrix $\Phi(t)$ is also a solution of the system (1.11), and if $X(t)$ is a solution of the system (1.11), Such that $X(t_0) = X_0$, then $X(t) = \Phi(t)\, X_0$, where $X_0 = (x_{10}, x_{20}, ..., x_{n0})$

Definition (1.20):

Let $\phi_1(t)$, $\phi_2(t)$, ..., $\phi_n(t)$ be solutions of the system (1.11) such that

$$\phi_j = (\phi_{1j}(t), \phi_{2j}(t), ..., \phi_{nj}(t)).$$

The Wronskian of $\phi_1(t), ..., \phi_n(t)$ is given by

$$W(t) = \det \Phi(t). \qquad (1.16)$$

Theorem (1.9) [2]:

$\Phi(t)$ of the system (1.11) is a fundamental matrix if and only if $W(t) \neq 0$ for $t \in I$.

Theorem (1.10) [2]:

If $\Phi(t)$ is a fundamental matrix of the system (1.11), then $\Phi(t).C$ also is a fundamental matrix of the system (1.11), where C is any constant non singular matrix.

Definition (1.21):

Let A be an *nxn* matrix, then *a matrix exponential* of A is given by

$$e^{tA} = \sum_{n=0}^{\infty} A^n \frac{t^n}{n!}.$$

Remark:

$$e^{tA} = \Phi(t)\,\Phi^{-1}(0) \quad \text{if A is a constant matrix.}$$

Definition (1.23):

The polynomial in λ of degree n

$$f(\lambda) = \det(A - \lambda I) = a_0 \lambda^n + a_1 \lambda^{n-1} + ... + a_n \ ,$$

is called *characteristic polynomial* of A where A is an *nxn* matrix, I is the identity matrix.

Definition (1.23):

Let $\phi(t) = Ce^{\lambda t}$ is a solution of the system (1.13), where λ is a scalar constant, C is nonzero constant vector, then

$$f(\lambda) = \det\,(A - \lambda I) = 0, \tag{1.17}$$

is called the *characteristic equation* of A and it's n roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ are called the *eigenvalues* of A. For given *eigenvalues* the nonzero vectors $V_1, V_2, \ldots, V_n$ are called *eigenvectors.*

Theorem (1.11) [11]:

The general solution of the system (1.13) on $(-\infty, +\infty)$ is given by

$$\phi(t) = c_1 V_1\, e^{\lambda_1 t} + c_2 V_2\, e^{\lambda_2 t} + \ldots + c_n V_n\, e^{\lambda_n t}, \tag{1.18}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are n distinct real *eigenvalues*, and $V_1, V_2, \ldots, V_n$ are their corresponding *eigenvectors*, $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Theorem (1.12) [11]:

If $\lambda_1 = \alpha + i\beta$, β is real then the general solution of the system (1.13) on $R = (-\infty, +\infty)$ is given by $\phi(t) = c_1 x_1 + c_2 x_2$, where c_1, c_2 are arbitrary constants, and

$$\begin{aligned} x_1 &= e^{\alpha t}\left[B_1 \cos\beta t - B_2 \sin\beta t\right], \\ x_2 &= e^{\alpha t}\left[B_2 \cos\beta t + B_1 \sin\beta t\right], \end{aligned} \tag{1.19}$$

B_1 = real parts (V_1), and B_2 = imaginary parts (V_1).

Theorem (1.13) [2]:

The general solution of the system (1.9) is given by

$$\phi(t) = \Phi(t)\,\phi(t_0) + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\,B(s)\,ds, \qquad (1.20)$$

where $\Phi(t)$ is the fundamental matrix of the system (1.11), $\phi(t_0) = X_0$

Remark:

If A is a constant matrix in the system (1.9), then

$$\phi(t) = \phi(t_0)\,e^{At} + \int_{t_0}^{t} e^{A(t-s)}\,f(s)\,ds. \qquad (1.21)$$

Definition (1.24):

The general solution of the system (1.13) can be written as

$$\phi(t) = \phi(t_0)\,e^{At}. \qquad (1.22)$$

Theorem (1.14) [2]:

If all *eigenvalues* of (1.17) have negative real parts, then

$$\left|\phi(t)\right| \le M\,e^{-\alpha t}, \; t \ge 0,$$

$$\lim_{t\to\infty}\left|\phi(t)\right| = 0,$$

where $\phi(t)$ is any solution of the system (1.13), α and M are positive numbers..

Theorem (1.15) [2]:

Let $\phi(t)$ be a solution of the system (1.13), and if

$$|\phi(t)| \le M \text{ for all } t \ge 0, \text{ where M is a positive constant,}$$

then $\phi(t)$ is bounded on $[0,\infty)$.

Definition (1.25):

The *norm* of any vector $X \in R^n$ is given by

$$\|X\| = \sum_{i=1}^{n} |X_i|, \tag{1.23}$$

Which satisfy

(i) $\|X\| > 0$ with $\|X\| = 0$ if and if $X = 0$,

(ii) $\|KX\| = |K|\,\|X\|$ for any constant k,

(iii) $\|X + Y\| \le \|X\| + \|Y\|$, $X, Y \in R^n$.

Definition (1.26):

The norm of the *nxn* matrix A is given by

$$\|A\| = \sum_{i,j=1}^{n} |a_{ij}| . \tag{1.24}$$

Definition (1.27):

Suppose that $\frac{dX}{dt} = F(X)$, the point X_0 such that $F(X_0) = 0$ is called equilibrium point (or equilibrium solution, critical point).

Definition (1.28):

Equilibrium point is called *isolated* if there exists a neighborhood about it which does not contain any other equilibrium points.

Definition (1.29):

Let two dimensional linear autonomous system.

$$\frac{dx}{dt} = ax + by,$$
$$\frac{dy}{dt} = cx + dy, \tag{1.25}$$

or
$$\begin{bmatrix} x^\bullet \\ y^\bullet \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad x^\bullet = \frac{dx}{dt}, \quad y^\bullet = \frac{dy}{dt}, \tag{1.26}$$

where a, b, c and d are real constants.

Remark:

(i) The point $(0,0)$ is the only critical point of the system (1.25) if $ad - bc \neq 0$, and the trajectory of the system (1.25) approaches the critical point $(0,0)$ as $t \to \infty$ if

$$\lim_{t \to \pm\infty} \phi_1(t) = 0,$$
$$\lim_{t \to \pm\infty} \phi_2(t) = 0,$$

where $\phi_1(t)$, $\phi_2(t)$ are solutions of (1.25).

(ii) Let the trajectories $\phi_1(t)$ and $\phi_2(t)$ of the system (1.25) approache $(0,0)$ as $t \to \pm\infty$, we say that the trajectories enter the critical point $(0,0)$ as $t \to \pm\infty$ if $\lim_{t \to \pm\infty} \frac{\phi_2(t)}{\phi_1(t)}$ exists.

Chapter Two

Some concept of stability, asymptotically stable and two dimensional linear autonomous systems.

1. Some concept of stability and asymptotically stable:

Consider the system

$$\frac{dx}{dt} = P(x,y),$$
$$\frac{dy}{dt} = Q(x,y), \qquad (2.1.1)$$

where $P(x,y)$ and $Q(x,y)$ are real functions which have continuous first partial derivatives for all (x,y).

However, the point $(0,0)$ is an isolated critical point for this system.

Definition (2.1.1):

The critical point of the system (2.1.1) is said to be *stable* if for every neigh bourhood δ of $(0,0)$, there is a smaller neigh- bourhood $\varepsilon \subseteq \delta$ of $(0,0)$, such that every trajectory which passes through ε remains in δ as t increases.

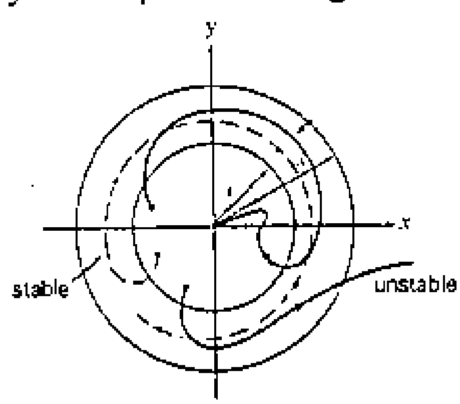

Figure. (2.1.1)

Definition (2.1.2):

The critical point $(0,0)$ of the system (2.1.1) is *asymptotically stable* if there is a neighborhood ε approaches $(0,0)$ as t tends to infinity.

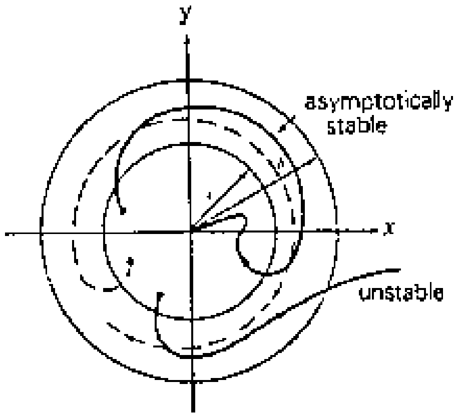

Figure (2.1.2)

We shall consider the system

$$\frac{dX}{dt} = F(t,x), \qquad (2.1.2)$$

with initial condition $\phi(t_0) = \phi_0$.

Definition (2.1.3):

Let $\phi(t)$ be a solution of the system (2.1.2), then $\phi(t)$ is called *Laplace stable* if there exists a constant M such that $|\phi(t)| \le M$.

Definition (2.1.4):

A solution $\phi(t)$ of the system (2.1.2) is called *Liapunov stable* if given $\varepsilon > 0$, then we can choose $\delta(\varepsilon)$ such that for any other solution $\psi(t)$ and $|\phi(t_0) - \psi(t_0)| < \delta$ we have $|\phi(t) - \psi(t)| < \varepsilon$ for all $t \ge t_0$.

Note that if we put $t_0 = 0$, then we get $| \phi(0) - \psi(0) | < \delta$ this implies that $| \phi(t) - \psi(t) | < \varepsilon$ for all $t \geq 0$.

Definition (2.1.5):

The solution $\phi(t)$ of the system (2.1.2) is called *asymptotically stable* if

(i) $\phi(t)$ is stable.

(ii) $| \phi(t) - \psi(t) |$ tends to zero as t tends to infinity.

Observe that every asymptotically stable equilibrium solution is stable. However the converse is not true.

For example the critical point $(0,0)$ of the differential equation $x'' + (x')^3 + x = 0$ is stable, but is not asymptotically stable, we will explain the solution of this example in the next chapter.

Definition (2.1.6):

The solution $\phi(t)$ of the system (2.1.2) is called *unstable* if there exists at least one solution $h(t)$ of (2.1.2) such that

$$|\phi(t_0) - h(t_0)| < \delta \text{, but } |\phi(t) - h(t)| > \varepsilon_0 \text{ for all } t \geq t_0$$

Now we shall consider the autonomous system

$$\frac{dX}{dt} = AX \quad , \qquad (2.1.3)$$

Theorem (2.1.1) [4]:

(i) Every solution $\phi(t)$ of the system (2.1.3) is *stable* if equilibrium solution is *stable*.

(ii) Every solution $\phi(t)$ of the system (2.1.3) is *asymptotically stable* if equilibrium solution is *asymptotically stable.*

Proof:

(i) Let $\psi(t)$ be another solution of the autonomous system.

$$\frac{dX}{dt} = AX,$$

such that $|\phi(0) - \psi(0)| < \delta$, and $x(t) = 0$ is equilibrium solution.

To show that $|\phi(t) - \psi(t)| < \varepsilon$ for all $t \geq 0$.

Since $x(t) = 0$ is stable.

Let $z(t)$ be another solution such that

$|z(0)| < \delta$ and we have $|z(t)| < \varepsilon$ for all $t \geq 0$.

Since $\phi(t)$ is a solution, $\psi(t)$ is a solution, and also $z(t)$ is a solution, therefore $z(t) = \phi(t) - \psi(t)$, $z(0) = \phi(0) - \psi(0)$.

And we get $|z(0)| = |\phi(0) - \psi(0)| < \delta$

therefore $|\phi(t) - \psi(t)| = |z(t)| < \varepsilon$

By definition (2.1.4) the solution $\phi(t)$ is *stable*.

(ii) Let $\psi(t)$ be another solution of the system (2.1.3) such that

$|\phi(0) - \psi(0)| < \delta$ for all $t \geq 0$.

To show that $|\phi(t) - \psi(t)|$ tends to zero as t tends to infinity.

Since $x(t) = 0$ is *asymptotically stable* then we have

$|z(t)| = |\phi(t) - \psi(t)|$, $|z(0)| = |\phi(0) - \psi(0)| < \delta$

we have $|\phi(t) - \psi(t)| = |z(t)|$ tends to zero as t tends to infinity.

Theorem (2.1.2) [4]:

Every solution $\phi(t) \neq 0$ of the system (2.1.3) is *unstable* if the equilibrium solution is *unstable*.

Proof:

Given $x(t) = 0$ is unstable, by definition (2.1.7) there exists another solution $h(t)$ such that $|h(0)| < \delta$ but $|h(t)| > \varepsilon$ for all $t \geq 0$.

Let $\psi(t) = \phi(t) + h(t)$ be a solution of the system (2.1.3) such that

$$|\psi(0) - \phi(0)| = |h(0)| < \delta.$$

But $|\psi(t) - \phi(t)| = |h(t)| < \varepsilon$ for all $t \geq 0$.

Therefore the solution $\phi(t)$ is *unstable* by definition (2.1.7).

Theorem (2.1.3) [4]:

Every solution of the system

$$\frac{dX}{dt} = AX, \tag{2.1.4}$$

(where $A = (a_{ij})$ is constant matrix), is *asymptotically stable* if all eigen values of the matrix A have negative real parts.

Proof:

Let $\phi(t)$ is solution of the system (2.1.3) given by

$$\begin{bmatrix} \phi_1(t) \\ \phi_2(t) \\ \vdots \\ \phi_n(t) \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ & & & \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{bmatrix} \begin{bmatrix} \phi_1(0) \\ \phi_2(0) \\ \vdots \\ \phi_n(0) \end{bmatrix}$$

or $\quad \phi(t) = e^{At}\,\phi(0)$,

or $\quad \phi_i(t) = \sum_{j=1}^{n} \phi_{ij}(t)\,\phi_j(0)$

Since all eigen values of the matrix A have negative real parts by theorem (1.14), there exists positive constant K and α such that

$$|\phi_{ij}(t)| \le k e^{-\alpha t}$$

$$|\phi_i(t)| = \left|\sum_{j=1}^{n} \phi_{ij}(t)\ \phi_j(0)\right|$$

$$|\phi_i(t)| \le \left|\sum_{j=1}^{n} \phi_{ij}(t)\right|\ |\phi_i(0)|$$

$$|\phi_i(t)| \le \sum_{j=1}^{n} k\ e^{-\alpha t}|\phi_j(0)|$$

Define $\quad \|\phi(t)\| = \max\ \{|\phi_1(t)|\ ,|\phi_2(t)|\,,\ \ldots\ldots, |\phi_n(t)|\}$

$\quad\quad\quad \|\phi(0)\| = \max\ \{|\phi_1(0)|\ ,|\phi_2(0)|\,,\ \ldots\ldots, |\phi_n(0)|\}$

Then we have $\quad |\phi_i(t)| \le \|\phi(t)\|$ for all i.

$\quad\quad\quad |\phi_i(0)| \le \|\phi(0)\|$ for all i.

$$\therefore\ \|\phi(t)\| \le \sum_{j=1}^{n} k\ e^{-\alpha t}\|\phi(0)\|$$

Let $\quad \|\phi(0)\| < \delta = \dfrac{\varepsilon}{k.n}$

$$\|\phi(t)\| \le k\ e^{-\alpha t}.\,n\,.\,\|\phi\ (0)\|$$

which tends to zero as t tends to infinity.

Therefore by definition (2.1.5) $X(t) = 0$ is *asymptotically stable*, this implies that the solution $\phi(t)$ of the system (2.1.3) is *asymptotically stable* by theorem (2.1.1).

Theorem (2.1.4) [4]:

Every solution of the system (2.1.4) is *unstable* if at least one eigen values of the matrix A has positive real part.

Proof:

Let λ be *eigenvalues* of the matrix A with positive real part, and V are the corresponding *eigenvectors*.

$\phi(t) = c \, . \, e^{\lambda t} \, . \, V$ is solution of (2.1.3) for any constant *c*.

We have $\|\phi(t)\| = |c| \, e^{\lambda t} \, \|v\|$

If $\lambda > 0$ then $\|\phi(t)\|$ tends to infinity as t tends infinity.

If $\lambda = a + ib$, $a > 0$, then $\|\phi(t)\|$ tends to infinity as t tens to infinity.

Therefore $x(t) = 0$ is *unstable*, and every solution $\phi(t)$ of (2.1.3) is *unstable* by theorem (2.1.2).

Now, consider the linear non-homogeneous system.

$$\frac{dX}{dt} = AX + F(t), \qquad (2.1.6)$$

where $A = (a_{ij})$ is a constant matrix and $F(t)$ are continuous n-vector for all t.

Theorem (2.1.5) [2]:

Any solution $\phi(t)$ of the system (2.1.6) is *stable* if and only if equilibrium solution of the system (2.1.4) is *stable*.

Proof:

Let $\phi(t)$ be any solution of (2.1.6) given by

$$\phi(t) = e^{At}\ \phi(0) + \int_0^t e^{A(t-s)}\, F(s)\, ds$$

and $\psi(t)$ be another solution of (2.1.6) given by

$$\psi(t) = e^{At}\, \psi(0) + \int_0^t e^{A(t-s)}\, F(s)\, ds$$

since $\phi(t)$ is stable.

Therefore by definition (2.1.7).

$$|\phi(t) - \psi(t)| < \varepsilon \text{ , where every } |\phi(0) - \psi(0)| < \delta$$

$$|\phi(t) - \psi(t)| = |e^{At}\phi(0) - e^{At}\psi(0)|$$

$$|\phi(t) - \psi(t)| = e^{At}|\phi(0) - \psi(0)|$$

$$|\phi(t) - \psi(t)| \le e^{At}\ .\ \delta < \varepsilon \quad \text{this implies} \quad e^{At} < \frac{\varepsilon}{\delta}.$$

To show that $x(t) = 0$ is stable.

i.e. to show that $|z(t) < \varepsilon|$ where every $|z(0)| < \delta$, where $z(t)$ is any solution of (2.1.5) given by

$$z(t) = e^{At}\, z(0),$$

$$|z(t)| = |e^{At}\, z(0)|$$

$$|z(t)| < e^{At}\,\delta < \frac{\varepsilon}{\delta}\,\delta = \varepsilon$$

this implies that $x(t) = 0$ is *stable*.

Conversely given $x(t) = 0$ is stable for the system (2.1.4).

There exists $z(t) = e^{At} z(0)$ such that $|z(t)| < \varepsilon$ there every $|z(0)| < \delta$.

To show that $\phi(t) \neq 0$ is stable for (2.1.6).

Let $\psi(t)$ be another solution of (2.1.6).

$\phi(t) = z(t) + \psi(t)$ this implies that

$|\phi(t) - \psi(t)| = |z(t)| < \varepsilon$ and $|\phi(0) - \psi(0)| = |z(0)| < \delta$

or Let $|\phi(0) - \psi(0)| = |z(0)| < \delta$

$|\phi(t) - \psi(t)| = |e^{At}\phi(0) - e^{At}\psi(0)| = e^{At}|\phi(0) - \psi(0)|$

$|\phi(t) - \psi(t)| \leq e^{At}\,|z(0)| = |e^{At} z(0)| < \varepsilon$

Therefore $\phi(t) \neq 0$ of (2.1.6) is *stable*.

We shall consider the system

$$\frac{dX}{dt} = A(t)\,X, \tag{2.1.7}$$

where $A(t)$ is an nxn continuous matrix on $[0,\infty)$, and $X = \phi(t) = (\phi_1(t), \ldots, \phi_n(t))$ is an unknown n-dimensional vector functions.

Theorem (2.1.6) [2]:

All the solutions of the system (2.1.7) are *stable* if and only if these solutions are *bounded*.

Definition (2.1.7):

Let

$$f(\lambda) = \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots + a_n = 0, \qquad (2.1.8)$$

be *a polynomial in* λ *of degree* n, where $a_1, \dots, a_n$ are real constants coefficients, and

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = 0, \qquad (2.1.9)$$

is the *n-th order linear homogeneous differential equation* related to (2.1.8).

Theorem (2.1.7) [6]:

All the roots of (2.1.8) have negative real parts if and only if all the principal diagonal minors of *Hurewitz's matrix* H_n are positive, where

$$H_n = \begin{bmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix}.$$

A Hurewitz matrix is constructed in the following way. We place the coefficients of (2.1.8) a_1 to a_n on the principle diagonal.

The colums are then filled with coefficients that have only odd indices or only even (including $a_0 = 1$).

Hence, the elements of the Hurewitz matrix $H_n = (h_{ij})$ are given $h_{i,j} = a_{2i-j}$, $(i, j = 1, 2, 3, \dots, n)$. All the missing elements (coefficients with indices greater than n or less than zero) being replaced by zeros.

Denote the principal diagonal minors of the Hurewitz matrix by

$$H_1 = [a_1],\ \det(H_1) > 0,$$

$$H_2 = \begin{bmatrix} a_1 & 1 \\ 0 & a_2 \end{bmatrix},\quad \det(H_2) > 0,$$

$$H_3 = \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ 0 & 0 & a_3 \end{bmatrix},\quad \det(H_3) > 0,$$

$$H_4 = \begin{bmatrix} a_1 & 1 & 0 & 0 \\ a_3 & a_2 & a_1 & a_2 \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_4 \end{bmatrix},\quad \det(H_4) > 0, \ldots$$

Theorem (2.1.8): [2]

Let A be a matrix of the system (2.1.4) such that all *eigenvalues* of the matrix A have negative real parts. The matrix A is stable.

Theorem (2.1.9): [6]

All the roots of the equation (2.1.8) have negative real parts if and only if

(i) $a_0 > 0,\ a_1 > 0,\ \ldots,\ a_n > 0,$

(ii) $\Delta_{n-1} > 0,\ \Delta_{n-3} > 0,\ \ldots\ldots$

The theorem (2.1.9) is called *Lienard – Chaipart test*.

Theorem (2.1.10): [6]

Let all roots of the equation (2.1.8) have negative real parts. The equation (2.1.8) is stable.

Theorem (2.1.11) [2]:

If all *eigenvalue* of the matrix A have negative real parts, then every solution $\phi(t)$ of the system (2.1.5) is *asymptotically stable*.

2. Two dimensional linear autonomous systems:

Consider the linear autonomous system

$$\frac{dx}{dt} = ax + by,$$
$$\frac{dy}{dt} = cx + dy, \qquad (2.2.1)$$

where a, b, c and d are real constants, the point $(0,0)$ is the only critical point, and λ_1, λ_2 are *eigenvalues* of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and $ad - bc \neq 0$.

Theorem (2.2.1):

If λ_1 and λ_2 are real and distinct *eigenvalues*, then the general solution of the system (2.2.1) is given by

$$x = c_1 A_1 e^{\lambda_1 t} + c_2 A_2 e^{\lambda_2 t},$$
$$y = c_1 B_1 e^{\lambda_1 t} + c_2 B_2 e^{\lambda_2 t},$$

where A_1, A_2, B_1, B_2 are constants, c_1, c_2 are arbitrary constants, and $v_1 = \begin{bmatrix} A_1 \\ B_1 \end{bmatrix}, v_2 = \begin{bmatrix} A_2 \\ B_2 \end{bmatrix}$ are correspoind *eigenvectors.*

If λ_1 and λ_2 are real and equal *eigenvalues*, such that there exists one linearly independent *eigenvector*, then the general solution of the system (2.2.1) is given by

$$x = c_1 A_1 e^{\lambda t} + c_2 (A_1 t + A_2) e^{\lambda t},$$
$$y = c_1 B_1 e^{\lambda t} + c_2 (B_1 t + B_2) e^{\lambda t},$$

where c_1 c_2 are arbitrary constants, A_1, A_2, B_1, B_2 are constants,

and $v_1 = \begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$, $v_2 = \begin{bmatrix} A_2 \\ B_2 \end{bmatrix}$ are correspoind *eigenvectors*.

The *characteristic equation* of the system (2.2.1) is given by

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0,$$

Definition (2.2.1):

The critical point $(0,0)$ of the system (2.2.1) is called *center* if there exists $N(0,0)$ (a neighborhood of $(0,0)$) Such that this neighborhood contains infinite number of closed trajectories, and $(0,0)$ is not approached by any trajectory as t tends to ∞ or t tends to $-\infty$. (Figure 2.2.1).

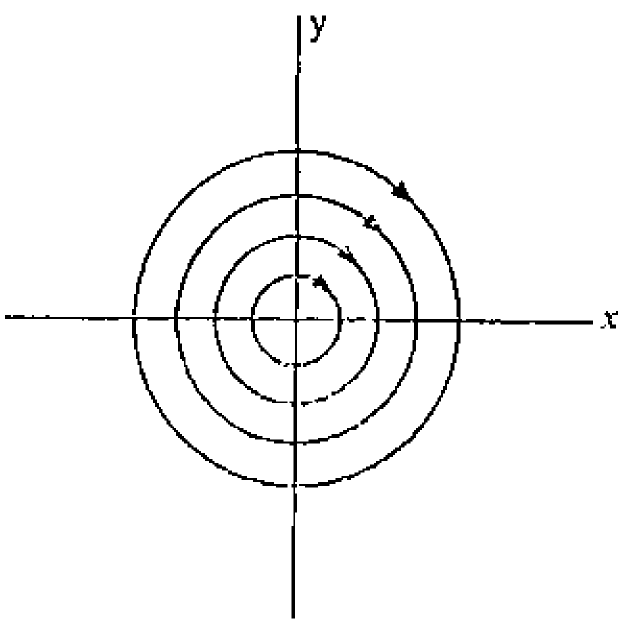

Figure (2.2.1)

Definition (2.2.2):

The critical point $(0,0)$ of the system (2.2.1) is called saddle point if there exists $N(0,0)$ (a neighborhood of $(0,0)$) such that:

1. There exists two trajectories which approach and enter $(0,0)$ from a pair opposite directions as t tends to ∞, and there exists two trajectories which approach and enter $(0,0)$ from different pair of opposite directions as t tends to $-\infty$.

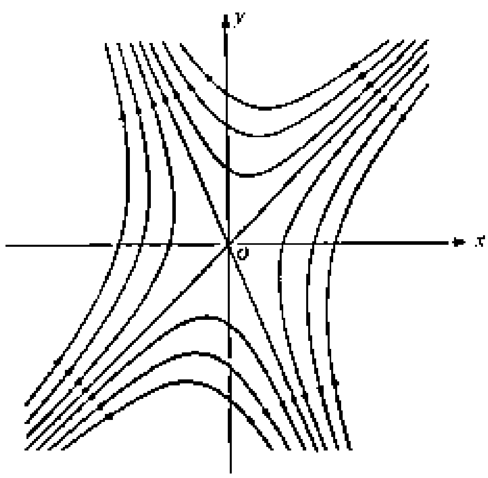

Figure (2.2.2)

2. In each of the four domains between any two of the four directions in (1) there are infinitely many trajectories which don't approach to $(0,0)$ as t tends to ∞ or t tends to $-\infty$. (Figure 2.2.2).

Definition (2.2.3):

The critical point $(0,0)$ of the system (2.2.1) is called *a spiral point* if there exists a neighborhood about $(0,0)$ such that every trajectory inside this neighborhood is defined for all $t>t_0$ and it approaches $(0,0)$ as t tends to ∞ or t tends to $-\infty$, and approaches $(0,0)$ in a spiral like manner, winding around $(0,0)$ an infinite number of times as t tends to ∞ or t tends to $-\infty$. (Figure 2.2.3).

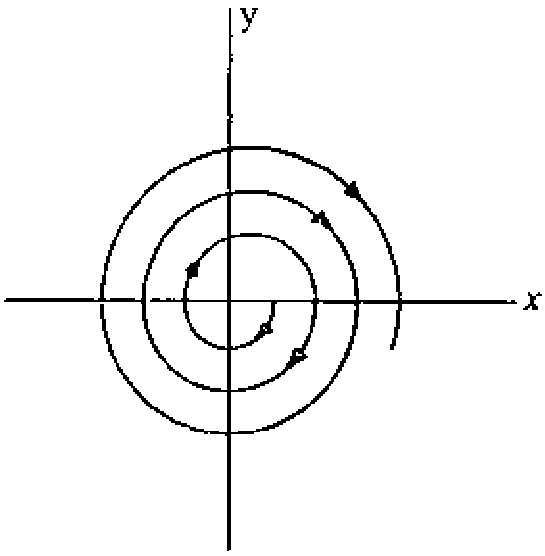

Figure (2.2.3)

Definition (2.2.4):

The critical point $(0,0)$ of the system (2.2.1) is called *improper node* if there exists a neighborhood about $(0,0)$ such that every trajectory inside a neighborhood is defined for all $t > t_0$, and it approaches and enters $(0,0)$ as t tends to ∞ or t tends to $-\infty$ and enters $(0,0)$ as t tends ∞ or t tends to $-\infty$. (Figure 2.2.4).

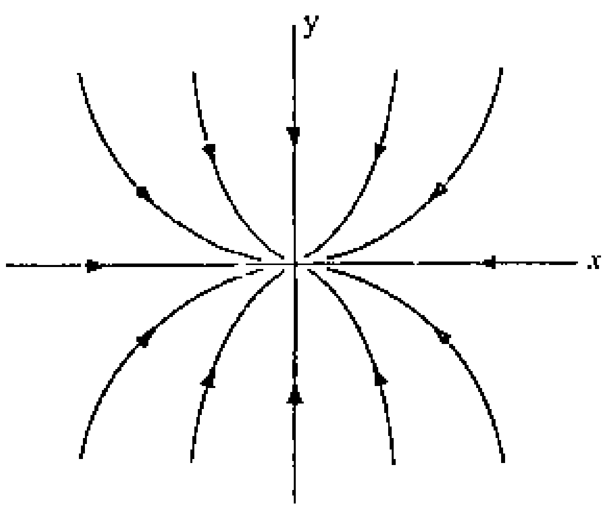

Figure (2.2.4)

Theorem (2.2.2) [1]:

1. If $\lambda_1 \neq \lambda_2$ and $\lambda_1 > 0$, $\lambda_2 > 0$ or ($\lambda_1 < 0$, $\lambda_2 < 0$), then the critical point $(0,0)$ of the system (2.2.1) is improper node.

2. If $\lambda_1 \neq \lambda_2$ and $\lambda_1 > 0$, $\lambda_2 < 0$ or ($\lambda_1 < 0$, $\lambda_2 > 0$), then the critical point $(0,0)$ of the system (2.2.1) is saddle point.

3. If $\lambda_1 = \lambda_2$, then the critical point $(0,0)$ of the system (2.2.1) is improper node if there exists one eigenvector, and proper node if there exist two corresponding eigenvectors.

4. If $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$ such that $\alpha \neq 0$, then the critical point $(0,0)$ of the system (2.2.1) is a spiral point, and if $\alpha = 0$, then the critical point $(0,0)$ of the system (2.2.1) is *a center*.

Theorem (2.2.3) [2]:

1. If $\lambda_2 < \lambda_1 < 0$, then the critical point of the system (2.2.1) is *asymptotically stable*, and the phase portrait is an *improper node*. (Figure 2.2.5).

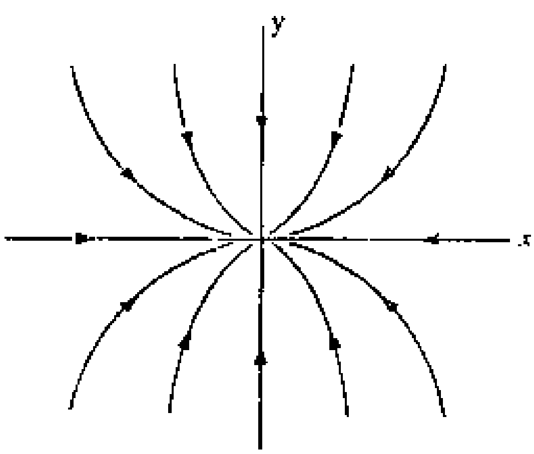

Figure (2.2.5)

2. If $\lambda_1 > \lambda_2 > 0$, then the critical point $(0,0)$ of the system (2.2.1) is *unstable*, the phase portrait is *improper node*. (Figure 2.2.6).

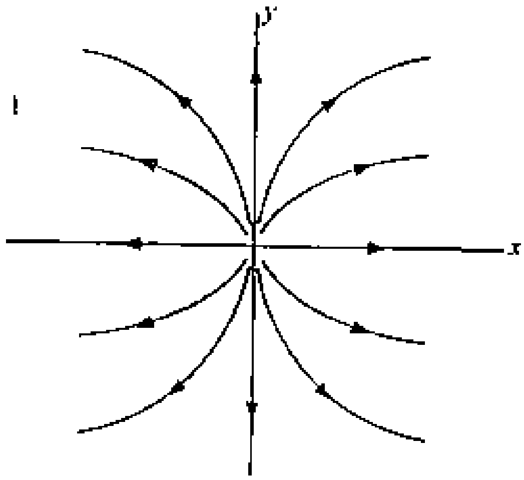

Figure (2.2.6)

3. If $\lambda_1 < 0$ and $\lambda_2 > 0$, then the critical point of the system (2.2.1) is *unstable*, the phase portrait is a *saddle point*. (Figure 2.2.7).

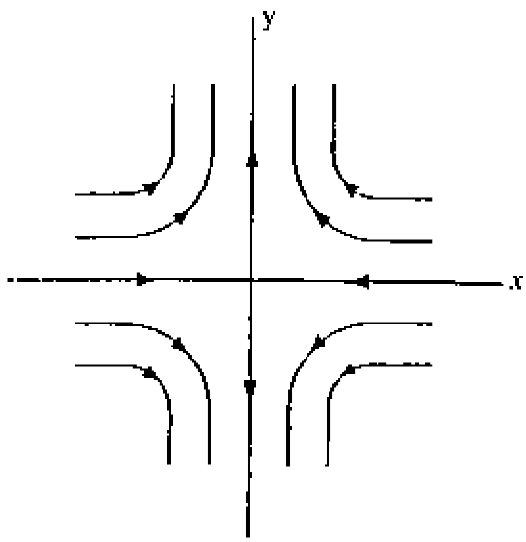

Figure (2.2.7)

4. If $\lambda_1 = 0$ and $\lambda_2 < 0$, then the critical point $(0,0)$ of the system (2.2.1) is *stable*, the phase portrait is a *proper node*. (Figure 2.2.8).

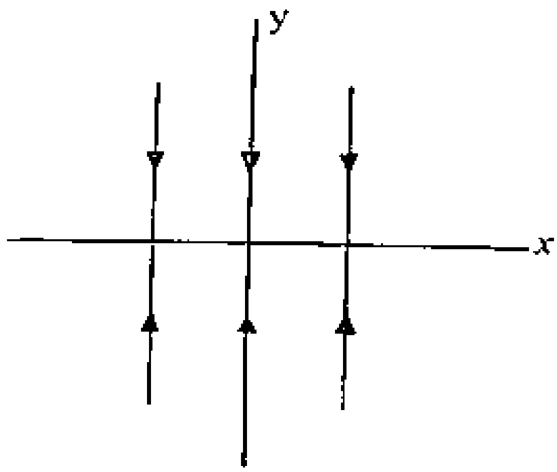

Figure (2.2.8)

5. If $\lambda_1 = 0$ and $\lambda_2 > 0$, then the critical point $(0,0)$ of the system (2.2.1) is *unstable*, the phase portrait is *proper node*. (Figure 2.2.9)

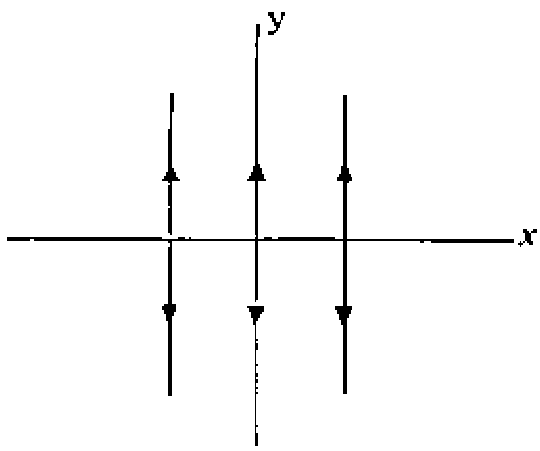

Figure (2.2.9)

Theorem (2.2.4) [2]:

If $\lambda_1 = \lambda_2$ we have two cases:

Case I: linearly dependent eigenvectors:

1. If $\lambda_1 = \lambda_2 < 0$, then the critical point $(0,0)$ of the system (2.2.1) is *asymptotically stable*, the phase portrait is *improper node*. (Figure 2.2.10).

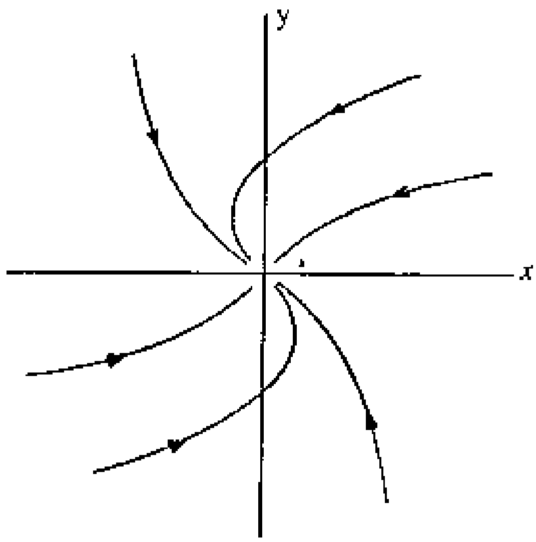

Figure (2.2.10)

2. If $\lambda_1 = \lambda_2 > 0$, then the critical point $(0,0)$ of the system (2.2.1) is *unstable*, the phase portrait is *improper node*. (Figure 2.2.11).

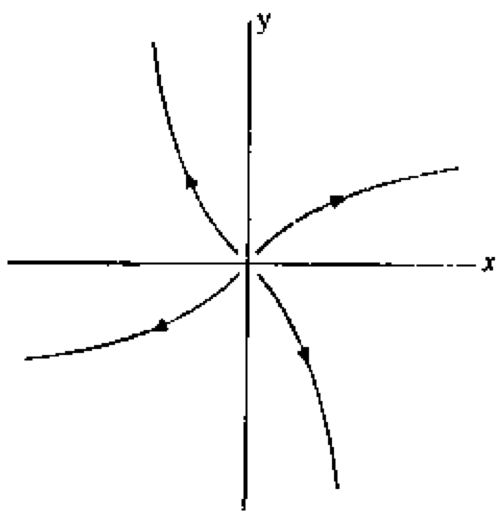

Figure (2.2.11)

3. If $\lambda_1 = \lambda_2 = 0$, then the critical point $(0,0)$ of the system (2.2.1) is *unstable*, the phase portrait is *a proper node*. (Figure 2.2.12).

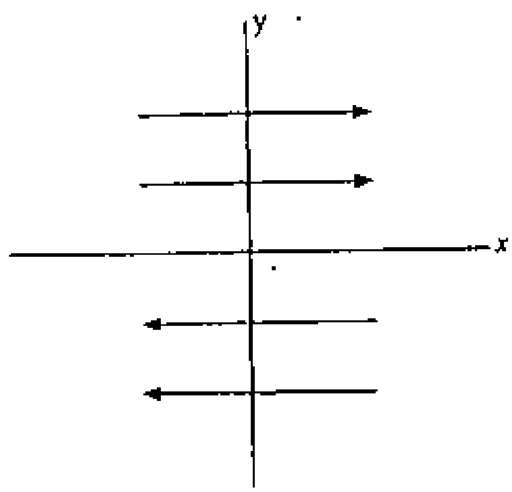

Figure (2.2.12)

Case II: linearly independent eigenvectors:

1. If $\lambda_1 = \lambda_2 < 0$, then the critical point of the system (2.2.1) is *asymptotically stable*, the phase portrait is *a proper node*. (Figure 2.2.13).

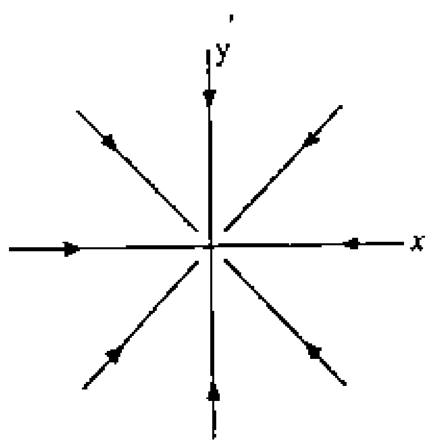

Figure (2.2.13)

2. If $\lambda_1 = \lambda_2 > 0$, then the critical point (0,0) of the system (2.2.1) is *unstable*, the phase pretrial is *a proper node*. (Figure 2.2.14).

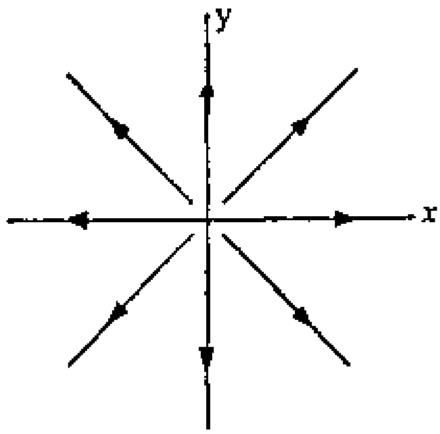

Figure (2.2.14)

3. If $\lambda_1 = \lambda_2 = 0$, then the critical point $(0,0)$ of the system (2.2.1) is *unstable*, the phase portrait is *a proper node*. (Figure 2.2.12).

Theorem (2.2.5) [2]:

Consider the system (2.2.1), where $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, then we have the following cases:

1. If $\alpha < 0$, $\beta > 0$, then the critical point $(0,0)$ of the system (2.2.1) is *asymptotically stable*, the phase portrait is *a spiral*. (Figure 2.2.15).

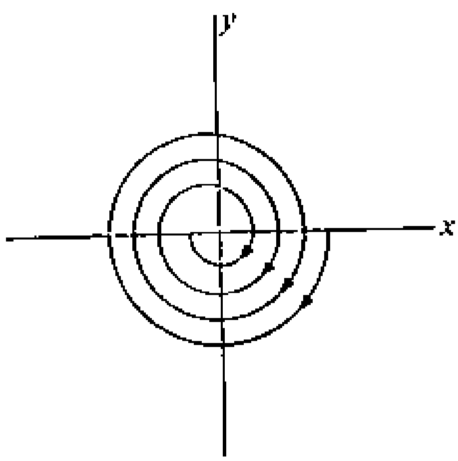

Figure (2.2.15)

2. If $\alpha > 0$, $\beta > 0$, then the critical point $(0,0)$ of the system (2.2.1) is *unstable*, the phase portrait is *a spiral*. (Figure 2.2.16).

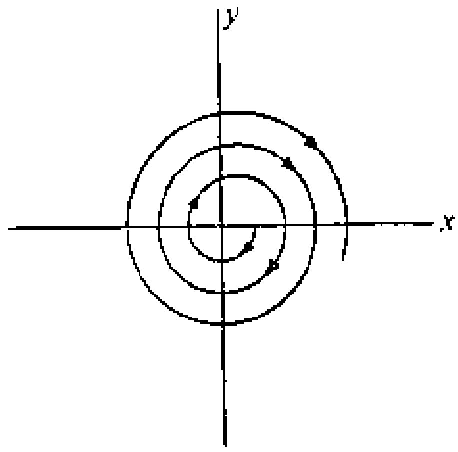

Figure (2.2.16)

3. If $\alpha = 0$, $\beta > 0$, then the critical point $(0,0)$ of the system (2.2.1) is *stable*, the phase portrait is *a center*. (Figure 2.2.17).

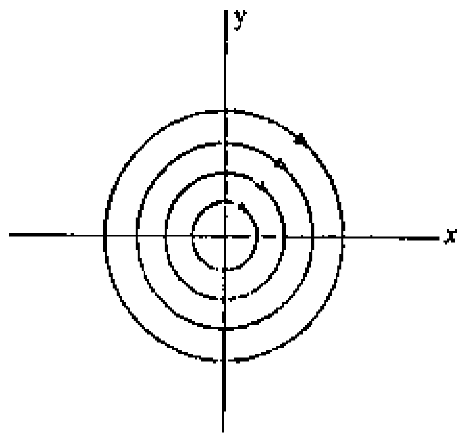

Figure (2.2.17)

4. If $\alpha < 0$, $\beta < 0$, then the critical point $(0,0)$ of the system (2.2.1) is *asymptotically stable*, and the phase portrait is a spiral. (Figure 2.2.15).
5. If $\alpha = 0$, $\beta < 0$, then the critical point $(0,0)$ of the system (2.2.1) is *stable*, and the phase portrait is *a center*. (Figure 2.2.17)
6. If $\alpha > 0$, $\beta < 0$, then the critical point $(0,0)$ of the system (2.2.1) is *unstable*, and the phase portrait is *a spiral*. (Figure 2.2.16)

Definition (2.2.5):

Consider the non linear system

$$\frac{dx}{dt} = P(x,y),$$
$$\frac{dy}{dt} = Q(x,y), \qquad (2.2.2)$$

where P and Q are real functions which have continuous first partial derivatives for all (x,y) and system (2.2.2) has an isolated critical point $(0,0)$.

If $P(x,y)$ and $Q(x,y)$ can be expanded in power series about $(0,0)$ such that

$$P(x,y) = P_x(0,0)\,x + P_y(0,0)\,y + R_1,$$
$$Q(x,y) = Q_x(0,0)\,x + Q_y(0,0)\,y + R_2,$$

where R_1 and R_2 are terms of order 2 or higher in x and y, such that

$$\lim_{(x,y)\to(0,0)} \frac{R_1}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{R_2}{\sqrt{x^2+y^2}} = 0,$$

then system (2.2.2) can be written as

$$x' = P_x(0,0)x + P_y(0,0)y + R_1\,,$$
$$y' = Q_x(0,0)x + Q_y(0,0)y + R_2\,, \qquad (2.2.3)$$

and for the system (2.2.3), the system of equations of first approximation is

$$\begin{cases} x^{\cdot} = P_x(0,0)x + P_y(0,0)y \ , \\ y^{\cdot} = Q_x(0,0)x + Q_y(0,0)y \ , \end{cases} \qquad (2.2.4)$$

Theorem (2.2.6) [1]:

Consider the system (2.2.3) where R_1 and R_2 have continuous first partial derivatives for all (x, y) such that

$$\lim_{(x,y)\to(0,0)} \frac{R_1}{\sqrt{x^2 + y^2}} = \lim_{(x,y)\to(0,0)} \frac{R_2}{\sqrt{x^2 + y^2}} = 0,$$

If the system of equations of first approximation (2.2.4), where the system (2.2.3) and the system (2.2.4) have an isolated critical point $(0,0)$, and λ_1 and λ_2 are *eigenvalues* of the system (2.2.4), such that:

1. If $\lambda_1 > 0$, $\lambda_2 > 0$, then the critical point $(0,0)$ of the system (2.2.2) is *a node.*
2. If $\lambda_1 > 0$, $\lambda_2 < 0$, then the critical point $(0,0)$ of the system (2.2.2) is *a point saddle.*
3. If $\lambda_1 = \lambda_2$, such that $a = d = 0$, $b = c \neq 0$, then the critical point $(0,0)$ of the system (2.2.2) is *a node.*
4. If $\lambda_1 = \alpha + i\beta$, and $\alpha \neq 0$, then the critical point $(0,0)$ of the system (2.2.2) is *a center.*

5. If $\lambda_1 = \alpha + i\beta$, and $\alpha = 0$, then the critical point $(0,0)$ of the system (2.2.2) maybe *a center* or *a spiral point.*
6. If $\lambda_1 = \lambda_2$, such that $P_x(0,0) = Q_y(0,0) \neq 0$ and $P_y(0,0) = Q_x(0,0) = 0$ then the critical point $(0,0)$ of the system (2.2.2) maybe *a node* or *a spiral point.*

Theorem (2.2.7) [1]:

Let λ_1 and λ_2 are *eigenvalues* of the system (2.2.4), and definition (2.2.5) hold: -

1. If $\lambda_1 < 0$, $\lambda_2 < 0$, then the critical point $(0,0)$ of the system (2.2.2) is *asymptotically stable.*
2. If $\lambda_1 > 0$, $\lambda_2 < 0$, then the critical point $(0,0)$ of the system (2.2.2) is *unstable.*
3. If, $\lambda_1 = \alpha + i\beta$ such that $\alpha < 0$, then the critical point $(0,0)$ of the system (2.2.2) is *asymptotically stable.*
4. If $\lambda_1 = \alpha + i\beta$, such that $\alpha > 0$, then the critical point $(0,0)$ of the system (2.2.2) is *unstable.*
5. If $\lambda_1 = \alpha + i\beta$, such that $\alpha = 0$, then theorem is not applicable.

Chapter Three

Stability by Liapunov Direct Method.

There is a large number of differential equations systems which are not easy to find solutions for them by simple methods, and even if solutions were found they could not be efficiently applied in studying the characteristics of stability.

The study of stability concepts through finding solutions as we have seen in chapter two was used by the scientist "*Liapunov*" and was named the first method of *Liapunov*, but later he found another more effective method which was called the direct method because it did not depend on previously knowing the solutions. The method in fact dealt directly with the differential systems through functions of special nature, which were named *Liapunov* functions.

These functions and this method will be the subject of this chapter.

Let us start with the following definitions.

Definition (3.1):

Let Ω be an open set in R^n containing the point $(0,0)$, and let $V(X) = V(x_1, x_2, ..., x_n)$, be valued continuous scalar function defined on $\Omega \subseteq R^n \to R$.

1. Let $V(X)$ be a scalar function such that $V(0) = 0$, $V(X) > 0$ for all $X \neq 0 \in \Omega$, then $V(X)$ is called *positive definite* on Ω.
2. Let $V(X)$ be a scalar function such that $V(0) = 0$, $V(X) < 0$ for all $X \neq 0 \in \Omega$, then $V(X)$ is called *negative definite* on Ω.

3. Let $V(X)$ be a scalar function such that $V(0)=0, V(X)\geq 0$ for all $X\neq 0 \in \Omega$, then $V(X)$ is called *positive semi definite* on Ω.
4. Let $V(X)$ be a scalar function such that $V(0)=0, V(X)\leq 0$ for all $X\neq 0 \in \Omega$, then $V(X)$ is called *negative semi definite* on Ω.

Example (3.1):

The function $V(X)=x_1^2+x_2^2$ is *positive definite* on R^2, but the function $V(X)=x_1^2$ is *positive semi definite* on R^2.

Definition (3.2):

The derivative of the function $V(X)$ with respect to the system.

$$\frac{dX}{dt}=F(X), \tag{3.1}$$

where $X=(x_1, x_2, ..., x_n)\in R^n$ is denoted by $V^*(X)$, and defined as

$$V^*(X)=grad\,V(X).F(X),$$

or

$$V^*(X)=(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, ..., \frac{\partial v}{\partial x_n})\ (f_1, f_2, ..., f_n)$$

$$V^*(X)=\frac{\partial v}{\partial x_1}\ f_1+\frac{\partial v}{\partial x_2}\ f_2+......+\frac{\partial v}{\partial x_n}\ f_n,$$

or by the chain rule

$$V^*(X)=\frac{dv}{dt}=\frac{\partial v}{\partial x_1}\frac{dx_1}{dt}+\frac{\partial v}{\partial x_2}\frac{dx_2}{dt}+.....+\frac{\partial v}{\partial x_n}\frac{dx_n}{dt}.$$

Definition (3.3):

A scalar function $V(X) \in \Omega$ is called *Liapunov function* for the system (3.1) if $V(X)$ satifies:

i) $V(X)$ is positive definite on Ω,

ii) $\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, \ldots, \frac{\partial v}{\partial x_n}$, exists and continuous on Ω,

iii) $\frac{dv}{dt}$ is negative semi definite on Ω.

Theorem (3.1) [3]:

If there exists a *Liapunov* function $V(x_1, x_2)$ for the system.

$$\frac{dx_1}{dt} = P(x_1, x_2),$$

$$\frac{dx_2}{dt} = Q(x_1, x_2), \tag{3.2}$$

on Ω, then the critical point $(0,0)$ of the system (3.2) is *stable.*

Theorem (3.2) [3]:

If there exists a *Liapunov* function $V(x_1, x_2)$ for the system (3.2) on Ω, and $\frac{dv}{dt} \le -\beta < 0$ for points out side of a neighborhood of $(0,0)$ (denoted by $N(0,0)$), then the critical point $(0,0)$ of the system (3.2) is *asymptotically stable.*

Proof:

Since $v(x_1, x_2)$ is define function, and bounded below by zero.

$$\therefore \lim_{t\to\infty} v(x_1, x_2) \text{ exists.}$$

Let

$$\lim_{t\to\infty} v(x_1, x_2) = L.$$

Case (i):-

$$L = 0 \quad \therefore \lim_{t \to \infty} (x_1(t), x_2(t)) = 0 = v(0,0).$$

This implies that $(x_1(t), x_2(t))$ tends to $(0,0)$ as t tends to infinity.

$\therefore$ by definition (2.1.5) the critical point $(0,0)$ of the system (3.2) is asymptotically stable.

Case (ii):-

$$L < 0.$$

Since $v(x_1, x_2)$ is positive definite.

This implies that this case is not possible .

Case (iii):-

$L > 0$ we have

$$\lim_{t \to \infty} v(x_1, x_2) = L > 0.$$

This implies that trajectory $(x_1(t), x_2(t))$ it can not be inside $N(0,0)$.

$$\therefore \frac{dv}{dt} \leq -\beta < 0.$$

$$\int_0^t dv \leq \int_0^t -\beta \, dt .$$

$$v(x_1(t), x_2(t)) - v(x_1(0), x_2(0)) \leq -\beta (t-0).$$

$$\therefore v(x_1(t), x_2(t)) \leq v(x_1(0), x_2(0)) - \beta t .$$

This implies that $v(x_1(t), x_2(t)) < 0$ for all $t \geq t_0$.

But it is a contradiction.

$\therefore L$ must be equal zero.

$\therefore$ The critical point $(0,0)$ of the system (3.2) is asymptotically stable.

Example (3.2):

Consider the differential equation

$$x^{\bullet\bullet} + x^{\bullet 3} + x = 0.$$

The *Liapunov* function is $v(x_1, x_2) = x_1^2 + x_2^2$, and the critical point $(0,0)$ is *stable*, but it is not *asymptotically stable*, since $\frac{dv}{dt} = -x_2^4 \le 0$.

Example (3.3):

Consider the system

$$x_1^{\bullet} = -x_2 - x_1^3,$$

$$x_2^{\bullet} = x_1 - x_2^3,$$

The *Liapunov* function is:

$$V(x_1, x_2) = x_1^2 + x_2^2.$$

The critical point $(0,0)$ of the system is *asymptotically stable*.

Example (3.4) [3]:

Consider the differential equation

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0, \qquad (3.3)$$

where $f(x)$ and $g(x)$ are continuous function on the given interval.

The differential equation (3.3) is called *Lienard differential equation* which is equivalent to the system.

$$\frac{dx}{dt} = y - F(x),$$
$$\frac{dy}{dt} = -g(x), \tag{3.4}$$

where $F(x) = \int_0^x f(s)\,ds$. Since $\frac{d^2x}{dt^2} = \frac{dy}{dt} - F'(x)\frac{dx}{dt}$, and $F'(x) = f(x)$

therefore $\frac{d^2x}{dt^2} = -g(x) - f(x)\frac{dx}{dt}$,

and we have $\frac{d^2x}{dt^2} + f(x_1)\frac{dx}{dt} + g(x) = 0$ which is equation (3.3).

The point $(x,y) = (0,0)$ is the only critical point of the system (3.4) if $g(0) = 0$.

The *Liapunov* function for the system (3.4) is

$$V(x,y) = \frac{y^2}{2} + G(x),$$

where $G(x) = \int_0^x g(s)\,ds$, $V(x,y) > 0$ for all $(x,y) \neq 0$, and $V(x,y)$ is positive definite, this implies that $\int_0^x g(s)\,ds > 0$, then $g(x) > 0$ *and* $x > 0$ or $g(x) < 0$ *and* $x < 0$ this implies that $xg(x) > 0$ for all $x \neq 0$.

Therefore $V(x,y) = \frac{y^2}{2} + G(x)$ is positive definite provided $xg(x) > 0$.

Now, we shall compute

$$\frac{dv}{dt}=\frac{\partial v}{\partial x}\frac{dx}{dt}+\frac{\partial v}{\partial y}\frac{dy}{dt}.$$

$$\frac{dv}{dt}=g(x)\,(y-F(x))+y\,(-g(x)).$$

$$\frac{dv}{dt}=-F(x)\;g(x).$$

We require to be $F(x)\,g(x)>0$, this implies that $x\,g(x)>0$ or $xF(x)>0$.

Therefore the critical point of the system (3.4) is asymptotically stable provided $g(0)=0$, $x\,f(x)>0$ and $x\,g(x)>0$.

Now we shall consider some methods to construct a *Liapunov* function for some linear systems.

Theorem (3.3) [2]:

A scalar function

$$V(X)=X^{T}BX=\sum_{i,j=1}^{n}b_{ij}\,x_i\,x_j, \qquad (3.5)$$

is called *a quadratic form* with the real (*nxn*) symmetric matrix $B=(b_{ij})=(b_{ji})$, $i,j=1,2,\ldots,n$.

$V(X)$ is *positive definite* if and only if for the matrix $B=(b_{ij})$

$$\det(B)=\det\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}>0,$$

Consider the autonomous system

$$X^{\bullet} = F(X) \; , \qquad (3.6)$$

where $F: R^n \rightarrow R^n$, $F(0) = 0$ for $X \neq 0$ in $N(0,0)$, and $F(X)$ is differentiable with respect to x_i.

Definition (3.4):

The real symmetric nxn matrix B is said to be

(i) *Positive definite* if the quadratic from $X^T BX$ is *positive definite.*

(ii) *Negative definite* if $-B$ is *positive definite.*

Definition (3.5):

The *Jacobian matrix* of the system (3.6) is define by

$$J(X) = \frac{\partial F}{\partial X} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix},$$

Theorem (3.4) [2]:

Consider a matrix $M(X) = J^T(X) + J(X)$, where $J^T(X)$ is the transpose of $J(X)$, and define a *Liapunov* function for the system (3.6) by $V(X) = F^T(X)\, F(X)$.

If a matrix $M(x)$ is negative definite in $N(0,0)$, then the critical point $(0,0)$ of the system (3.6) is *asymptotically stable.*

Proof:

$V(X) = F^T(X) \cdot F(X)$

$V(X) = F^T(X) . I . F(X)$

$\therefore V(X)$ is positive definite in $N(0,0)$, where I is the identity matrix.

$\therefore \frac{\partial v}{\partial F^T}$ and $\frac{\partial v}{\partial F}$ are exists and continuous on $\Omega \subseteq R^n$.

$\therefore V^{\cdot}(X) = (F^{\cdot}(X))^T . F(X) + F^T(X) . F^{\cdot}(X).$

Since $\frac{dF}{dt} = \frac{\partial F}{\partial X} . \frac{dX}{dt} = J(X) F(X).$

Therefore $V^{\cdot}(X) = F^T(X) J^T(X) F(X) + F^T(X) J(X) F(X).$

$V^{\cdot}(X) = F^T(X) (J^J(X) + J(X)) F(X).$

$V^{\cdot}(X) = F^T(X) M(X) F(X).$

We require $M(X)$ to be negative.

$\therefore$ The critical point $(0,0)$ of the system (3.6) is *asymptotically stable.*

This theorem is called *Krasovskii's method*, and it is not true for the *n-th* order $(n \geq 2)$ differential equation $x^{(n)} + g(x, x^{\cdot}, ..., x^{(n-1)}) = 0$.

Example (3.5):

Determine the stability of the zero solution of

$$x_1^{\cdot} = -x_1 ,$$

$$x_2^{\cdot} = x_1 - x_2 - x_2^3 .$$

For this system, $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $F(X) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}$,

Where $\quad f_1(x) = -x_1 ,$

$$f_2(x) = x_1 - x_2 - x_2^3 .$$

Therefore,

$$J(x) = \begin{bmatrix} -1 & 0 \\ 1 & -1-3x_2^3 \end{bmatrix},$$

and hence

$$M(x) = \begin{bmatrix} -2 & 1 \\ 1 & -2-6x_2^2 \end{bmatrix},$$

Since $M(x)$ is negative definite for all $x \in R^2$, krasovskii's method ensures that the zero solution of the given system is asymptotically stable.

Now we shall construct a *Liapunov* function for some linear system with constant coefficients.

Definition (3.6):

Two matrices A and B are said to be *similar* if these exists an invertible matrix T such that $A = T^{-1} B T$.

Theorem (3.5) [2]:

Similar matrices have the same characteristic polynomials.

Theorem (3.6) [2]:

If the *eigenvalues* of the matrix A are real and distinct, then there exists an invertible matrix T such that

$$T^{-1}AT = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

Remark:

The linear autonomous system

$$\frac{dX}{dt} = AX, \tag{3.7}$$

where X is an n-vector, A is nxn constant matrix, and it has real and distinct *eigenvalues* can be transformed by $X = TY$, (where T is a real constant non singular matrix) to the system

$$\frac{dY}{dt} = DY, \tag{3.8}$$

where $D = T^{-1}AT$.

Definition (3.7):

Let u, v be vectors in R^n, the function $<u,v>$ defined by $<u,v> = u^T v$ is called an *inner product* on R^n.

Theorem (3.7):

Let the matrix A of the system (3.7) be stable and its *eigenvalues* are real, distinct, and negative.

Let $V(Y) = \langle Y, BY \rangle$, (3.9)

be the *Liapunov* function for the system (3.8), where is a real nxn constant symmetric matrix, then

$$B = \begin{bmatrix} \frac{-1}{2\lambda_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{-1}{2\lambda_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & \frac{-1}{2\lambda_n} \end{bmatrix} > 0,$$

and *Liapunov* function in (3.9) is given by

$$V(Y) = \frac{-1}{2\lambda_1} y_1^2 - \frac{-1}{2\lambda_2} y_2^2 - \ldots\ldots - \frac{-1}{2\lambda_n} y_n^2 .$$

Proof:

Since $\lambda_1, \lambda_2, \ldots, \lambda_n$ are negative this implies that all the diagonal elements of D are negative therefore the system $Y' = DY$ is asymptotically stable, where $D = T^{-1}AT$.

Since $V(Y) = \langle \dot{Y}, BY \rangle$

$\therefore v(Y) = Y^T BY$ is positive definite.

The matrix B must be positive.

$$V^{\bullet}(Y) = Y^{\bullet T} BY + Y^T BY^{\bullet},$$

$$V^{\bullet}(Y) = < Y^{\bullet}, BY > + < Y, BY^{\bullet} >;$$

$$V^{\bullet}(Y) = < DY, BY > + < Y, B\,DY >,$$

$$V^{\bullet}(Y) = Y^T D^T BY + Y^T BDY,$$

$$V^{\bullet}(Y) = Y^T (D^T B + BD) < 0,$$

We require $V^{\bullet}(Y) < 0$, this implies that

$D^T B + BD = -I$, and we get

$$B = \begin{bmatrix} \frac{-1}{2\lambda_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{-1}{2\lambda_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & \frac{-1}{2\lambda_n} \end{bmatrix}$$

Since $\lambda_1, \lambda_2, \ldots, \lambda_n$ are negative.

Therefore $B > 0$ and $\quad V(Y) = Y^T BY = \sum_{i,j=1}^{n} b_{ij}\, y_i\, y_j$.

$$V(Y) = -\frac{1}{2\lambda_1} y_1^2 - \frac{1}{2\lambda_2} y_2^2 - \ldots\ldots - \frac{1}{2\lambda_n} y_n^2.$$

Example (3.6):

Construct a *Liapunov* function for $\frac{dX}{dt} = AX$,

where,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -20 & -9 \end{bmatrix}. \qquad (3.10)$$

The characteristic equation $\det(A - \lambda I) = 0$ has roots $\lambda_1 = -1$, $\lambda_2 = -2$, and $\lambda_3 = -6$. Then, it follows that

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -6 \\ 36 \end{bmatrix}.$$

Therefore,

$$T = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -6 \\ 1 & 4 & 36 \end{bmatrix}.$$

It can be easily shown that

$$T^{-1}AT = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{bmatrix}.$$

The transformation $X = TY$ reduces (3.10) to

$$\frac{dY}{dt} = DY, \qquad (3.11)$$

where $D = T^{-1}AT$.

To find a *Liapunov* function for system (3.11), we look for a matrix B such that

$D^T B + BD = -I$, we get

$$B = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{12} \end{bmatrix}.$$

Thus, the *Liapunov* function for (3.11) is

$$V(Y) = \langle Y, BY \rangle = \frac{1}{2} y_1^2 + \frac{1}{4} y_2^2 + \frac{1}{12} y_3^2.$$

To get a *Liapunov* function for (3.10), we transform variable Y back into the variable X.

REFERENCES

[1] Shepley L. Ross, Differential Equations, John Wiley, Sons, New York, 1984.

[2] M. Rama Mohana Rao, Ordinary Differential Equations, Affiliated East- West Press Pvt Ltd, New Delhi- Madras, 1980.

[3] Paul D. Ritget, Nicholas J. Rosse, Differential Equations With Applications, McGraw- Hill Book Company, New York, 1968.

[4] M. Braun, Differentianl Equations and Their Applications, Springer- Verlag, New York Inc, 1975.

[5] Earl. A Coddington and Norman Levinson, Theory of Ordinary Differentail Equations, McGraw- Hill, New York, 1955.

[6] M. L. Krasnov, A. I Kiselev, and G. I. Makarenko, Functions of Complex Variable, Operational Calculus, and Stability Theory, Translated from the Russian by Eugene Yankovsky, Mir Publishers, Moscow, 1984.

[7] H. K. Wilson, Ordinary Differential Equations, Addison- Wesley Publishing Company, 1971.

[8] Raimond A. Strubel, Non Linear Differential Equations, McGraw- Hill Book Company, New York, 1962.

[9] Wit old Hurewicz, Lectures on Ordinary Differential Equations, Massachusetts Institute of Technology, 1958.

[10] L. El sgolts, Differential Equations and The Calculus of Variations, Translated form the Russian by George Yankovsky, Mir Publishers, Moscow, 1973.

[11] Dennis G. Zill, A First Course In Differatial Equations With Modeling Applications, Brooks / Col. Publishing Company, 1997.

مقدمة

يتناول هذا البحث دراسة بعض سلوك الاستقرار للمعادلات التفاضلية الذاتية الخطية والغير خطية والتي نجد لها تطبيقات كثيرة في مجالات عديدة .

وقد كانت نقطة الانطلاقة لهذا المفهوم رسالة الماجستير للعالم الروسي ليابونوف "المسألة العامة لاستقرار الحركة" والتي نشرت بالروسية عام 1907 . وتتابعت بعدها الدراسات والبحوث والمحاولات التي كانت تعالج في كل مرة مشكلة ما .

إن الهدف الرئيسي لهذه الدراسة تبسيط بعض المفاهيم وتقديمها في إطار سهل وميسر وذلك حتى تمكن الدارسين من استيعاب الأسس الجوهرية لمثل هذه المفاهيم وقد اخترنا مواضيع معينة من مرجع أو أكثر اكتسبت صفة الجمود وقدمناها في أسلوب مناسب .

ولقد عالجنا هذه المفاهيم برؤية تحليلية مبتعدين بذلك كل البعد عن الرؤية الهندسية لهذه المفاهيم .

خُصّص الفصل الأول لتقديم بعض التعاريف والنظريات التي نحتاجها في الفصول اللاحقة، وخلال هذا الفصل تم تقديم وعرض برهان نظرية الوجود والوحدانية لنظام المعادلات التفاضلية العادية .

يتعامل الفصل الثاني مع بعض مفاهيم الاستقرار والاستقرار التقاربي وكان التركيز الرئيسي في هذا الفصل دراسة العلاقة بين الحل الحرج وباقي الحلول الأخرى لبعض الأنظمة التفاضلية .

واستمراراً في الرؤية التحليلية لمفاهيم الاستقرار تناولنا في الفصل الثالث الطريقة المباشرة للعالم ليابونوف وذلك بعرض براهين بعض النظريات التي تعالج هذا المفهوم .

وفي نهاية الفصل تم تقديم بعض الطرق التي يمكن من خلالها اشتقاق الدوال الليابونوفية لبعض الأنظمة التفاضلية .

الجماهيرية العربية الليبية
الشعبية الاشتراكية العظمى
شعبية سرت
جامعة التحدي

ان الدراسة ليست غاية في حد ذاتها
وانما هي حلق الإنسان نموذجي الجديد

G.S.P.L.A.J.
AL - TAHDI UNIVERSITY

التاريخ : ....
الموافق : ....17 / 4 / 2009 ف........

الرقم الاشاري : ....

كلية العلوم
قسم الرياضيات

عنوان البحث

((بعض سلوك الإستقرار للمعادلات التفاضلية الذاتية الخطية وغير خطية))

سرت / ليبيا

مقدمة من الطالب

فرج محمد عبد الله نجم

**لجنة المناقشة

الدكتور. أبو القاسم علي محمد الذرعاني ....2009 / 4 / 17 ف
(مشرف الرسالة)

(ممتحن خارجي)
الدكتور . فتح الله صالح المستيري ....

(ممتحن داخلي)
الدكتور . نبيل زكي فريد ....

يعتمد :-
د. محمد علي سالم الفرجاني
أمين اللجنة الشعبية لكلية العلوم

كلية العلوم

جامعة التحدي

قسم الرياضيات

بعض سلوك الاستقرار للانظمة الذاتية الخطية
والغير خطية

دراسة مقدمة من الطالب / **فرج محمد عبد الله نجم**

لأستكمال المقررات المطلوبة لنيل درجة
الماجستير في الرياضيات

تحت إشراف الدكتور / **با القاسم على محمد**

سرت – ليبيا

2005