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Study of Mathematical Morphology for Decision Tables – Using Rough Set Theory

THIS THESIS SUBMITTED AS PARTIAL FULLFILLMENTS FOR THE REQUIRMENTS OF THE MASTER DEGREE OF SCIENCE IN MATHEMATICS

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بسم الله الرحمن الرحيم

﴿ الْحَمْدُ لِلَّهِ الَّذِي أَنزَلَ عَلَى عَبْدِهِ الْكِتَابَ وَلَمْ يَجْعَل لَّهُ عِوَجًا ۞ قَيِّمًا لِّيُنذِرَ بَأْسًا شَدِيدًا مِّن لَّدُنْهُ وَيُبَشِّرَ الْمُؤْمِنِينَ الَّذِينَ يَعْمَلُونَ الصَّالِحَاتِ أَنَّ لَهُمْ أَجْرًا حَسَنًا ﴾

صدق الله العظيم

(الكهف 1 – 2)

Dedication

To my family

Father, Mother, Brothers and

Sisters

Also to my friends

Ahmed

ACKNOWLEDGMENT

I praise Allah Almighty; I must as well express my deep gratitude, to my family and to my friends for all the support and encouragement they extended to me, throughout my study.

I would like to express my deeper gratitude and thanks to my supervisor Dr. Ibrahim Abdalla Tentush , for having faith in my ability . I have great respect for his knowledge and criticism that were essential to me and for his invaluable assistance patient guidance and constant encouragement during the preparation of thesis.

THE CONTENTS

INTRODUCTION

In many application there is a need for efficient methods of data filtering even if then is no any preassumed geometrical structure in the set of (condition) attributes on the basis of which the decisions should be taken.

This situation often occurs when experimental data are recorded in decision tables. The problem arises of providing a frame work for data filtering based on non-geometric idea.

In this study we will see who to extract from local relations among data some near-to-functional relations. From these relations we get the so called approximation functions. in this study we will see a special mechanism for applying of these approximation functions to data in order to produce their appropriate filtration.

The thesis is divided to four chapters, chapter zero which gives some concepts in knowledge theory. Chapter one which gives some concepts in rough set theory. Chapter two which gives some concepts in information system. Chapter three which presents some ideas related to data filtration based on rough set approach. Chapter four concerns with mathematical morphology.

Abstract

We study a method called analytical morphology for data filtering. The method was created on the basis of some ideas of rough set theory and mathematical morphology. Mathematical morphology makes on essential use of geometric structure of objects.
While the aim of the introduced method in to provide tools for data filtering where is no directly available geometric structure in the set of data.

CHAPTER ZERO

Back Ground Material

Knowledge theory0.1

We present here a concise introduction, based on literature to the notation of knowledge, our approach in that of rough set theory as proposed in [6].

Introduction

Knowledge theory has a long and rich history for understanding, representing, and manipulating knowledge. There is a variety of opinions and approaches in this area.

One can understand knowledge as a body of an information about some parts of reality, which form the domain of interest. But this definition fails to meet precision standards and on closer inspection has multiple meaning tending to mean one of several things depending on the context and the area of interest.

We well give the formal definition of term " knowledge " proposed by Pawlak [6] and some of its basic properties.

The concept of knowledge presented here is general enough to cover various understanding of this concept in the current literature.

We advocate here a rough set concept as a theoretical fromwork for discussion about knowledge .

Definition 0.1.1

Let U be a set of objects we are interested in such that $\{ U \} \geq 1$ (the universe).

Definition 0.1.2

Any subset $X \leq U$ of the universe will be called (a concept or a category) in U.

Definition 0.1.3

Any family $\{X_i\}_{i=1}^{i=n}$ of concepts in U will be referred to as an abstract knowledge (or in a short knowledge) . For formal reason we can also admit the null set $\varnothing$ as a category.

Definition 0.1.4

Any knowledge $\{X_i\}_{i=1}^{i=n}$ of a certain universe U such that

$Xi \subseteq U$, $X_i \neq \varnothing$ $X_i \cap X_j = \varnothing$ for $i \neq j$ $i,j = 1, 2, \ldots n$

And $\bigcup X_i = U$ will form a partition of U and will be called a classification of U. Since we usually deal, not with a single classification.

But with some families of classifications over U , we will submit the following definition.

Definition 0.1.5

Any family $\{C_i\}_{i=1}^{n}$ of classification over U, will be called a knowledge base over U. thus knowledge base represents a variety of basic classification skills (e.g. according to color, shape and size) of an "intelligent" agent or group of agents (e.g. cars, toys).

Now since the concept of classification (partitions) and equivalence relation are mutually interchangeable and relation are easier to deal, with we will aften use equivalence relations. Let us now give some necessary definition using equivalence relations.

Definition 0.1.6

1. $U/R = \{ [\chi]_R : \chi \in U \}$ is that family of all equivalence classes of the equivalence relation R. i.e U/R is a classification of U and $[\chi]_R$ (the equivalence class of χ) is a concept or a category in R containing $\chi \in U$.

2. Let U be a universe such that $\neq |U| \geq 1$.

   let R = { R : R is an equivalence relation on U }. Then we define a relation system $K = (U, R)$ as a knowledge base.

Proposition 0.1.1

If $IP \subseteq IR$, $IP \neq \varnothing$, where IR is a family of equivalence relations over a non-empty finite universe U, then $\bigcap_{R \in IR} R$ is (an equivalence relation) and denoted by $IND(IP)$, i.e. $IND(IP) = \bigcap_{R \in IP} IP$, and we call it an indiscrnibility relation over P.

Moreover,

$$[\chi]_{IND(P)} = \bigcap_{R \in P} [\chi]_R$$

Proof

1. Reflexivity : since $\forall \; R \in IP \; (\chi, \chi) \in R$,

   $$(\chi,\chi) \in \bigcap_{R \in IP} R.$$

2. Symmetry : $(\chi,\gamma) \in \bigcap IP$ implies $(\chi , y) \in R$ for all $R \in IP$,

   But each R is symmetric so $(y,\chi) \in R$

   For all $R \in IP$ i.e. $(y,\chi) \in \bigcap IP$.

3. Transitivity : let (χ,y), $(y, z) \in \bigcap IP$ which implies

   $\forall R \in IP \; (\chi, y), (y, z) \in R$, but each
   R is transitive, therefore $(\chi, y) \in \bigcap IP$.

Definition 0.1.7

1. The family $U/IND(IP)$ will be called the P-basic knowledge about U in $K= (U,IP)$.

2. Equivalence classes of $IND(IP)$ are called basic concepts (categories) of knowledge IP.

   In particular if $Q \in IR$, then Q will be called a Q - elementary knowledge (about U in K), and equivalence classes of Q are referred to as Q - elementary concepts of knowledge IR.

3. We define the minimal set of all equivalence relations defined in K, to be

$$IND(K) = \{ IND(IP) : \phi \neq IP \subseteq IR \}.$$

Example 0.1.1

Let $U = \{ \chi_1, \chi_2, \chi_3, \chi_4, \chi_5 \}$

Where χ_i is a student in Tahddi university at Libya, for each i = 1,.......5. these students have different sex, nationality and specialty. Suppose U is a classification according to sex, nationality and specialty for example as shown below

χ_1, χ_3, χ_5 female
$\chi_2, \chi_4,$ male
} According to sex

χ_1 , χ_3 Polish
χ_2 , χ_4 Libyan
χ_5 German
} According to nationality

χ_1 , χ_2	mathematics	According to specialty
χ_3	physics	
χ_4 , χ_5	chemistry	

By these three classification we define three equivalence relation R_1, R_2, R_3 respectively having the following equivalence classes .

$$U/R_1 = \{ \{\chi_1, \chi_3, \chi_5\}, \{\chi_2, \chi_4\}\}$$

$$U/R_2 = \{ \{\chi_1, \chi_3\}, \{\chi_2, \chi_4\}, \{\chi_5\} \}$$

$$U/R_3 = \{ \{\chi_1, \chi_2\}, \{\chi_3\}, \{\chi_4, \chi_5\} \}$$

These are elementary concepts in the knowledge base

$$K = (U , \{ R_1, R_2, R_3 \}).$$

Examples of basic concepts

$$\{ \chi_1, \chi_3, \chi_5\} \cap \{ \chi_1, \chi_3\} = \{ \chi_1, \chi_3\}$$

This set is $\{ R_1, R_2 \}$- basic concept female and Polish. The set

$$\{ \chi_1, \chi_3, \chi_5\} \cap \{ \chi_1, \chi_3\} \cap \{ \chi_1, \chi_2\} = \{ \chi_1\}$$

Is $\{ R_1, R_2, R_3 \}$ - basic category female, Polish and mathematics.

Definition 0.1.8

1. Let K = (U , IP) and K' = (U, Q) be two knowledge bases. We will say that K and K' (IP and Q) are equivalent, denoted $K \cong K'$, $(IP \cong Q)$, if $IND(P) = IND(Q)$ or $U/IP = U/Q$

2. Let $K = (U, IP)$ and $K' = (U,Q)$ be two knowledge bases. If $IND(IP) \subset IND(Q)$,

   We say that knowledge IP (knowledge base K) is finer than knowledge Q (knowledge base K') or Q is coarser than P.

3. We will say that if IP is finer than Q then IP is a specialization of Q and Q is a generalization of IP.

Example 0.1.2

Let $U = \{ \chi_1, \chi_2, \chi_3, \chi_4, \chi_5\}$, let

$$R_1 = \{(\chi_1, \chi_1), (\chi_2, \chi_2), (\chi_3, \chi_3), (\chi_4, \chi_4), (\chi_5, \chi_5), (\chi_1, \chi_2), (\chi_2, \chi_1), (\chi_3, \chi_4), (\chi_4, \chi_3) \}$$

$$R_2 = \{ (\chi_1, \chi_1), (\chi_2, \chi_2), (\chi_3, \chi_3), (\chi_4, \chi_4), (\chi_5, \chi_5), (\chi_1, \chi_3), (\chi_3, \chi_1), (\chi_2, \chi_4), (\chi_4, \chi_2)(\chi_2, \chi_3), (\chi_3, \chi_2), (\chi_4, \chi_5), (\chi_5, \chi_4) \}$$

$$R_3 = \{ (\chi_1, \chi_1), (\chi_2, \chi_2), (\chi_3, \chi_3), (\chi_4, \chi_4), (\chi_5, \chi_5), (\chi_1, \chi_5), (\chi_5, \chi_1), (\chi_1, \chi_2), (\chi_2, \chi_1)(\chi_5, \chi_2), (\chi_2, \chi_5) \}$$

$$R_4 = \{ (\chi_1, \chi_1), (\chi_2, \chi_2), (\chi_3, \chi_3), (\chi_4, \chi_4), (\chi_5, \chi_5), (\chi_1, \chi_3), (\chi_3, \chi_1), (\chi_2, \chi_5), (\chi_5, \chi_2) \}$$

$$R_5 = \{ (\chi_1, \chi_1), (\chi_2, \chi_2), (\chi_3, \chi_3), (\chi_4, \chi_4), (\chi_5, \chi_5), (\chi_2, \chi_3), (\chi_3, \chi_2) \}$$

$$R_6 = \{ (\chi_1, \chi_1), (\chi_2, \chi_2), (\chi_3, \chi_3), (\chi_4, \chi_4), (\chi_5, \chi_5), (\chi_1, \chi_2),$$

$$(\chi_2, \chi_1), (\chi_1, \chi_3), (\chi_3, \chi_1)(\chi_1, \chi_4), (\chi_4, \chi_1), (\chi_2, \chi_3),$$

$$(\chi_3, \chi_2)(\chi_2, \chi_4), (\chi_4, \chi_2), (\chi_3, \chi_4), (\chi_4, \chi_3) \}$$

$$R_7 = \{ (\chi_1, \chi_1), (\chi_2, \chi_2), (\chi_3, \chi_3), (\chi_4, \chi_4), (\chi_5, \chi_5)(\chi_1, \chi_4), (\chi_4, \chi_1),$$

$$(\chi_2, \chi_5), (\chi_5, \chi_2),(\chi_2, \chi_3), (\chi_3, \chi_2)(\chi_5, \chi_3), (\chi_3, \chi_5) \}$$

$$U/R_1 = \{\{\chi_1, \chi_2\}, \{\chi_3, \chi_4\}, \{\chi_5\}\}$$

$$U/R_2 = \{\{\chi_1, \chi_3\},\{\chi_2, \chi_4, \chi_5\}\}$$

$$U/R_3 = \{\{\chi_1, \chi_2, \chi_5\}, \{\chi_3\}, \{\chi_4\}\}$$

$$U/R_4 = \{\{\chi_1, \chi_3\},\{\chi_2, \chi_5\}, \{\chi_4\}\}$$

$$U/R_5 = \{\{\chi_1\},\{\chi_2, \chi_3\}, \{\chi_4\}, \{\chi_5\}\}$$

$$U/R_6 = \{\{\chi_1, \chi_2, \chi_3, \chi_4\}, \{\chi_5\}\}$$

$$U/R_7 = \{\{\chi_1, \chi_4\}, \{\chi_2, \chi_5\}, \{\chi_3\}\}$$

Now take

$IP = \{R_1, R_2, R_3, R_4\}$ and $Q = \{R_5, R_6, R_7\}$

$$IND(IP) = R_1 \cap R_2 \cap R_3 \cap R_4$$

$$= \{(\chi_1, \chi_1), (\chi_2, \chi_2), (\chi_3, \chi_3), (\chi_4, \chi_4), (\chi_5, \chi_5)\}$$

$IND(Q) = R_5 \cap R_6 \cap R_7$

$$= \{(\chi_1, \chi_1), (\chi_2, \chi_2), (\chi_3, \chi_3), (\chi_4, \chi_4), (\chi_5, \chi_5), (\chi_2, \chi_3), (\chi_3, \chi_2)\}$$

It is clear that $IND(IP) \subset IND(Q)$

Therefore *IP* is a specialization of *Q* or *Q* is a generalization of *IP*.

Boolean Algebra 0.2

Definition 0.2.1

A Boolean Algebra is a set B together with two binary operations $\vee$ and $\wedge$ on *B* such that each of the following axioms is satisfied (for all $a,b,c \in B$) :

B_1 . commutative laws .

$$a \vee b = b \vee a \quad , \quad a \wedge b = b \wedge a,$$

B_2 . associative laws .

$$a \vee (b \vee c) = (a \vee b) \vee c \quad , \quad a \wedge (b \wedge c) = (a \wedge b) \wedge c,$$

B_3 . distributive laws .

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c),$$

B_4 . existence of zero and unity there are element 0 and 1 in B such that

$a \vee 0 = a,$ $a \wedge 1 = a,$

B_5 . existence of complements .

for each a in B there is an element a` in B such that

$a \vee a` = 1,$ and $a \wedge a` = 0,$

Theorem 0.2.1

If B is a Boolean algebra and $a,b,c \in B$, then:

1. $a \vee a = a,$ $a \wedge a = a,$
2. $a \vee 1 = 1,$ $a \wedge 0 = 0,$
3. $a \vee (a \wedge b) = a,$ $a \wedge (a \vee b) = a,$

proof :

1. $a = a \vee 0$ B_4

$= a \vee (a \wedge a`)$ B_5

$= (a \vee a) \wedge (a \vee a`)$ B_3

$= (a \vee a) \wedge 1$ B_5

$= (a \vee a)$ B_4

and

$a = a \wedge 1$ B_4

$= a \wedge (a \vee a`)$ B_5

$= (a \wedge a) \vee (a \wedge a`)$ B_3

$= (a \wedge a) \vee 0$ B_5

$= (a \wedge a)$ B_4

2. $a\vee 1 = (a\vee 1)\wedge 1$ B_4

$= (a\vee 1)\wedge(a\vee a')$ B_5

$= a\vee(1\wedge a')$ B_3

$= a\vee a'$ B_4

$= 1$ B_5

and

$a\wedge 0 = (a\wedge 0)\vee 0$ B_4

$= (a\wedge 0)\vee(a\wedge a')$ B_5

$= a\wedge(0\vee a')$ B_3

$= a\wedge a'$ B_4

$= 0$ B_5

3. $a\vee(a\wedge b) = (a\wedge 1)\vee(a\wedge b)$ B_4

$= a\wedge(1\vee b)$ B_3

$= a\wedge 1$ (2)

$= a$ B_4

and

$a\wedge(a\vee b) = (a\vee 0)\wedge(a\vee b)$ B_4

$= a\vee(0\wedge b)$ B_3

$= a\vee 0$ (2)

$= a$ B_4

Remark 0.2.1

The laws (1) and (2) of the above theorem colled respectively the idempotent and absorption laws .

Definition 0.2.2

Any expression as $a \vee a'$, $a \vee b'$, $[\, a \wedge (b \vee c') \,] \vee (\, a' \wedge b' \wedge c)$ consisting of combinations by $\vee$ and $\wedge$ of a finite number of elements of a Boolean algebra B will be called a Boolean function f .

Definition 0.2.3

An implicant of a Boolean function f is any conjunction of variables, and a prime implicant is a minimal implicant .

Example 0.2.1

assume that two Boolean function f_1 and f_2 are defined to be

$$f_1 = (\, a \vee b \vee c \,) \wedge (\, a \vee b \,) \text{ and } f_2 = (\, a \vee b \vee c \,) \wedge (\, a \vee d \,)$$

The laws of a Boolean algebra can be used to simplify this expression as follows :

$$f_1 = (\, a \vee b \vee c \,) \wedge (\, a \vee b \,)$$

$$= [\, a \wedge (a \vee b)] \vee [b \wedge (a \vee b)] \vee [\, c \wedge (a \vee b)]$$

$$= a \vee [b \wedge (b \vee a)] \vee [c \wedge (b \vee a)]$$

$$= a \vee b \vee [(\, c \wedge b) \vee (\, c \wedge a \,)]$$

$$= a \vee [b \vee (\, b \wedge c) \vee (\, a \wedge c \,)$$

$= (a \vee b) \vee (a \wedge c)$

$= (b \vee a) \vee (a \wedge c) = b \vee (a \vee (a \wedge c))$

$= b \vee a = a \vee b$

and

$f_2 = (a \vee b \vee c) \wedge (a \vee d)$

$= [a \wedge (a \vee d)] \vee [b \wedge (a \vee d)] \vee [c \wedge (a \vee b)]$

$= a \vee [(b \wedge a) \vee (b \wedge d)] \vee [c \wedge (a \vee d)]$

$= a \vee [(a \wedge b) \vee \{b \wedge d\}] \vee [(c \wedge a) \vee (c \wedge d)]$

$= [a \vee (a \wedge b)] \vee (b \wedge d) \vee [(c \wedge a) \vee (c \wedge d)]$

$= a \vee [(b \wedge d) \vee (a \wedge c)] \vee (c \wedge d)$

$= [a \vee (a \wedge c)] \vee (b \wedge d) \vee (c \wedge d)$

$= a \vee (b \wedge d) \vee (c \wedge d)$

Hence, there are two and three prime implicants to the functions f_1 and f_2 respectively , namely :

p-imp $(f_1) = \{ \{a\}, \{b\} \}$ and

p-imp $(f_2) = \{ \{a\}, \{b,d\}, \{c,d\} \}$.

Chapter one
Rough Sets

Imprecise categories
Approximations and rough sets

Introduction

We introduce the idea of rough sets in order to use it as an approach for defining approximately some categories (subsets of the finite universe U), which may not be defined in a give knowledge base.

In other words, we want to address here the central point of the approach, the vague categories, the idea of the rough set was proposed by Pawlak [6]

Rough sets 1.1

Definition 1.1.1

Let $X \subseteq U$, and R be an equivalence relation then we say that X is R-definable (or R-exact set), if X is the union of some R-basic categories, otherwise X is R-undefinable (or R-inexact , or R-rough set).

Approximation of set 1.2

Definition 1.2.1

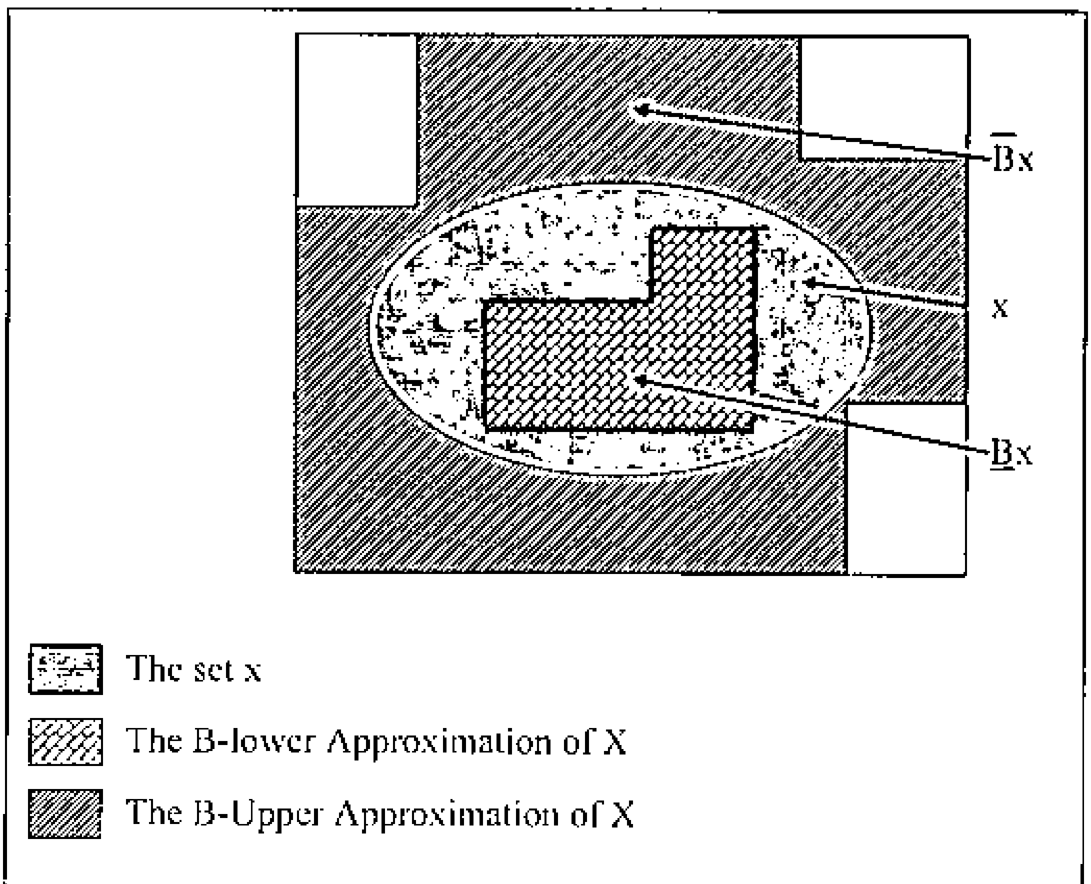
Let U be a universe , X be a concept (category) , $K = (U , R)$ be relation system and $B \in IND(K)$ then we define :

$\underline{B}X = \bigcup \{ [\chi]_B : [\chi]_B \subseteq X \}$ and

$\overline{B}X = \bigcup \{ [\chi]_B : [\chi]_B \cap X \neq \phi \}$

The former set is the *B*-lower approximation of X and the latter set is the *B*-upper approximation of X.

Notice that : $\underline{B}X \subseteq X \subseteq \overline{B}X$ and this double inclusion provides a description of X in terms *B*.



Definition 1.2.2

The set $BN_B(X) = \overline{B}X - \underline{B}X$ will be called the B-boundary of X.

Remark 1.2.1

The set $\underline{B}X$ is the set of all elements from *U* which can be with certainty classified as elements of X in the knowledge R.

The set $\overline{B}X$ is the set of all elements from *U* which can be possibly classified as elements of X, employing knowledge *R*. The set BN_B (X) is the set of elements which cannot be classified either to X or -X having knowledge R.

Definition 1.2.3

$POS_B(X) = \underline{B}X$, B- positive region of X.

The positive region $POS_B(X)$ or the lower approximation of X is the collection of those objects which can be classified with full certainty as members of the set X , using knowledge R .

Definition 1.2.4

$NEG_B(X) = U - \overline{B}X$, B- negative region of X.

The negative region $NEG_B(X)$ is the collection of objects with which it can be determined with out any ambiguity employing knowledge R. that they, do not belong to the set X, that is the belong to the complement of X.

Proposition 1.2.1

1. X is R- definable (exact) iff $\underline{B}X = \overline{B}X$
2. X is rough with respect to B iff $\underline{B}X \neq \overline{B}X$

Proof (2)

Follows from 1 and X is the exact iff

$X = \bigcup\{ [\chi]_B : X\in U \} \Leftrightarrow \quad X = \{ \chi \in U : [\chi]_B \subseteq X \}$

$\Leftrightarrow \quad X = \{ \chi \in U : [\chi]_B \cap X = [\chi]_B \}$

$\Leftrightarrow \quad X = \underline{B}X = \overline{B}X$

Example (4) 1.2.1

Assume the knowledge base k= (U,R), where $U = \{ \chi_1, \chi_2, \ldots, \chi_8 \}$ and the equivalence relation $R \in$ IND(k) with the following equivalence classes.

$E_1 = \{ \chi_1, \chi_4, \chi_8 \}$

$E_2 = \{ \chi_2, \chi_5, \chi_7 \}$

$E_3 = \{ \chi_3 \}$

$E_4 = \{ \chi_6 \}$

And we have these three sets

$X_1 = \{ \chi_1, \chi_4, \chi_5 \}$

$X_2 = \{ \chi_3, \chi_5 \}$

$X_3 = \{ \chi_3, \chi_6, \chi_8 \}$

$\underline{B}X_1 = \Phi = POS_B(X)$

$\overline{B}X_1 = E_1 \cup E_2 = \{ \chi_1, \chi_2, \chi_4, \chi_5, \chi_7, \chi_8 \}$

$BN_B(X_1) = E_1 \cup E_2 = \{ \chi_1, \chi_2, \chi_4, \chi_5, \chi_7, \chi_8 \}$

$NEG_B(X_1) = E_3 \cup E_4 = \{ \chi_3, \chi_6 \}$

$\underline{B}X_2 = E_3 = \{ \chi_3 \}$

$\overline{B}X_2 = E_2 \cup E_3 = \{ \chi_2, \chi_3, \chi_6, \chi_7\}$

$BN_B(X_2) = E_2 = \{ \chi_2, \chi_5, \chi_7\}$

$NEG_B(X_2) = E_1 \cup E_4 = \{ \chi_1, \chi_4, \chi_6, \chi_8\}$

$\underline{B}X_3 = E_3 \cup E_4 = \{ \chi_3, \chi_6\}$

$\overline{B}X_3 = E_1 \cup E_3 \cup E_4 = \{ \chi_1, \chi_3, \chi_4, \chi_6, \chi_8\}$

$BN_B(X_3) = E_1 = \{ \chi_1, \chi_4, \chi_8\}$

$NEG_B(X_3) = E_2 = \{ \chi_2, \chi_5, \chi_7\}$

Properties of approximation 1.3

Proposition 1.3.1

1. $\underline{B}X \subseteq X \subseteq \overline{B}X$
2. $\underline{B}\Phi = \overline{B}\Phi = \Phi$, $\underline{B}U = \overline{B}U = U$
3. $\overline{B}(X \cup Y) = \overline{B}X \cup \overline{B}Y$
4. $\underline{B}(X \cap Y) = \underline{B}X \cap \underline{B}Y$
5. $X \subseteq Y$ implies $\underline{B}X \subset \underline{B}Y$
6. $X \subseteq Y$ implies $\overline{B}X \subseteq \overline{B}Y$
7. $\underline{B}(X \cup Y) \supseteq \underline{B}X \cup \underline{B}Y$
8. $\overline{B}(X \cap Y) \subseteq \overline{B}X \cap \overline{B}Y$
9. $\underline{B}(-X) = X / \overline{B}X$
10. $\overline{B}(-X) = X / \underline{B}X$
11. $\underline{B}\,\underline{B}X = \overline{B}\,\underline{B}X = \underline{B}X$

12. $\overline{B}\overline{B}X = \underline{B}\,\overline{B}X = \overline{B}X$

Proof :

1a) If $\chi \in \underline{B}X$, then $[\chi] \subseteq X$, But $\chi \in [\chi]$

Hence $\chi \in X$ and $\underline{B}X \subseteq X$

1b) If $\chi \in X$, then $[\chi] \cap X \neq \Phi$, (because $\chi \in [\chi] \cap X$)

Hence $\chi \in \overline{B}X$ and $X \subseteq \overline{B}X$

2a) $\underline{B}\Phi = \Phi$

From (1) $\underline{B}\,\Phi \subseteq \Phi$ and $\phi \subseteq \underline{B}\Phi$ (because the empty set is a subset of every set

Thus $\underline{B}\,\Phi = \Phi$

Assume $\overline{B}\Phi \neq \Phi$, Then there exists χ such that $\chi \in \overline{B}\Phi$.

Hence $[\chi] \cap \Phi = \Phi$, but $[\chi] \cap \Phi = \Phi$, which contradicts the assumption, thus $\overline{B}\,\Phi = \Phi$.

2b) From (1) $\underline{B}U \subseteq U$, now let $\chi \in U$, then $[\chi] \subseteq U$, hence $\chi \in \underline{B}\,U$, thus $\underline{B}U = U$.

From (1) $\overline{B}U \supseteq U$, and clearly $\overline{B}U \subseteq U$, thus $\overline{B}U = U$.

3) $\chi \in \overline{B}(X \cup Y)$ iff $[X] \cap (X \cup Y) \neq \Phi$ iff

$[\chi] \cap X \cup [\chi] \cap Y \neq \Phi$ iff $[\chi] \cap X \neq \Phi$ OR

$[\chi] \cap Y \neq \Phi$ iff $\chi \in \overline{B}X$ OR $\chi \in \overline{B}Y$ iff $\chi \in \overline{B}X \cup \overline{B}Y$

Thus $\overline{B}(X \cup Y) = \overline{B}X \cup \overline{B}Y$.

4) $\chi \in \underline{B}(X \cap Y)$ iff $[\chi] \subseteq X \cap Y$ iff $[\chi] \subseteq X \wedge [\chi] \subseteq Y$

iff $\chi \in \underline{B}X \cap \underline{B}Y$.

5) Because $X \subseteq Y$ iff $X \cap Y = X$, then by (4) we have

$\underline{B}(X \cap Y) = \underline{B}X$ iff $\underline{B}X \cap \underline{B}Y = \underline{B}X$ which implies that

$\underline{B}X \subset \underline{B}Y$

6) Because $X \subseteq Y$ iff $X \cup Y = Y$, hence $\overline{B}(X \cup Y) = \overline{B}Y$ and

by (3) we have $\overline{B}X \cup \overline{B}Y = \overline{B}Y$ and hence $\overline{B}X \subseteq \overline{B}Y$

7) since $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$, we have $\underline{B}X \subset \underline{B}(X \cup Y)$

and $\underline{B}Y \subseteq \underline{B}(X \cup Y)$which implies that $\underline{B}X \cup \underline{B}Y \subseteq \underline{B}(X \cup Y)$.

8) since $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$, we have $\overline{B}(X \cap Y) \subseteq \overline{B}X$

and $\overline{B}(X \cap Y) \subseteq \overline{B}Y$ hence $\overline{B}(X \cap Y) \subseteq \overline{B}X \cap \overline{B}Y$.

9) $\chi \in \underline{B}X$ iff $[\chi] \subseteq X$ iff $[\chi] \cap X^c = \phi$ iff $\chi \notin \overline{B}(X^c)$

iff $\chi \in [(\overline{B}(X^c)]^c$, hence $\underline{B}X = (\overline{B}(X^c))^c$.

10) we substitute X^c for X in (9) We get $\overline{B}X$ ($\underline{B}X^c)^c$.

11a) From (1) $\underline{BB}X \subseteq \underline{B}X$, thus we have to show $\underline{B}X \subseteq \underline{BB}X$.

If $\chi \subseteq \underline{B}X$ then $[\chi] \subseteq X$ hence $\underline{B}[\chi] \subseteq \underline{B}X$ But $\underline{B}[\chi] = [\chi]$, Thus

$[\chi] \subseteq \underline{B}X$ and $\chi \in \underline{BB}X$, that is $\underline{B}X \subseteq \underline{BB}X$.

11b) From (1) again $\underline{B}X \subseteq \overline{B}\underline{B}X$. Thus it is enough to show that $\underline{B}X \supseteq \overline{B}\underline{B}X$. if $\chi \in \overline{B}\underline{B}X$, then $[\chi] \cap \underline{B}X \neq \phi$, i.e. there exists $y \in [\chi]$, S.T $y \in \underline{B}X$, hence $[y] \subseteq X$, But $[\chi] = [y]$

Thus $[\chi] \subseteq X$ and $\chi \in \underline{B}X$, which implies that $\underline{B}X \supseteq \overline{B}\underline{B}X$.

12a) From (1) $\overline{B}X \subseteq \overline{BB}X$. we have to show , that $\overline{B}X \supseteq \overline{BB}X$. That $\overline{B}X \supseteq \overline{BB}X$. if $\chi \in \overline{BB}X$. then $\chi \cap \overline{B}X \neq \phi$ and for some $y \in [\chi]$, $y \in \overline{B}X$, hence $[y] \cap X \neq \phi$ But $[\chi] = [y]$, thus $[\chi] \cap X \neq \phi$ i.e. $\chi \in \overline{B}X$, which implies that $\overline{B}X \supseteq \overline{BB}X$.

12b) From (1) $\underline{B}\,\overline{B}X \subseteq \overline{B}X$. we have to show , that $\underline{B}\overline{B}X \supseteq \overline{B}X$. if $\chi \in \overline{B}X$ then $[\chi] \cap X \neq \phi$. Hence $[\chi] \subseteq \overline{B}X$ (because if $y \in [\chi]$, then $[y] \cap X = [\chi] \cap X \neq \phi$ i.e. $y \in \overline{B}X$) and $\chi \in \underline{B}\overline{B}X$, which gives $\underline{B}\overline{B}X \supseteq \overline{B}X$.

Reduction of knowledge 1.4

(Two fundamental concepts) - a reduct and core - are considered in connection with knowledge reduction. A reduct of knowledge is its essential part, which suffices to define all basic concepts occurring in the considered knowledge where as the core is the common part of all reducsts i.e. in certain sense the core is the most important part [6].

Definition 1.4.1

Let $R = \{B_\alpha\}$ be a family of equivalence relation . We say that $B_\alpha \in R$ for some α is dispersible in R if :

$$IND(R) = IND\ (R - \{B_\alpha\}\ )$$

Otherwise B_α is indispensable in R. the family R is independent if B_α for each α is indispensable in R ; otherwise R is dependent.

Definition 1.4.2

Let A and B be families of equivalence relation such that $A \subseteq B$, then A is a reduct of B , if A is independent and

$$IND(A) = IND(B)$$

The set of all reducts in A is denoted by RED(A).

Definition 1.4.3

The set of all indispensable relation in B will be called the core of B , and will be denoted by CORE (B) i.e

$$CORE(B) = \{\ B{:}B \text{ is indispensable in } B\}$$

Proposition 1.4.1

If R is independent and $IP \subseteq IR$, then IP is also independent

Proof :

The proof is by contradiction. Suppose $IP \subseteq IR$ and IP is dependent , then there exists $S \subset IP$ such that IND(S) = $IND(IR)$

which implies IND(S $\cup$(/R -/P))= IND(/R) and S$\cup$(/R -/P) $\subset$ /R, hence /R is dependent, which is a contradiction.

Proposition 1.4.2

$$CORE(A) = \bigcap RED(A)$$

Where *Red(A)* = { B:B is a reduct of A}.

Proof .

If B is areduct of A and $R \in A\text{-}B$,

Then $IND(A) = IND(B)$; and $B \subseteq A\text{-}\{R\}$.

Note that , if *A,B,R* are sets of equivalence relation,

IND(A) = IND (B) , and $B \subsetneq R \subseteq A$, then

IND (B) = IND (*R*)

Assuming that $R= A\text{-}\{R\}$ we conclude that $R \notin$ CORE (*A*) , and

$$CORE(A) \subseteq \bigcap \{B{:}B \in RED(A)\}$$

Now suppose $R \notin$ CORE (*A*) i.e. $IND(A) = IND(A\text{-}\{R\})$

Which implies that there exists an independent subset $C \subseteq A - \{R\}$ such that

$$IND(C) = IND(A\text{-}\{R\})$$

It is clear that C is a reduct of *A* and $R \not\subseteq C$

This shows that

$$CORE(A) \supseteq \bigcap \{B{:}B \in RED(A)\}.$$

Example 1.4.1

Suppose we are given a family */R* = { *P*, Q, *R* } of equivalence relations P, Q and *R* with the following equivalence classes:

$$U/P = \{\{\chi_1, \chi_4, \chi_5\}, \{\chi_1\}\ \{\chi_2, \chi_8\}, \{\chi_6, \chi_7\}\}$$

$$U/Q = \{\{\chi_1, \chi_3, \chi_5\}, \{\chi_6\}, \{\chi_2, \chi_4, \chi_7, \chi_8\}\}$$

$$U/R = \{\{\chi_1, \chi_5\}, \{\chi_6\}, \{\chi_2, \chi_7, \chi_8\}, \{\chi_3, \chi_4\}\}$$

Thus the relation *IND(/R)* has the equivalence classes

$$U/IND(/R) = \{\{\chi_1, \chi_5\}, \{\chi_2, \chi_8\}, \{\chi_3\}, \{\chi_4\}, \{\chi_6\}, \{\chi_7\}\}$$

The relation P is indispensable in R , since

$$U/IND(/R-\{P\}) = \{\{\chi_1, \chi_5\}, \{\chi_2, \chi_7, \chi_8\}, \{\chi_3\}, \{\chi_4\}, \{\chi_6\}\} \neq /IND(/R)$$

For relation Q we have

$$U/IND(Q) = \{\{\chi_1, \chi_5\}, \{\chi_2, \chi_8\}, \{\chi_3\}, \{\chi_4\}, \{\chi_6\}, \{\chi_7\}\} = U/IND(/R)$$

Thus the relation Q is dispensable in */R*.

Similarly for the relation */R*

$$U/IND(/R-\{R\}) = \{\{\chi_1, \chi_5\}, \{\chi_2, \chi_8\}, \{\chi_3\}, \{\chi_4\}, \{\chi_6\}, \{\chi_7\}\} = U/IND(/R)$$

Hence the relation *R* is also dispensable in */R*.

In order to find reducts of the family $IR = \{P, Q, R\}$ we have to check whether pairs of relation P,Q and P,R are independent or not.

Because $$U/IND(\{P,Q\}) \neq U/IND(Q)$$

and $$U/IND(\{P,Q\}) \neq U/IND(P)$$

Hence the relation P and Q are independent, and consequently {P,Q} is a reduct of R, proceeding in the same way we find $\{P,R\}$ Is also a reduct of R. thus there are two reducts of the family R, namely $\{P,Q\}$ and $\{P,R\}$ and $\{P,Q\} \cap \{P,R\} = \{P\}$ is the core of IR.

Relative reducts and relative core of knowledge and categories 1.5

In this section we give the generalization of the concepts of a reduct and a core. To do this we need first to define the concepts of the positive region of a classification with respect to another classification.

Definition 1.5.1

1. Let IA and IB be two equivalence relations over U, then the set

$$POS_A(IB) = \cup_{X \in U/B} (\underline{A}X)$$

This set will be understand as the IA-positive region of IB.

2. Let /A and /B be two families of equivalence relations over the universe U . then A $\in$ /A is B- dispensable in /A if

$POS_{IND(A)}$ (IND(/B)) = $POS_{IND(A/\{A\})}$ (IND(/B))

Otherwise A is /B- indispensable in /A.

3. If $\forall$ A $\in$ /A is /B- indispensable, then we say that /A is /B-independent (or /A is independent with respect to /B).

4. the family C $\subseteq$ /A, is called a /B-reduct of /A, iff C is a /B independent subfamily of A and POS_C (/B) = POS_A (/B)

5. the set of all /B- indispensable elementary relation in /A will be called the /B-Core of /A , and denoted by : $CORE_B$(/A)

6. let F = $\{x_1, x_2, \ldots, x_n\}$ where each $X_i \subseteq U$ and let $Y \subseteq U$.

$\cap F \subseteq Y$. then

I) X_i is Y – dispensable in $\cap$ F , if $\cap (F - \{ X_i \}) \subseteq Y$.

otherwise the set X_i is Y – indispensable in $\cap$ F .

II) The family F is Y– independent in $\cap$ F if all of its Components are Y– indispensable in $\cap$ F . otherwise F is Y– dependent in $\cap$F .

III) The family $K \subseteq F$ is a reduct of $\cap$ F , if K is Y– indispensable in $\cap$ F and $\cap K \subseteq Y$.

IV) The family of all Y– indispensable sets in $\bigcap$ F will be called the Y-Core of F , and will be denoted by : $CORE_y(F)$

We will also say that a Y-reduct (Y- Core) is a relative reduct (core) with respect to Y .

Proposition 1.5.1

I) $CORE_B(A) = \bigcap RED_B(A)$.

II) CORE (F) = $\bigcap$RED (F) .

III) $CORE_y(F) = \bigcap RED_y(F)$.

Proof :

The proof of (I) , (II) and (III) is a Copy of The proof of proposition (1.4.2)

Dependencies in knowledge base 1.5

If knowledge Q is derivable from knowledge *IP* , If all elementary categories of Q can be define in term of some elementary categories of knowledge *IP*. And we say that Q is depends on *IP* . And can be written in the form *IP* ⇨ Q .

Dependency of knowledge .

Formally , the dependency can be defined as shown below

Let K= (U, R) be a knowledge base and let $P,Q \subseteq R$.

1. knowledge Q depends on knowledge P iff

$IND(P) \subseteq IND(Q)$.

2. knowledge P and Q are equivalent , denoted as

$$p \equiv Q \text{ iff } \quad P \Rightarrow Q \quad \text{and} \quad Q \Rightarrow P.$$

3. knowledge P and Q are independent , denoted as

$p \neq Q$ iff neither $P \Rightarrow Q$ nor $Q \Rightarrow P$ hold .

Obviously $p \equiv Q$ iff $IND(P) = IND(Q)$.

The following example will demonstrate the definition of dependency .

Example 1.5.1

Suppose we are given knowledge, P and Q with the following partitions .

$U/P = \{\{1,5\}, \{2,8\}, \{3\}, \{4\}, \{6\}, \{7\}\}$ and

$U/Q = \{\{1,5\}, \{2,7,8\}, \{3,4,6\}\}$

Hence $IND(P) \subseteq IND(Q)$ and onsequently

$$P \Rightarrow Q$$

Partial Dependency of knowledge 1.6

The partial derivability can be defined using the notion of the positive region of knowledge .

We will define the partial derivability formally .

Let K = (U,/R) be the knowledge base and P, Q, ⊂ /R. we say that knowledge Q depends in a degree K ($0 \le K \le 1$) from knowledge P.

Symbolically $P \Rightarrow_K Q$, if

$$K = \gamma_p(Q) = \left[\frac{card Pos_p(Q)}{card \quad U} \right]$$

Where card denotes cardinality of the set

If (K = 1), we will say that Q totally depends from P , if ($0 < K < 1$) , we say that Q roughly (partially) depends from P , if (K=0) we say that Q is totally independent from P.

Example 1.6.1

Compute the degree of dependency of knowledge Q from knowledge P, where the corresponding partitions are the following

$U/Q = \{ X_1, X_2, X_3, X_4, X_5\}$ and

$U/P = \{ Y_1, Y_2, Y_3, Y_4, Y_5, Y_6\}$

$X_1=\{ 1 \} X_2 = \{2,7\}$, $X_3 = \{3,6\}$, $X_4 = \{4\}$, $X_5 = \{5,8\}$ and

$Y_1 = \{1,5\}$, $Y_2 = \{2,8\}$, $Y_3 =\{3\}$, $Y_4 =\{4\}$, $Y_5 = \{6\}$, $Y_6 = \{7\}$

Because $\underline{P} X_1 = \emptyset$, $\underline{P} X_2 = Y_6$, $\underline{P} X_3 = Y_3 \cup Y_5$,

$\underline{P} X_4 = Y_4$ and $\underline{P} X_5 = \emptyset$ thus

$POS_P(Q) = Y_3 \cup Y_4 \cup Y_5 \cup Y_6 = \{3,4,6,7\}$. That is to say that only these elements can be classified into blocks of the partition U/Q employing the knowledge P .

Hence the degree of dependency between Q and P is

$$k = \gamma_P(Q) = 4/8 = 0.5$$

Knowledge Representation 1.7

Introduction

The issue of knowledge representation is of a primary importance in current research in AI and variety of approaches. Our major concern in this section is to discuss the issue of knowledge representation in the framework of concept introduced so far , i.e knowledge understood as partition (classification), which can be viewed as semantic definition of knowledge .

A data table will be called knowledge representation system (KR – system or KRS). (Sometimes also called knowledge information system or attribute – value system).

Definition 1.7.1

The knowledge representation system can be perceived as a data table, column of which are labelled by attributes, rows are labelled by objects (states, processes, ...,etc) and each row represents a piece of information about the corresponding object.

Example 1.7.1

In this example a characterization of animals in terms of size, Animality and color .

Kinds	Size	Animality	color
A_1	Small	Bear	Black
A_2	Medium	Bear	Black
A_3	Large	Dog	Brown
A_4	Small	Cat	Black
A_5	Medium	Horse	Black
A_6	Large	horse	Black
A_7	Large	horse	Brown

Objects in the system are kinds A_1 , ... ,A_7 and attributes are size, animality and color.

Chapter Two
Information systems

Information systems 2.1

An information system can be reviewed as a description of a knowledge base, sometimes the information system is called data tables, attribute-value system table, knowledge representation system etc. the notation of information system presented here is taken from [6].

Definition 2.1.1

Formally, a knowledge representation system can be formulated as follows

Knowledge representation system (information system) is a pair $IA = (U, A)$, where

U – Is a nonempty, finite set called the universe.

A- is a nonempty, finite set of attributes. Each attribute $a \in A$ is a function a: $U \rightarrow V_a$ Where V_a is the set of values of a.

Hence, the expression $a(\chi)$ denotes the value of attribute for a object $\chi \in U$.

	a_i	...	a_j	...	a_m
χ_1	$a_1(\chi_1)$				$A_m(\chi_1)$
		.			
χ_i			$a_j(\chi_i)$		$A_m(\chi_i)$
				.	
χ_n					$A_m(\chi_n)$

TABLE (2.1.1) : AN INFORMATION SYSTEM

Definition 2.1.2

Let *IA* = (*U*,A) be an information system and B $\subseteq$ A, then by *IA*/*IB* we denotes the information system (*U*,B), called the restriction of *IA* To B.

Definition : 2.1.3

Let *IA* = (*U*,A) be an information system. Every subset of attributes B $\subseteq$ A, defines an equivalence relation $IND_A(B)$ or IND(B) called the B- indiscernibility relation defined as follows

$$IND(B) = \{(\chi_1, \chi_2) \in U^2 : a(\chi_1) = a(\chi_2) \text{ for every } a \in B\}$$

The object χ_1, χ_2 satisfying the relation IND(B) are indiscernible by attributes from B.

Definition 2.1.4

Every subset $B \subseteq A$ will be called an attribute. If B is a single element set, then B is called primitive, otherwise the attribute is said to be compound.

Remark 2.1.1

1. Notice that for each $a \in A$, IND({a}) is an equivalence relation.
2. For any $B \subseteq A$, $\bigcap_{a \in B} IND(a)$ is an equivalence relation .

Properties of indiscernibility relation 2.1.1

1. $IND(B) = \bigcap_{a \in B} IND(a)$.
2. $IND(B \cup C) = IND(B) \cap IND(C)$.
3. If $C \subseteq B$ then $IND(B) \subseteq IND(C)$.

Example 2.1.1

Let us consider a simple example of an information system $IA = (U,A)$. where $U = \{\chi_1, \chi_2, \chi_3, \ldots, \chi_8\}$, $A = \{a,b,c,d,e\}$ and the values of the attributes are defined as in the table below (2.1.2)

U	a	b	c	d	e
χ_1	57	37	90	130/90	32
χ_2	30	39	110	100/70	37
χ_3	57	40	110	95/80	26
χ_4	30	37	90	95/80	37
χ_5	57	37	75	130/90	26
χ_6	22	40	110	95/80	26
χ_7	30	40	110	100/70	37
χ_8	30	39	75	100/70	26

Where

a- Age

b- Temperature

c- Pulse

d- Blood pressure

e- Resp. valary

Let B = { a,b,c} $\subseteq$ A. now we want to compute IND(B)

IND({a}) = $\{ (\chi_1, \chi_1), (\chi_2, \chi_2), (\chi_3, \chi_3), (\chi_4, \chi_4), (\chi_5, \chi_5), (\chi_6, \chi_6),$
$(\chi_7, \chi_7), (\chi_8, \chi_8), (\chi_1, \chi_3), (\chi_3, \chi_1), (\chi_3, \chi_5), (\chi_5, \chi_1),$
$(\chi_5, \chi_3), (\chi_2, \chi_4), (\chi_2, \chi_7), (\chi_2, \chi_8), (\chi_4, \chi_8), (\chi_4, \chi_7),$
$(\chi_4, \chi_8), (\chi_7, \chi_2), (\chi_7, \chi_4), (\chi_7, \chi_8), (\chi_8, \chi_2), (\chi_8, \chi_4),$
$(\chi_8, \chi_7)\}$.

IND({b}) = $\{ (\chi_1, \chi_1), (\chi_2, \chi_2), (\chi_3, \chi_3), (\chi_4, \chi_4), (\chi_5, \chi_5), (\chi_6, \chi_6),$
$(\chi_7, \chi_7), (\chi_8, \chi_8), (\chi_1, \chi_4), (\chi_1, \chi_5), (\chi_4, \chi_1), (\chi_4, \chi_5),$
$(\chi_5, \chi_1), (\chi_5, \chi_4), (\chi_3, \chi_6), (\chi_3, \chi_7), (\chi_6, \chi_3), (\chi_6, \chi_7),$
$(\chi_7, \chi_3), (\chi_7, \chi_6), (\chi_2, \chi_8), (\chi_8, \chi_2)\}$.

IND({c}) = $\{ (\chi_1, \chi_1), (\chi_2, \chi_2), (\chi_3, \chi_3), (\chi_4, \chi_4), (\chi_5, \chi_5), (\chi_6, \chi_6),$
$(\chi_7, \chi_7), (\chi_8, \chi_8), (\chi_1, \chi_4), (\chi_4, \chi_1), (\chi_2, \chi_3), (\chi_2, \chi_6),$
$(\chi_2, \chi_7), (\chi_3, \chi_2), (\chi_3, \chi_6), (\chi_3, \chi_7), (\chi_6, \chi_2), (\chi_6, \chi_3),$
$(\chi_6, \chi_7), (\chi_7, \chi_2), (\chi_7, \chi_3), (\chi_7, \chi_6), (\chi_5, \chi_8), (\chi_8, \chi_5)\}$.

$IND(B) = \bigcap_{a \in B} IND(a) = IND(a) \cap IND(b) \cap IND(c)$

$IND(B) = \{(\chi_1, \chi_1), (\chi_2, \chi_2), (\chi_3, \chi_3), (\chi_4, \chi_4), (\chi_5, \chi_5), (\chi_6, \chi_6), (\chi_7, \chi_7), (\chi_8, \chi_8)\}.$

Decision tables 2.2

Introduction 2.2.1

In this section we will consider a special, important class of knowledge information system, called decision tables, which play an important part in many applications.

A decision table is a kind of prescription, which specifies what the decision (action) should be undertaken when some conditions are satisfied. Most decision problems can be formulated employing decision table formalism; therefore, this tool is particularly useful in decision making.

In this section we wish to discuss some basic problems of decision tables theory in terms of rough sets philosophy.

Formal definition and some properties 2.2.2

(1) **The decision table in KRS**

decision tables can be defined in terms of KR-system as follows/ let $K=(U,A)$ be a knowledge representation system and let $C, D \subset A$ be two subsets of attributes, called condition and decision attributes respectively. KR-system with distinguished condition and decision attributes will be called a decision table and will be denoted $T = (U, A, C, D)$, or in short CD-decision table [6] .

Definition 2.2.2.1

Equivalence classes of the relation $IND(C)$ and $IND(D)$ will be called **condition and decision** classes, respectively.

Definition 2.2.2.2

For every $\chi \in U$ we will associate a function $d_\chi : A \rightarrow V$ such that $d_\chi(a) = a(\chi)$ for every $a \in C \cup D$; the function d_χ will be called a decision rule (in T) and χ will be referred to as label of the decision rule d_χ .

Definition 2.2.2.3

IF d_χ is a decision rule, then the restriction of d_χ to C, denoted $d_\chi | C$, and the restriction of d_χ To D denoted $d_\chi | D$ will be called condition and decisions (actions) of d_χ respectively.

Definition 2.2.2.4

1- The decision rule d_χ is consistent (in T), if for every $\chi \neq y$, $d_\chi | C = d_y | C$ impels $d_\chi | D = d_y | D$; other wise the decision rule is inconsistent.

2- A decision table is consistent if all it's decision rule are consistent, otherwise the decision table in inconsistent.

The following is the important properties that establish the relationship between consistency and dependency of attributes in a decision Table.

Proposition 2.2.2.1

A decision table T= (U,A,C,D) is consistent iff $C \Rightarrow D$.

Proposition 2.2.2.2

Each decision table $T=(U,A,C,D)$ can be uniquely decomposed into two decision tables $T_1=(U,A,C,D)$ and $T_2 = (U,A,C,D)$ so that $C \Rightarrow_1 D$ in T_1 and $C \Rightarrow_0 D$ in T_2, where

$T_1 = POS_c(D)$and

$$T_2 = \bigcup_{X \in U/IND(D)} BN_C(X)$$

Example (2.2.2.1) :

let us consider table (2.2.2.1) given below

U	a	b	c	d	e
1	1	0	2	2	0
2	0	1	1	1	2
3	2	0	0	1	1
4	1	1	0	2	2
5	1	0	2	0	1
6	2	2	0	1	1
7	2	1	1	1	2
8	0	1	1	0	1

Assume that *a*, *b*, and c are condition attributes, *d* and *e* are decision attributes. In this table, for instance, the decision rule 1 is inconsistent where as decision rule 3 is consistent. by employing proposition (2.2.2.2) we can decompose the table (2.2.2.1) into the following two tables .

Table (2.2.2.2) consistent

U_1	a	b	c	d	e
3	2	0	0	1	1
4	1	1	0	2	2
6	2	2	0	1	1
7	2	1	1	1	2

Table (2.2.2.3) Inconsistent

U_2	a	b	c	d	e
1	1	0	2	2	0
2	0	1	1	1	2
5	1	0	2	0	1
8	0	1	1	0	1

Table (2.2.2.2) is consistent where as table (2.2.2.3) is totally inconsistent, which means that all decision rules in table (2.2.2.2) are consistent, and table (2.2.2.3) all decision rules are inconsistent.

Simplification of decision tables 2.2.3

Simplification of decision tables is of a primary importance in many applications. An example of simplification is the reduction of condition attributes in a decision table. In the reduced decision table. The same decision can be based on a smaller number of conditions.

This kind of simplification eliminates the need for checking unnecessary condition or, in some applications for performing expensive test to arrive at a conclusion which eventually be achieved by simpler means.

The approach to table simplification presented here consist of the following steps

1- Computation of reducts of condition attributes which is equivalent to elimination of some column from the decision table.

2- Elimination of duplicate rows.

3- Elimination of superfluous value of attributes.

Remark 2.2.3.1

We should note that in contrast to the general notion of knowledge representation system rows do not represent here description on any real objects. Consequently duplicate rows can be eliminated as they correspond to the same decision.

In this way we obtain an "incomplete" decision tables, containing only those values of condition attributes which are necessary to make decisions. According to our definition of a decision table, the incomplete table, is not a decision table and can be treated as an abbreviation of such a table.

From mathematical point of view, removing attributes an removing values of attributes are a like and will be explained in what follows. For the sake of simplicity, we assume that the set of condition attributes is already reduced. I.e. there are not superfluous condition attributes in the decision table.

As we have already mentioned, with every subset of attributes B⊆A we can associate partition $U/_{IND(B)}$, and consequently the set of condition and decision attributes define partitions of objects into condition and decision classes.

Because we want to discern every decision class using a minimal number of condition – our problem can be reduced to searching for relative reducts of condition classes with respect to decision classes. To this end we can use method similar to that used in finding reducts of attributes.

Similarly we can reduce superfluous value of condition attributes from a decision table.

In "formal definition" we know that with every subset of attributes $B \subseteq A$ and object χ we may associate set $[\chi]_B$.

($[\chi]_B$ denotes an equivalence class of the relation *IND(B)* containing object χ).

i.e. $[\chi]_B$ is an abbreviation of $[\chi]_{IND(B)}$) thus with any set of condition attributes C in decision rule d_x we can associate set $[\chi]_c = \bigcap_{a \in c} [\chi]_a$.

But each set $[\chi]_a$ is uniquely determined by attributes value $a(\chi)$, hence in order to remove superfluous equivalence classes $[\chi]_a$ from the equivalence class $[\chi]_c$ as discussed in " significance of attributes" . Thus the problem of elimination of superfluous value of attributes and elimination of corresponding equivalence classes are equivalent.

Example 2.2.3.1

suppose we are given the following decision table

Table (2.2.3.1)

U	a	b	c	d	e
1	1	0	0	1	1
2	1	0	0	0	1
3	0	0	0	0	0
4	1	1	0	1	0
5	1	1	0	2	2
6	2	1	0	2	2
7	2	2	2	2	2

Where a, b, c and d are condition attributes and e is a decision attribute.

It is easy to compute that the only e- dispensable condition attribute is c, consequently, we can remove column C in table above, which yield table below

Table (2.2.3.2)

U	a	b	d	e
1	1	0	1	1
2	1	0	0	1
3	0	0	0	0
4	1	1	1	0
5	1	1	2	2
6	2	1	2	2
7	2	2	2	2

In the next step we have to reduce superfluous values of condition attributes, in every decision rule, to this end we have first to compute core values of condition attributes in every decision rule.

For the sake of illustration, let us compute the core values of condition attributes for the first decision rule i.e. the core of the family of sets

$F=\{ [1]_a, [1]_b, [1]_d \} = \{\{ 1,2,4,5\}, \{4,5,6\}, \{1,4\} \}$

From consideration in reduction of categories and formal definition we have

$[1]_{\{a,b,d\}} = [1]_a \cap [1]_b \cap [1]_d = \{1,2,4,5\} \cap \{4,5,6\} \cap \{1,4\} = \{4\}$

Moreover a (1) = 1 , b(1) = 0 and d(1) = 1.

IN order to find dispensable categories, we have to drop one category at a time and check whether the intersection of remaining categories is still included in the decision category $[1]_e = \{1,2\}$, i.e.

$[1]_b \cap [1]_d = \{4,5,6\} \cap \{1,4\} = \{4\}$

$[1]_a \cap [1]_d = \{1,2,4,5\} \cap \{1,4\} = \{1,4\}$

$[1]_a \cap [1]_b = \{1,2,4,5\} \cap \{4,5,6\} = \{4,5\}$

This means that the core value is b (1) = o . Similarly we can compute remaining core value of condition attribute in every decision rule and the final results are presented in table below

Table (2.2.3.3)

U	a	b	d	e
1	-	0	-	1
2	1	-	-	1
3	0	-	-	0
4	-	1	1	0
5	-	-	2	2
6	-	-	-	2
7	-	-	-	2

Now we can proceed to compute value reducts. As an example, let us compute value reducts for the first decision rule of the decision table.

$$F = \{ [1]_a, [1]_b, [1]_d \} = \{\{1,2,4,5\}, \{1,2,3\}, \{1,4\}\}$$

We have to find all subfamilies $G \subseteq F$ such that $\bigcap G \subseteq [1]_e = \{1,2\}$

There are three following subfamilies of F

$$[1]_b \cap [1]_d = \{1,2,3\} \cap \{1,4\} = \{1\}$$

$$[1]_a \cap [1]_d = \{1,2,4,5\} \cap \{1,4\} = \{1,4\}$$

$$[1]_a \cap [1]_b = \{1,2,4,5\} \cap \{1,2,3\} = \{1,2\}$$

And only two of them

$$[1]_b \cap [1]_d = \{1,2,3\} \cap \{1,4\} = \{1\} \subseteq [1]_e = \{1,2\}$$

$$[1]_a \cap \{1]_b = \{1,2,4,5\} \cap \{12,3\} = \{1,2\} \subseteq [1]_e = \{1,2\}$$

Are reducts of the family F. hence we have two value reducts:

$b(1) = 0$ and $d(1) = 1$ or $a(1) = 1$ and $b(1) = 0$.

This means that the attribute values of attribute a and b or d and b are characteristic for decision class 1 and do not occur in any other decision class in the decision table. We see also that the value of attribute b is the intersection of both values reducts, $b(1)=0$ i.e. it is the core value.

In table below (2.2.3.4) we list value reducts for all decision rule in table (2.2.2.1).

U	a	b	d	e
1	1	0	x	1
1'	x	0	1	1
2	1	0	x	1
2'	1	x	0	1
3	0	x	x	0
4	x	1	1	0
5	x	x	2	2
6	x	x	2	2
6'	2	x	x	2
7	x	x	2	2
7'	x	2	x	2
7"	2	x	x	2

As we can see from table (2.2.3.4), for decision rules 1 and 2 we have two value reducts of condition attributes. Decision rules 3,4 and 5 have only one value reduct of condition attributes for each decision rule row. The remaining decision rules 6 and 7 contain two and three values reducts respectively.

Hence there are two reduce forms of decision rule 1 and 2 , decision rules 3 , 4 and 5 have only one reduced form each, decision rule 6 have two reducts and decision rule 7 have three reducts.

Thus there are 4x2x3 = 24 (not necessarily different) solution to our problem. One such solution is presented in table below

Table (2.2.3.5)

U	a	b	d	e
1	1	0	x	1
2	1	x	0	1
3	0	x	x	0
4	x	1	1	0
5	x	x	2	2
6	x	x	2	2
7	2	x	x	2

Another solution is shown in table (2.2.3.6)

Table (2.2.3.6)

U	a	b	d	e
1	1	0	x	1
2	1	0	x	1
3	0	x	x	0
4	x	1	1	0
5	x	x	2	2
6	x	x	2	2
7	x	x	2	2

Because a decision rules 1 and 2 are identical, and so are rules 5, 6 and 7, we can represent our table in form.

Table (2.23.7)

U	a	b	d	e
1,2	1	0	x	1
3	0	x	x	0
4	x	1	1	0
5,6,7	x	x	2	2

In fact, enumeration of decision rules is not essential, so we can enumerate them arbitrarily and we get as final result table below (2.2.3.8)

U	a	b	d	e
1	1	0	x	1
2	0	x	x	0
3	x	1	1	0
4	x	x	2	2

This solution will be referred to as a minimal.

The presented method of decision table simplification can be semantic, since it refers to the meaning of information contained in the table.

(2) The decision table in information system

We consider a special case of an information systems called decision tables. a decision table is an information system of the form $IA = (U, A \cup \{d\})$, where $d \notin A$ is a distinguished attribute called the decision. The elements of A are called conditions.

Decision tables are called training sets of examples in machine learning [15] [16] [20].

Definition (2.2.3.1)

Every information system $IA = (U, A)$ and non- empty $B \subseteq A$ determine a B- information function

$$Inf_B^{IA} : U \rightarrow IP(B \times \bigcup_{a \in B} V_a)$$

Defined by

$Inf_B^{IA}(x) = \{ (a, a(\chi)) : a \in B \}$. We write $Inf_B(x)$ instead of $Inf_B^{IA}(x)$ when no confusion arises.

The set

$$\{ Inf_B(x) : \chi \in U)$$

Is called the B-information set and it is denoted by $INF(IA)\lceil B$. the set $INF(IA)|A$ will be denoted by $INF(IA)$. By $INF(A,V)$ we denote the set of all functions a from U into V satisfying $a(\chi) \in V_a$ for any $\chi \in U$ and $a \in A$.

Definition 2.2.3.2

The cardinality of the image

d (U) = { K : d(s) = K for some $s \in U$ } is called the rank of d and is denoted by r(d).

We assume that the set of V_d of values of the decision d is equal to { 1, ... ,r(d) }.

Let us observe that the decision d determines the partition $\{\chi_1, \ldots \chi_{r(d)}\}$ of the universe U, where $\chi_k = \{\chi \in U : d(\chi) = k\}$ for $1 \le k \le r(d)$. The set χ_i is called the i-th decision class of *IA*.

Discernibility matrix and the Boolean function 2.3

Definition 2.3.1

Let *IA* be an information system with n objects. By *M (IA)* we denotes an n x n matrix (C_{ij}) such that $U = \{\chi_1, \chi_2, \ldots, \chi_n\}$ and $A = \{a_1, a_2, \ldots, a_n\}$, called the discernibility matrix of *A* such that

$$C_{ij} = \{a \in A : a(\chi_i) \ne a(\chi_j)\} \qquad for \quad i,j = 1,2,\ldots n$$

Notice here that this matrix is symmetric and $C_{ii} = \phi$ for i = 1, 2, ...,n

For any attribute $a \in A$, we denote by $\overline{a}$ the Boolean variable corresponding to a [16].

We denote by $f_{/A}$ the Boolean function of m Boolean variables $\overline{a}_1,\overline{a}_2,\ldots\overline{a}_m$, where A = $\{a_1,a_2,\ldots a_n\}$ defined by letting

$$f_{/A}\ (\overline{a}_1,\overline{a}_2,\ldots\overline{a}_m) = \wedge\ \{\vee\overline{C}_{ij} :\quad 1\le j<i\le n \qquad C_{ij}\ne\emptyset\}$$

Where $\overline{C}_{ij} = \{\ \overline{a} : a\in C_{ij}\ \}$

Here $V(\overline{C}_{ij})$ is the disjunction of all variables $\overline{a}$ such that $a \in C_{ij}$. We denote by Val_m the set of all Boolean valuation on the set :

$$\{\ \overline{a}_1,\overline{a}_2,\ldots,\overline{a}_m\}$$

I.e. Any $v \in Val_m$ is a function

$$v=\{\ \overline{a}_1,\overline{a}_2,\ldots\overline{a}_m\} \to \{\ 0,1\ \}$$

For any $B \subseteq A$, we denote by $V_B \in Val_m$ the characteristic function of B *i.e.* $V_B\ (\overline{a}) = 1$ iff $a \in B$ we denote by MIN ($f_{/A}$) the set

$$\{\ B\subseteq A : \quad f_{/A}\ (V_B(\overline{a}_1), V_B(\overline{a}_2), \ldots V_B(\overline{a}_m)) = 1 \quad \text{and}$$
$$f_{/A}\ (V_B(\overline{a}_1), V_B(\overline{a}_2), \ldots V_B(\overline{a}_m)) = 0 \quad \text{and}$$
$$\text{For any} \quad B` \subseteq B \quad \text{and} \quad B` \ne B\ \}$$

Example 2.3.1

Let U = {1,2,3,4,5} and A={a,b,c,d} . Let the values of the attributes be defined as in table (2.3.1)

Theorem 2.3.1 [19]

Let /A =(U , A) be an information system with $U = \{\chi_1, \chi_2, \cdots \chi_n\}$, $A = \{a_1, a_2, \ldots a_m\}$, and let $\phi \neq B \subseteq A$. the following condition are equivalent:

(i) B contains a reduct from *RED(/A)* i.e. *IND(A) = IND(B)*.

(ii) f_A ($V_B(a_1), \ldots V_B(a_m)$) = 1;

(iii) for all i and j such that $c_{ij} \neq \phi$ and $1 \leq j < i \leq n$, $c_{ij} \cap B \neq \phi$

Proof the equivalence

(ii) ⇔ (iii) follows from the constructions of the discernibility function $f_{/A}$ and the discernibility matrix M (/A).

(iii) ⇔ (i) by our assumption we have : $c_{ij} \cap B \neq \phi$ for any c_{ij} such that $1 \leq j < i \leq n$. it means that in B we have enough attributes to discern between all these objects from U , which are to discernible with respect to all attribute in /A ; i.e. B contains a reduct from *RED(A)*.

(i) ⇔ (iii) If B contains reduct X from *RED(IA)* , then any two objects discernible with respect to some attributes from A are also discernible with respect to some

Table (2.3.1)

U	a	b	c	d
1	1	0	2	1
2	2	1	0	1
3	1	1	0	1
4	1	1	0	2
5	0	1	2	0

The discernibility matrix for this table is as follows:

Table (2.3.2)

U	1	2	3	4	5
1	O	abc	bc	bcd	abd
2	abc	O	a	ad	acd
3	bc	a	O	d	acd
4	bcd	ad	d	O	acd
5	abd	acd	acd	acd	O

The discernibility matrix M (A).

The discernibility function:

$$f_{IA}\ (a,b,c,d) = a \wedge d \wedge (b \vee c) \wedge (a \vee d) \wedge (a \vee b \vee c) \wedge$$

$$(b \vee C \vee d) \wedge (a \vee b \vee d) \wedge (a \vee c \vee d)$$

from attributes $B \supseteq X$. hence if $c_{ij} \neq \phi$ then $c_{ij} \cap B \neq \phi$ for any i,j .

Definition 2.3.2

If /A = (U , A $\cup$ {d}) is a decision table then we define a function ∂_A: U →/p ({ 1,..., r(d) }called the generalized decision in /A, by

$$\partial_A(\chi) = \{ i : \exists \chi' \in U \;\; \chi' \; IND\,(A)\, \chi \quad \text{and} \quad d(\chi) = i \}.$$

Definition 2.3.3

A decision table /A is called consistent (deterministic) if $|\partial_A(\chi)| = 1$ for any $\chi \in U$ other wise /A is inconsistent (non-deterministic). It is easy to see that a decision table /A is consistent iff $pos_a(d) = U$. Moreover, if $\partial_B = \partial_{B'}$ then $Pos_B\,(d) = Pos_{B'}\,(d)$ for any non-empty sets $B,B' \subseteq A$.

Definition 2.3.4

A subset B of the set A of attributes of decision table /A = (U ,A $\cup$ {d}) is a relative reduct of IA iff B is a minimal set with the following property: $\partial_B = \partial_A$. The set of all relative reducts in /A is denoted by RED(/A,d).

IF B$\in$ *RED(IA,d)* then the set $\{ (\{ a=a(\chi) : a \in B \},\; d = d(\chi)) : \chi \in U \}$ is called the trace of B in /A and is a denoted by $Trace_{/A}(B)$.

Rough membership function 2.4

One of the fundamental notions of set theory is the membership relation, usually denoted by $\in$.When one consider subsets of a given universe one usually applies characteristic function in order to express the fact that a given element belongs to a given set . We discuss the case when only partial information about objects is accessible. In this section we show that it is possible to extend the notion of a characteristic function to that case.

Definition 2.4.1

Let *I*A = (*U* , A) be an information system and $\phi \neq X \subseteq U$. the rough IA-membership function of the set X denoted by μ_x^{IA} , is defined by

$$\mu_X^{IA} : U \rightarrow [\,0,1\,] \text{ and } \quad \mu_X^{IA}(\chi) = \frac{|[\chi]_A \cap X|}{|[\chi]_A|}$$

Obviously $\mu_X^{IA}(\chi) \in [0,1]$. A value of membership function μ_X^{IA} is a kind of conditional probability, and can be interpreted as a degree of certainty to which X belongs to χ (or 1-$\mu_X^{IA}(\chi)$as a degree of uncertainty).

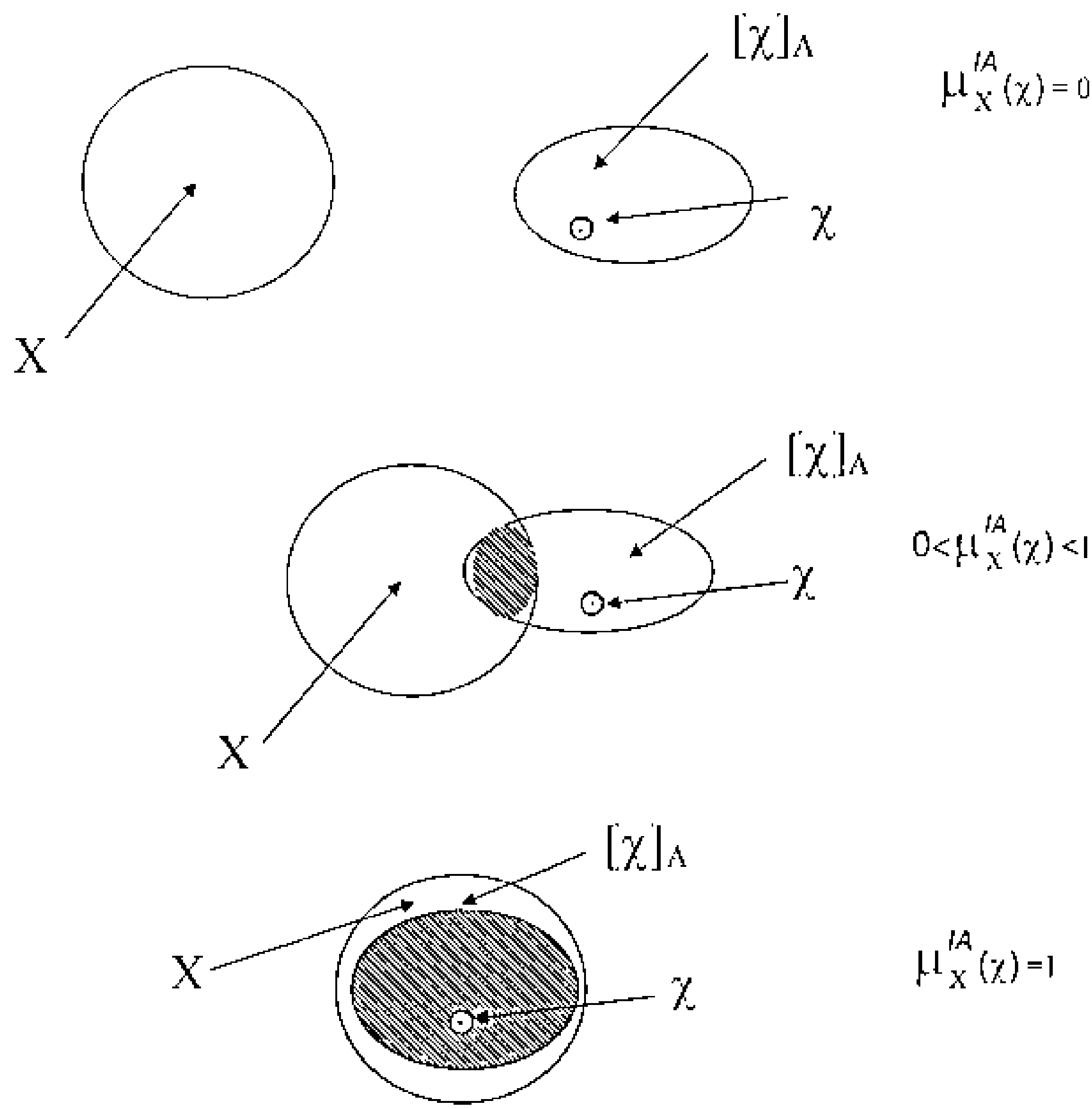
The rough membership function can be used to define approximations and the boundary region of a set, as shown below

$$\underline{A}(X) = \{\chi \in U : \mu_X^{IA}(\chi) = 1\},$$

$$\bar{A}(X) = \{\chi \in U: \mu_X^{IA}(\chi) > 0\},$$

$$BN_A(X) = \{\chi \in U: 0 < \mu_X^{IA}(\chi) < 1\}.$$

The meaning of rough membership function can be depicted as shown below [19]



Theorem 2.4.1 [6] [19][20]

Let /A = (U , A) be an information system and let $X,Y \subseteq U$, the rm-function μ_X^{IA} has the following properties:

(i) $\mu_X^{IA}(\chi) = 1$ iff $\chi \in \underline{A}X$

(ii) $\mu_X^{IA}(\chi) = 0$ iff $\chi \in U - \bar{A}X$,

(iii) $0 < \mu_X^{IA}(\chi) < 1$ iff $\chi \in BN_A(X)$

(iv) If *IND(A)* = { (χ,χ) : $\chi \in U$ } then μ_X^{IA} is the characteristic function of X ;

(v) If χ *IND(A)* y then $\mu_X^{IA}(\chi) = \mu_X^{IA}(\gamma)$

(vi) $\mu_{U-X}^{IA}(\chi) = 1 - \mu_X^{IA}(\chi)$ for any $\chi \in X_i$

(vii) $\mu_{X \cup Y}^{IA}(\chi) \geq Max(\mu_X^{IA}(\chi), \mu_Y^{IA}(\chi))$ for any $\chi \in U$,

(viii) $\mu_{X \cap Y}^{IA}(\chi) \leq Min(\mu_X^{IA}(\chi), \mu_Y^{IA}(\chi))$ for any $\chi \in U$,

(ix) if X is a family of pairwise disjoint subset s of U

Then $\mu_X^{IA}(\chi) = \sum_{x \in X} \mu_X^{IA}(\chi)$ for any $\chi \in U$

Deterministic decision rules 2.5 [19]

Now we can recall the definition of decision rules. Let /A = (U ,A$\cup${d}) be a decision table and let $V = \cup(V_a\ a \in A: \} \cup V_d$ or

$$V = \bigcup_{a \in A} V_a \cup V_d$$

Definition 2.5.1

The atomic formulas over $B \subseteq A \cup \{d\}$ and V expressions of the form a=v, called descriptors over B and V, where $a \in B$ and $v \in V_a$. the set *IF* (B,V) of formulas over B and V is the least set containing all atomic formulas over B and V and closed with respect to the classical propositional connectives V (disjunction), $\wedge$ (conjunction).

Definition 2.5.2

Let $\tau \in IF(B,V)$. Then by $\tau_{/A}$ we denote the meaning of τ in the decision table *IA*, i.e. the set of all objects in U with property τ, defined inductively as follows:

1- if τ is of the form a=V then $\tau_{/A} = \{ \chi \in U : a(\chi) = v \}$.

2- $(\tau \wedge \tau^{`})_{/A} = \tau_{/A} \cap \tau^{`}_{IA}$; $(\tau V \tau^{`})_{/A} = \tau_{/A} \cup \tau^{`}_{IA}$

The set *IF*(A,V) is called the set of conditional formulas of *IA* and is denoted by C(A,V).

A decision rule of /A is any expression of the form

$$\tau \Rightarrow d=v$$

Where $\tau \in C(A,V)$ and $v \in V_d$.

The decision rule $\tau \Rightarrow d=v$ for $/A$ is true in $/A$ iff $\tau_{/A} \subseteq (d=v)_{/A}$, if $\tau_{/A}= (d=v)$ then we say that the rule is $/A$-exact.

IF $/A$ = (U , A U {d}) is a decision table then by $/A_\partial$ we denote the decision table (U, A U $\{\partial\}$ where $\partial=\partial_A$ let us observe that any decision rule $\tau \Rightarrow \partial_A=\theta$ where $\tau \in C(A,V)$ and $\phi \neq 0 \subseteq V_d$ valid in $/A_\partial$ and having examples in $/A$, i.e. satisfying $\tau_{/A} \neq \phi$ determines a distribution of objects satisfying τ among elements of θ. this distribution is defined by

$$\mu_i(/A, \tau, \theta) = \frac{|Y \cap X_i|}{|Y|}, \quad for\ i \in 0$$

Where $Y = \tau_{/A}$

Definition 2.5.3

Any decision rule $\tau \Rightarrow \partial_A = 0$ valid in $/A_\partial$ and having examples in $/A$ with $|\theta|>1$ is called non-deterministic, otherwise it is deterministic.

Definition 2.5.4

The number $n(/A, \tau, i) = |\tau_{/A} \cap X_i|$ is called number of examples supporting τ in the i-th decision class X_i. Let us observe that for any formula τ over A and V with $\tau_{/A} \neq \phi$.

there exists exactly one subset θ_τ of θ such that $\tau \Rightarrow_{/A} \partial_A = \theta_\tau$ and $\mu_i(/A, \tau, \theta_\tau) > 0$ for any $i \in \theta_\tau$ in the sequel we write $\mu_i(/A, \tau)$ instead of $\mu_i\ (/A, \tau, \theta\tau)$.

Chapter Three
Data filtration

Data filtration 3

The aim of this section is to present some ideas related to data filtration based on rough set approach. We expect that by developing the ideas presented here and some related algorithms efficient tools for data filtration can be obtained.

A general searching scheme for decision tables filtration based on rough set approach 3.1

The decision table /A' = (U , A'U {d}) is compatible with

/A = (U,A U {d}) iff $A = \{a_1, \ldots a_m\}$, $A' = \{a'_1, \ldots a'_m\}$

And $V_{a'_i} \subseteq V_{a_i}$ for i = 1,...,m.

let /A= (U ,A U{d}) and /A' = (U, A'U{d}) be decision tables and let $\delta \in (0,1]$. We say that /A' is a δ -filtration of /A iff :

(i) /A' is compatible with /A.

(ii) $\partial_{/A} = \partial_{/A'}$

(iii) $|\{[\chi]_{/A'} : [\chi]_{/A'} \subseteq Z_\theta\}| < \delta\, |\{[\chi]_{/A} : [\chi]_{/A} \subseteq Z_\theta\}|$

for any $\theta \subseteq V_d$ with $\phi \neq Z_\theta = \partial^{-1}_{/A}(\theta)$

One can also consider another version of definition with condition (ii) substituted by a weaker condition specifying that ∂_A

and $\partial_{A'}$, should be sufficiently close with respect to some distance function.

let K be a real number from the interval (0,1], let /A =(U,A $\cup$ {d}) be a decision table and let τ be a formula over A and V. if α = {(a_1,v_1),(a_2,v_2),... ,(a_m,v_m)} then $\Lambda\alpha$ denotes the formula $a_1= V_1\Lambda$, $a_2= V_2\Lambda \ldots \Lambda a_m = V_m$ if τ is a formula over /A and V then by /A_τ we denote the restriction of /A to the set of all objects from U, satisfying τ. i.e. /A_τ = ($\tau_{/A}$,A$\cup${d}). by T(/A,τ,K) we denote the conjunction of the following conditions

(i) $\tau_{/A} \neq \phi$

(ii) for any $\alpha \in$ *INF* (/Aτ) ;

(*) $M_i ax\ \mu_i(/A\ \Lambda\ \alpha\Lambda\tau) > K$

And

(**) there exists exactly one i_0 with the following property

$\mu_{i0}(/A\ \Lambda\ \alpha\Lambda\tau) = M_i ax\ \mu_i(/A\ \Lambda, \alpha\Lambda\tau)$.

Conditions (*) and (**) denote that exactly one of distribution coefficient μ_i (/A,τ) ($i \in \theta$) exceeds the threshold K.

Let /A = (U,A⋃{d}) be a decision table and let δ be a reduction coefficient. We present a general procedure *F* of searching for a filtration of */A* when fixed threshold *K* and critical level 1 are given.

Step 3.1.1

For any $a \in A' \subseteq A$, where A', is a randomly chosen sample of conditions, apply the methods for decision rules synthesis (see [10],[11]). For the decision table $/A_a$ = (U, {A - A'} U {a}) (in particular apply also for synthesis of rules the so called semi-reducts). The output of this step is a set of decision rules of the form:

$$\tau \Rightarrow \partial_{A-A'} = \theta_\tau$$

Where $\partial_{A-A'}$ is the generalized decision corresponding to the condition $a \in A'$ and τ is a formula over A- A' and V Now some strategies should be applied to get from these decision rules a "global" decision rule for a

$$\tau_0 \Rightarrow \partial_{A-A'} = \theta$$

Such that T(/A_a, τ_0 ,K) holds and *Max* n (/A_a, $\Lambda\, \alpha_\Lambda\, \tau_0$, i) > 1 for $\alpha \in$ *INF* ($(/A_a)_{\tau 0}|B$) where B is the set of conditions occurring in τ_0 and $(/A_a)_{\tau_0}$ is the restriction of $/A_a$ to the set of objects satisfying τ_0

Step 3.1.2

From the global decision rules for conditions in A the approximation functions are built. In this way we obtain a set f of approximation functions.

Step 3.1.3

We apply to a some approximation functions from f (using some strategies, see (3.1.4.3) and in this way we obtain a new decision table /A'.

Step 3.1.4

If /A' is a δ-filtration of /A then stop else go to step 1 subsituting /A' for /A if /A' is a δ'-filtration of /A with $\delta' < 1$.

There are several problems to be solved when one would like to implement the above method. Among them are:

1. **SAMPLING PROBLEM 3.1.4.1**

how to choose /A' in step 1?

the random choice of /A' (proposed in step 1) can be not sufficient. some methods for the proper choice of /A' can be developed basing on properties of reducts [14]. one can apply also some analogies from mathematical morphology [12] which have been used in building to so called analytical morphology. here we would like only to add, by analogy with the mathematical morphology operations of erosions and dilation, that one randomly choose a subset /A" of /A – /A' as a new subset of /A after entering again step 1 where /A' is the previously chosen subset of /A. we shall now look for decision rules

$$\tau \Rightarrow \partial_{/A'} = \theta_{\tau}$$

Where τ is over /A' and V and $\partial_{/A'}$ is the generalized decision corresponding to a condition a $\in$ /A' .

2. GENERATION OF APPROXIMATION FUNCTIONS FOR DECISION RULES 3.1.4.2

In the next section we will present a definition of the approximation function.

3. CONFLICT PROBLEM 3.1.4.3

This problem arises when one would like to apply the parallel strategy in step 3. we discuss this problem later. in particular here arises the composition of decision rules problem. this problem is related to the synthesis of the global decision rule (see step 1).

4. APPLICATION OF APPROXIMATION FUNCTION PROBLEM 3.1.4.4

The problem is related to the question how to choose a proper subset of approximation functions from f and in what order to apply them to get the best filtration of /A. To solve this problem we would like to apply genetic algorithms.

In the above procedure f one can distinguish two strategies. The first one S procedure from an actual decision table A for any $a \in$ /A' a sequence of decision rules of the following form:

$\tau_1 \Rightarrow \partial_{A-\{a\}} = \theta_1, \ldots, \tau_p \Rightarrow \partial_{A-\{a\}} = \theta_p$

satisfying $T(/A_a, \tau_i, K)$ for $i = 1,\ldots,p$.

These rules are used next to generate approximation functions by the second strategy H. the strategy H produces from the above sequence a new sequence

$$(S_1, \ldots, S_r)$$

Where S_1 is a set of non-conflict decision rules of the form $\tau \Rightarrow \partial_{A-\{a\}} = \theta$ for some $a \in A$, $\theta \subseteq V_d$ and formula τ over A-{a} and V.

The approximation function corresponding to the decision rules from S_1 are chosen by H for simultaneous application at the i-th step of transformation of the actual decision table /A.

In this way our procedure f has parameters S, H, /A, δ (by assumption 1 and K are fixed). The proper values for K and 1 should be chosen by making experiments with A.

Now we can formulate a general version of filtration problems.

Filtration Problem 3.1.4.1

Input: A, and $\delta \in [0,1] \cap Q$, where Q is the set of rational numbers.

Output: strategies S and H such that a δ-filtration of /A is returned after calling f with parameters S, H, /A and δ if such strategies exists.

Optimal filtration problem 3.4.1.2

Input : A

Output: inf {$\delta \in Q$: there exists strategies S and H such that a δ-filteration of A is returned after calling f with parameters S, H, /A and δ }

one can not expect that the solution S and H for above filtration problems are of polynomial time complexity with respect to the size of A and δ because the strategies S and H are based, e.g on procedure for decision rules generation and these procedures are based on the reduct set generation [9] .

nevertheless, one can build efficient heuristics for solving the filtration problem for practical applications.

Approximation functions for non-deterministic decision rules 3.2

For any decision table /A = (U, A ⋃ {d}) , formula τ over A and V and a threshold K satisfying the condition T(A, τ, K) one can define a τ - approximation function in A with threshold K .

$$f\,(A, \tau, K) : INF\,(A_\tau)\,|B \rightarrow INF\,(\{d\}, V)$$

By $f\,(A, \tau, K)\,(\alpha) = \{(d, i_0)\}$ for any $\alpha \in INF\,(A\tau)|B$ where

$\mu_{i0}\,(/A, \wedge\,\alpha \wedge \tau) = Max\,\{\mu_i\,(/A, \wedge\,\alpha \wedge \tau) : i \in 0_\tau\}$ and B is the set of conditions occurring in τ .

We apply the above construction to decision tables derived from a given decision table A = (U, A ⋃{d}). these decision tables are of the form /B = (U,B⋃ {d}) and they are constructed from information system B = (U, B⋃C), where B,C ⊆ A and B⋂C=ϕ by representing c by means of decision attribute c. we denote by

$code_C$ (or, code, in short) a fixed coding function for information vectors restricted to C in V_c defining c by :

$c(\chi) = code_C(\{a, a(\chi)) : a \in C\})$ for any $\chi \in U$. for a given threshold K we consider decision tables /B = (U, BU {c}) with the property that there exists a formula τ over B and V such that

T (B, τ, K) holds.

These tables correspond to near-to-functional relations of data represented in A by which term we understand that only one decision is pointed out with the strength exceeding the threshold K.

Let us observe that by assumption we have $INF(/B\tau)|B = INF(/A\tau)|B$ let 1 by a positive integer called the critical level of example. we denote by f(/A ,K ,1) the family of all functions of the form f(/B,τ,K) such that T(B,τ,K) holds and Max n (B,τ,i)>1. By f (/A , 1) (or f (/A), in short) we denote the union of the family { f(/A, K, 1) : $0 < K \leq 1$ } .

Let us observe the function f (B, τ, K) produces from the decision table. A= (U , A U{d}) with A = {a_1, ...a_m } a new decision table /A' = (U , A'U {d}) with A = {a_1, ...a_m }

defined by { (C , $code_C$ {(a_i, $a_i(\chi)$) : $a_i \in C$})= f (B, τ, K) ($inf_{IB}^{IA}(\chi)$)}

and $a'_i(\chi) = a_i(\chi)$ when $a_i \notin C$, for any $\chi \in \tau_{/B}$; if $\chi \in U - \tau_{/B}$ then $a'_i(\chi) = a_i(\chi)$ for any i .

The definition of approximation functions presented here can be treated as of possible formalization of near-to-functional relations between data; e.g an approximation function could choose the mean value from the set of values exceeding a given threshold. We plan to test on various data tables the properties of different notions of approximation functions from the point of view of application to data filtering.

Conflict problem 3.3

Let us assume that a set f of approximation functions of A is given. Now we would like to transform rows in decision table in parallel i.e by simultaneous application of function chosen from f on the basis of some strategies. In this case it is necessary to solve the conflict problem created by functions from f.

Let f be a given set of approximation function and let $c \in A$. let f(C) be the set of all functions from f into INF({c}, V) . We say that f(C) is conflicting in A. iff there exist $\chi \in$ U and f, $f' \in f$ (c) such that

$$f(\inf_B(\chi) \neq f'(\inf_{B'}(\chi)$$

Where

$$f : \mathrm{INF}\,(A\tau)|B \to \mathrm{INF}\,(\{c\}, V)$$

$$f : \mathrm{INF}\,(A\tau')|B' \to \mathrm{INF}\,(\{c\}, V) \quad \text{for some } \tau, \tau'$$

We say that f is conflicting in A iff f(c) is conflicting in /A for some c $\in$ A .

If the value set of c is ordered then one of the possibilities to resolve the conflict (created by f(c) is to take as the new value of c for a given $\chi \in$ U a randomly chosen value from the interval [min, max], where Min and Max are the minimum and maximum of the values of function from f(c) at χ .

This is an analogy to morphological operation which in analytical from modify geometrical objects according to min and max functions. This will be explained in the sequel.

Another strategy to resolve the conflict created by f(c) is to define as the new value of c from a given $\chi \in$ U the value of an approximation function corresponding to a formula τ (f , c) representing in some sense the influence of all elements of f(c) on c and built from some formulas defining conflict functions from f(c). One can search for such a formula a mong disjunctions and conjunctions of formulas defining conflicting functions from f(c).

The main constraints in searching for the conflict resolving functions f : INF (/Aτ) |B $\rightarrow$ INF ({c} , V) where B = (U , B$\cup${c}) Are the necessities to maintain:

(1) Max$_i$ n (/A, $\wedge\, \chi \wedge \tau$, i) > 1 for $\alpha \in$ INF (/A$_\tau$) |B where B is the set of condition occurring in τ and 1 is the critical level of examples;

(2) The classification of objects by /A and /A' where /A' is obtained from /A by applying f, i.e. $\partial_{/A'} = \partial_{/A}$.

Chapter Four
Mathematical Morphology

Mathematical morphology 4

Introduction 4.1

Mathematical morphology started in 1964 with the work of Georges Matheron on geometry of porous media and the prediction of their permabilities and the work of Jean Serra on petrography of iron ores and the prediction of their milling properties. The theoretical and applied research in the field led to the emergence of mathematical morphology as a mathematical discipline in its own right, the first stage of development was delineated with monographs[6][9]

The idea of mathematical morphology have been assimilated and further developed by image and signal processing communities. As a result the mathematical morphology has been extended to grayscale object and signals. Mathematical morphology is an actively developed with main research directed towards new efficient algorithms, analysis of complex filters, and analysis of theoretical foundations of morphology.

A morphological action on an object results in a geometrical filtering of the object which permits to characterize it in terms of some qualitative attributes (Area, permits, number of connected components....)

This aim requires recognizing the structure of the object described by means of a Boolean algebra R of relations generated by the two primitive relations of set inclusion and set intersection. Experimental structure recognition is achieved for a relation in R by means of model pattern (S) representing this relation (structural elements (S)). Moving a structural element B about an object X defines the object ϕ (x) of elements where the relation represented by B is fulfilled. The operation ϕ is a morphological operation.

We begin with the binary case i.e. our objects will be subsets of either an Euclidean E or a digital space Z^2.

The binary case 4.2 [12]

Definition 4.2.1

Any subset $A \subseteq R^2$ we will call it an image. Besides dealing with the usual set-theoretic of union and intersection, morphology depends extensively on the translation operation.

Definition 4.2.2

The translation of A by a point χ is denoted by A+ χ or $(A)_\chi$ and is defined as follows:

$$A+\chi = \{ a+\chi : \text{for some } a \in A \}$$

Geometrically, A+χ results by translating every point of A along the vector χ. Figure (4.2.1) illustrates A+χ and B+χ , where A, B $\subset R^2$. Notice that if a point y of the input image A coincides with the origin, that this point in the translated image A+χ correspond to point χ.

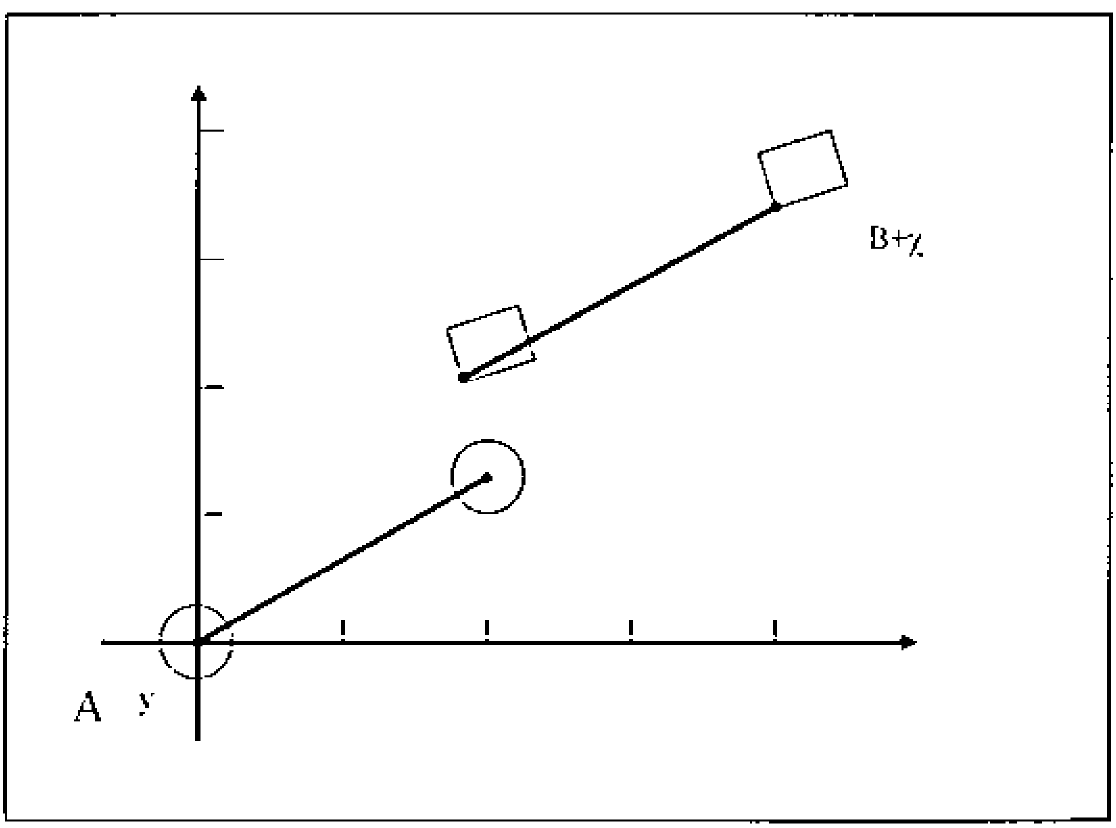


Figure (4.2.1)Translations of disk A and square B byχ (A,B$\subset R^2$)

Definition 4.2.3

The reflection of A is denoted by-A or A^s and is defined as follows:

$$-A = \{ -a : \text{for some } a \in A \}$$

Note that the use of the notation $-A = \{ -a : a \in A \}$ where $-A$ is the scalar multiple of the vector a by -1 . thus , $-A$ is simply A rotated 180^0 around the origin .

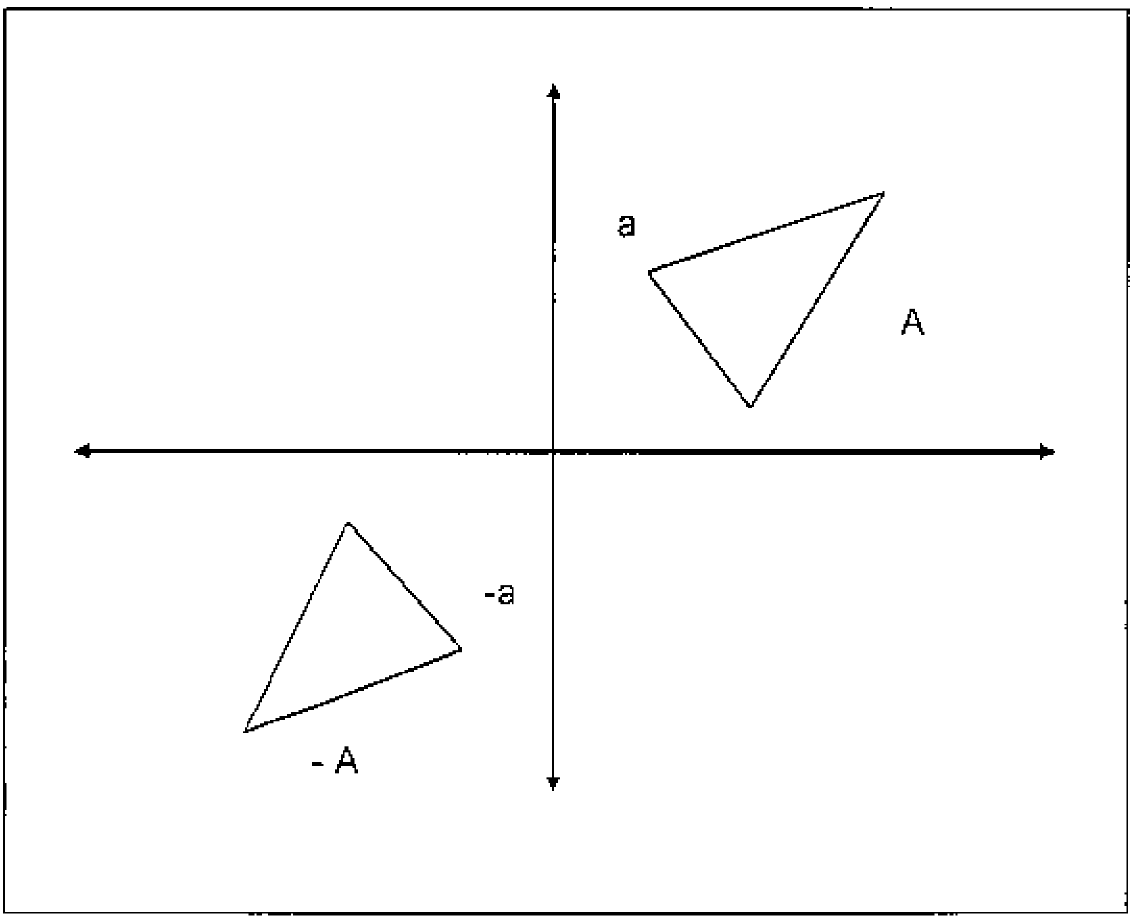


Figure (4.2.2)

Example 4

Let $A = \{ (0,0), (1,0), (0,1), (1,1), (2,2) \}$ and $\chi = (3,1)$

Then $A+\chi = \{ (3,1), (4,1), (3,2), (4,2), (5,3) \}$.

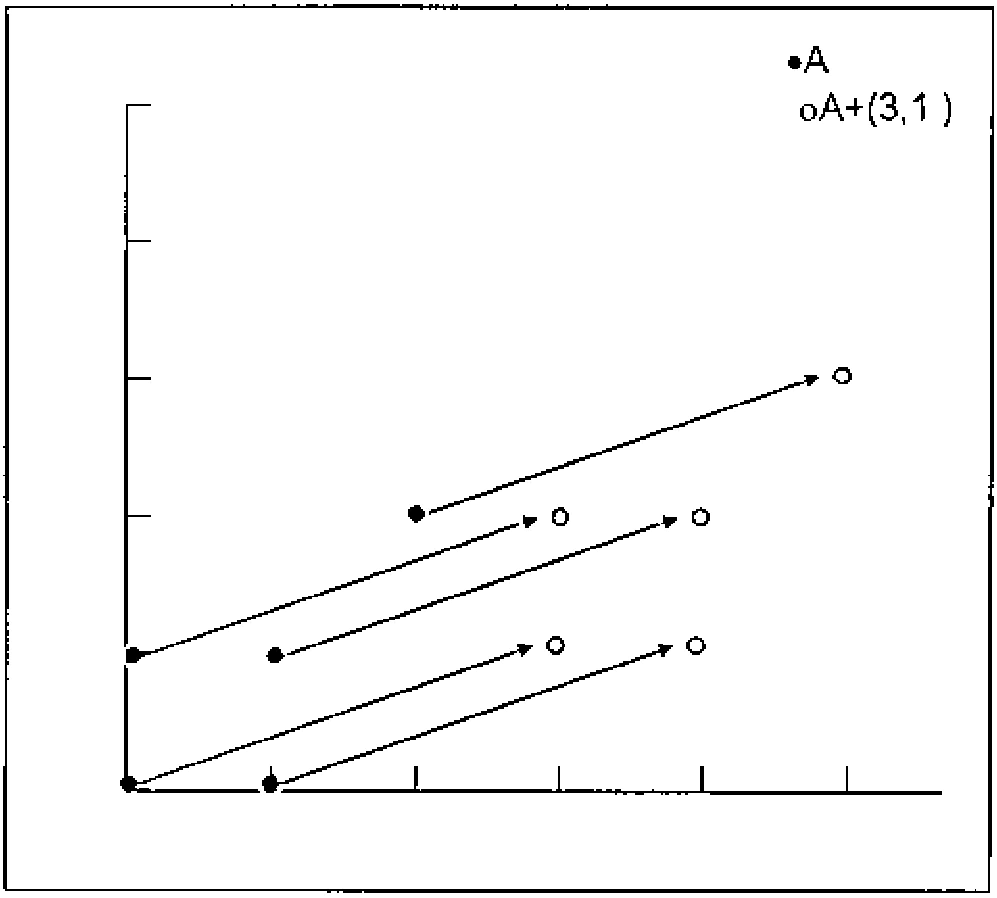


Figure (4.2.3) Translation of a discrete image

Remark 4.2.1

Note that the point z is in the translated set $A+\chi$ if and only if there exists some point a' in A such that z= a' $+\chi$

Also, because vector addition is commutative, we can write $\chi+A$ interchangeably with $A+\chi$.

Now we introduce the two fundamental operations which are utilized in the morphological analysis of two valued images, which are defined as follows:

Definition (Minkowski addition) 4.2.4

Given two images *A* and *B* in R^2 , we define the Minkowski sum $A \oplus B$ set-theoretically as :

$$A \oplus B = \bigcup_{b \in B} A + b$$

Where $A \oplus B$ is constructed by translating A by each element of *B* and then taking the union of all the resulting translates.

Example 4.2.4.1

Let *A* the unit disk centered at (2,2) and let $B = \{(4,1), (5,1), (5,2)\}$. Then $A \oplus B$ is the union of the sets *A*+ (4,1) , A+ (5,1) and *A*+ (5,2), *A*,*B* and $A \oplus B$ are depicted.

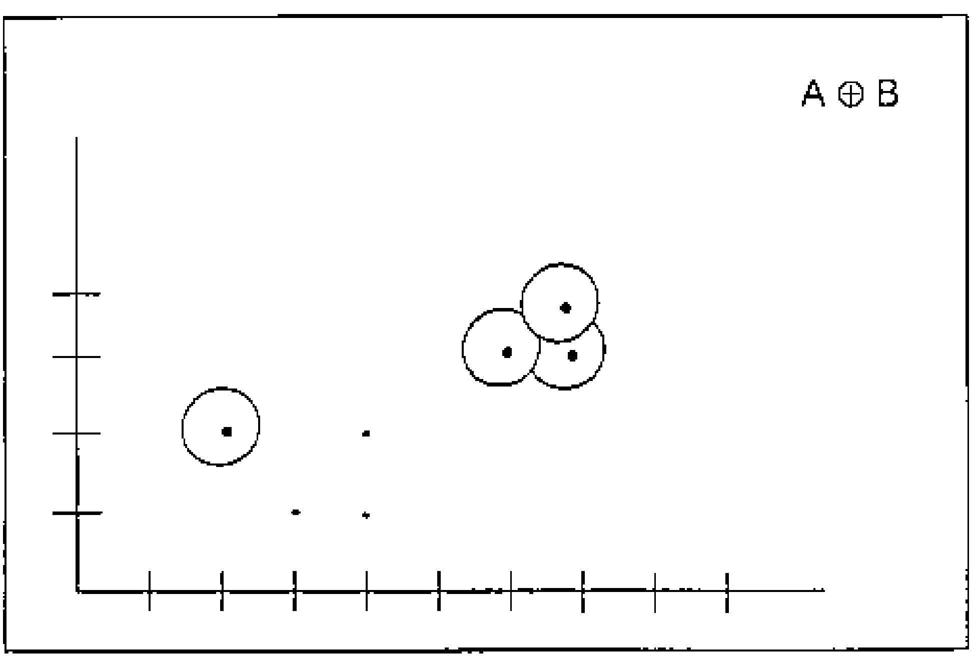


Figure (4.2.4.1) minkowski addition

A ⊕ B = { A+(4+1) } U { A+(5,1) } U { A+(5,2) }

Definition (Minkowski Subtraction) 4.2.5

Given images *A* and B in R^2 · We define the Minkowski difference

$$A \ominus B = \bigcap_{b \in B} A + b$$

In this operation , *A* is translated by every element of *B* and then the intersection is taken .

Example 4.2.5.1

Consider the 3 by 2 rectangle *A* in figure (4.2.5.1), let $B = \{ (4,0), (5,1) \}$.

There $A \ominus B$ is the intersection of the translates A+(4+0) and A + (5,1) . that is , $A \ominus B$ is the 2 by 1 rectangle.

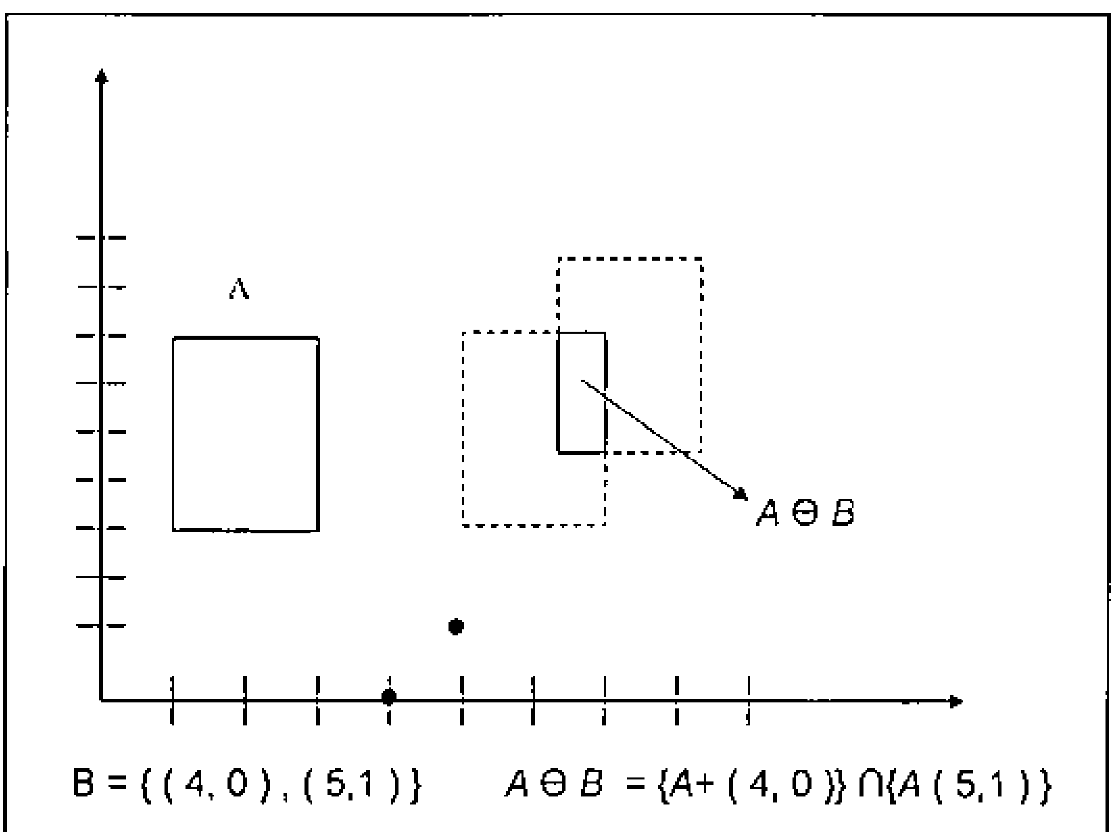


Figure (4.2.5.1) Minkoaisk subtraction

Theorem 4.2.5.1

If A and B are two images in R^2 , Then

$$A \ominus B = \{ \chi : -B + \chi \subset A \}$$

Proof

Since $A \ominus B = \bigcap_{y \in B} A + y = \bigcap_{y \in B} \{ \chi : \chi \in A + y \}$

$$= \bigcap_{y \in B} \{ \chi : -y + \chi \in A \} = \{ \chi : -B + \chi \subset A \}$$

Thane $A \ominus B = \{ \chi : -B + \chi \subset A \}$

where $A \ominus B$ can be found by first rotating B 180^0 around the origin and then all points χ such that the translate by χ of that rotated image is a subimage (subset) of A.

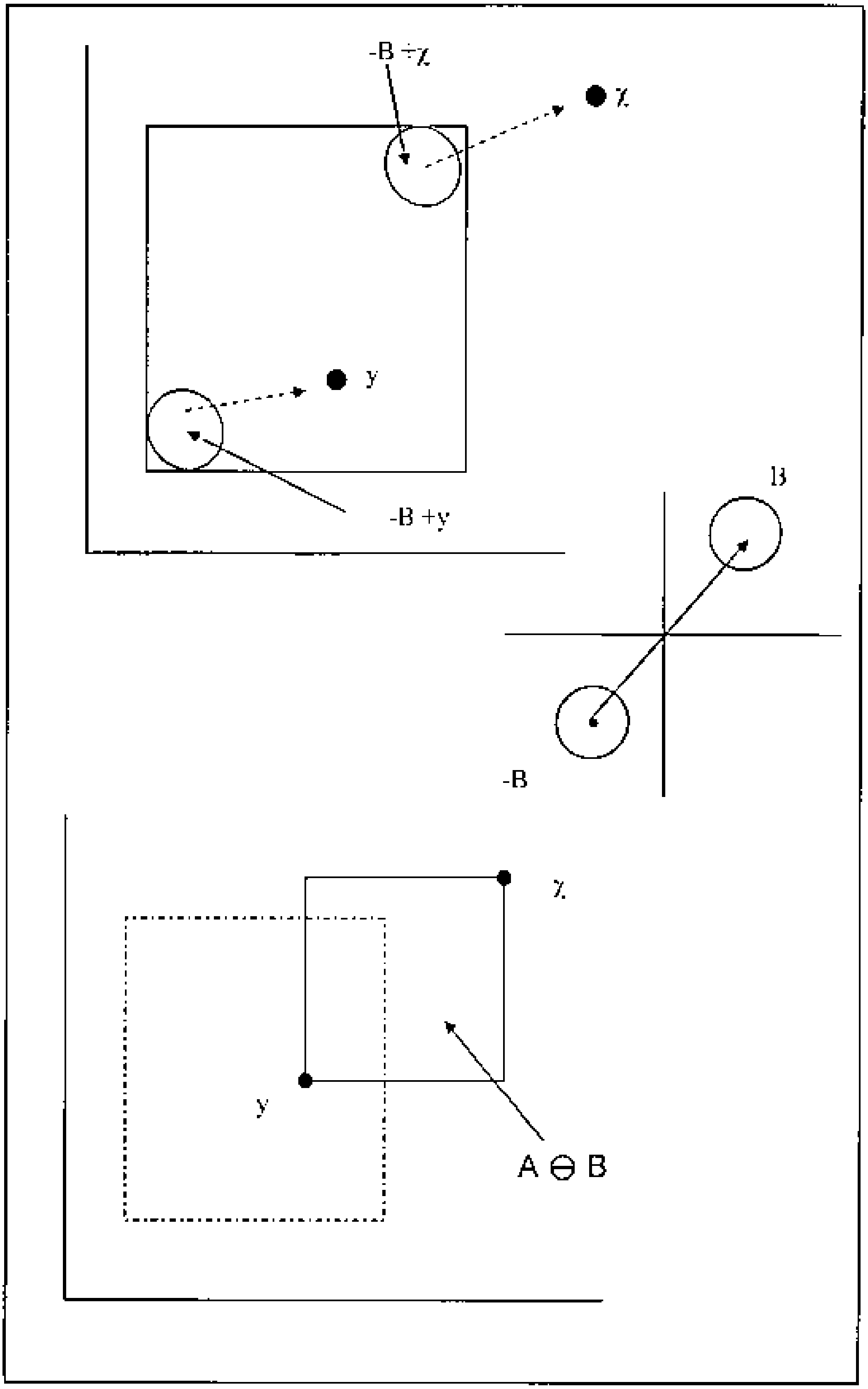


Figure (4.2.5.2) Minkowski subtraction by fitting

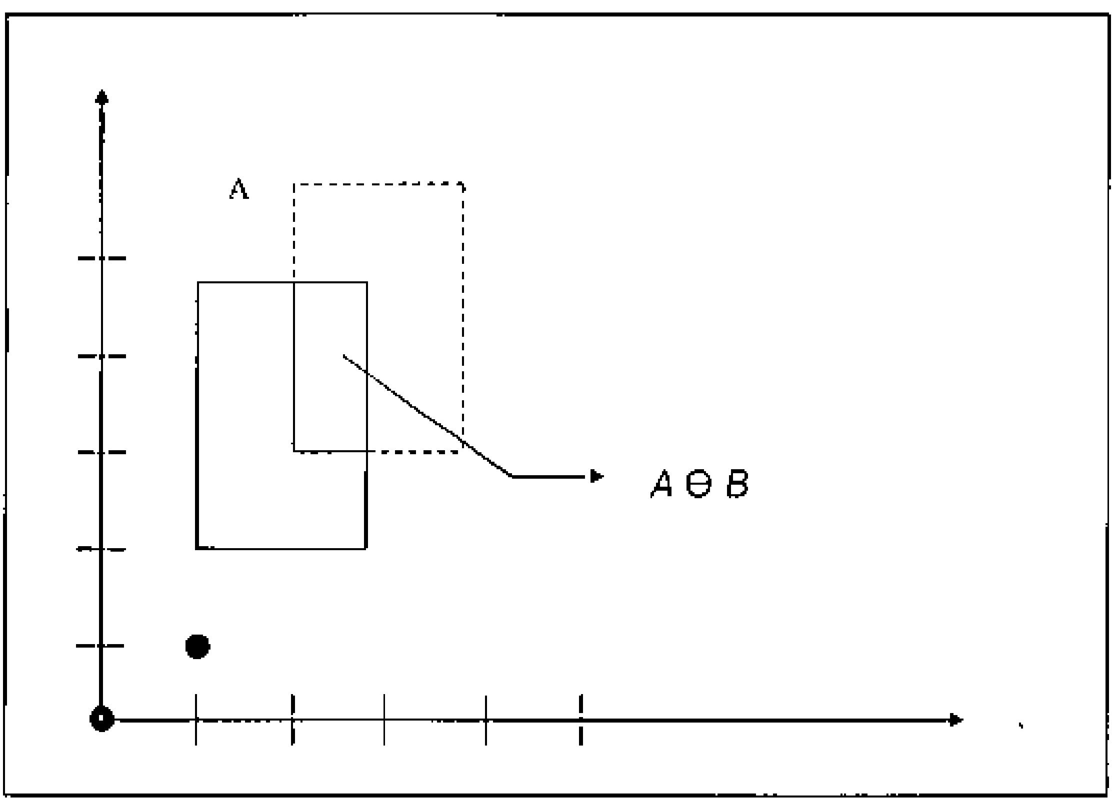
Remark 4.2.5.1

Note that the output image $A \ominus B$ is not necessarily a subimage of the original image A , we are assured that $A \ominus B$ is a subimage of A only if B contains the origin .

Example 4.2.5.2

Consider the 3 by 2 rectangle A in figure (4.2.5.3), let $B = \{ (0,0) , (1,1) \}$.

Then $A \ominus B$ is the intersection of the translates $A+(0,0)$ and $A + (1,1)$. That is, $A \ominus B$ is the 2 by 1 rectangle depicted in figure (4.2.5.3).



$$A \ominus B = \{ A + (0,0) \} \cap \{ A + (1,1) \}$$

Figure (4.2.5.3) Minkowski subtraction

Then $A \Theta B \subseteq A \Leftrightarrow (0, 0) \in B$.

Erosion and Dilation 4.2.6

Now we introduce tow primitive operations which will be used in many of the morphological algorithms that are discussed in this chapter .

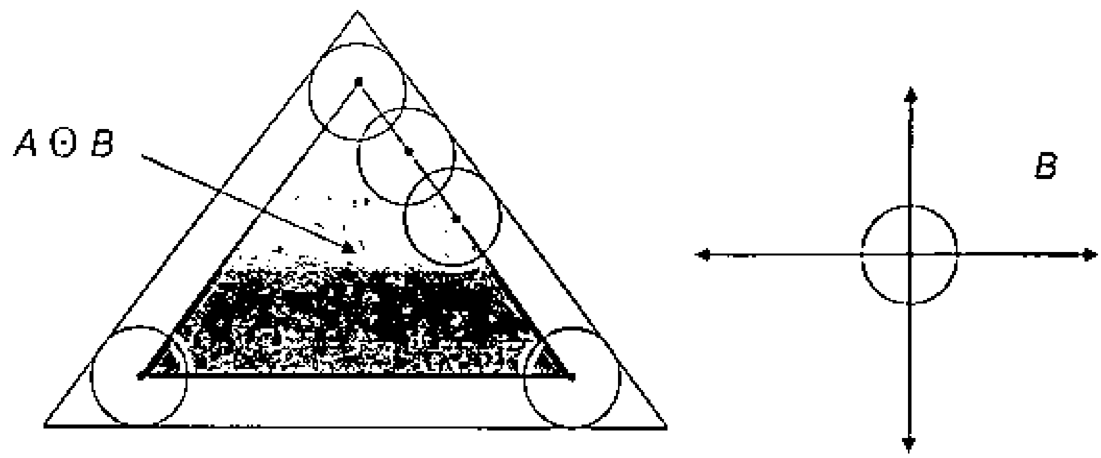
Erosion 4.2.6.1

The fundamental operation of mathematical morphology is erosion. All mathematical morphology depends to this notation. The erosion of an input image A by a structuring element B , is defined as follows:

$$A \Theta B = \{ \chi : B + \chi \subseteq A \} \quad \rightarrow ❶$$

This means that in order to perform the erosion of A by B we translate B by χ so that this lies inside A. the set of all points χ satisfying this condition constitutes $A \Theta B$

Figure (4.2.6.1) illustrates the erosion of a triangle by a disk



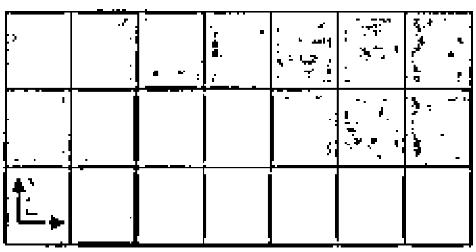
Figure(4.2.6.1)$A \Theta B$ is the internal triangle according To equation ❶

The erosion of an image can also be found by intersecting all translates of the input image by the reflection of structuring element:

$$A\Theta B = \bigcap \{ A+b : b \in -B \} \rightarrow \text{❷}$$

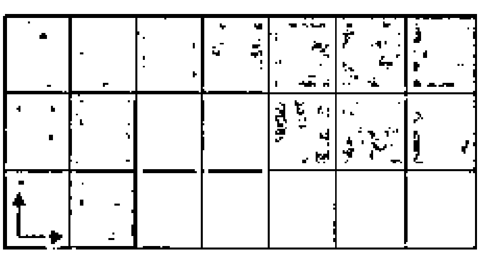
An example of performing erosion using equation ❷ is illustrated in figure below

The arrows denote the origin and the shaded area represents the points.

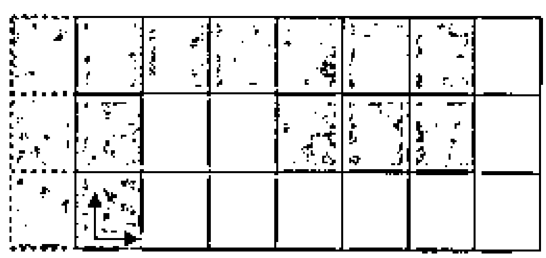


A

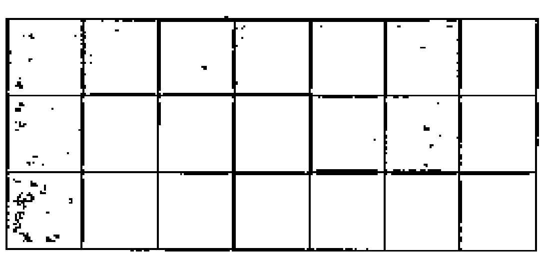
$B = \{(0,0),(0,1)\}$



A+(0,0)



A+(0,-1)



$A\Theta B = \cap \{ A+(0,0), A+(0,-1) \}$

figure (4.2.6.2) AΘB results from A+ (0,0) ∩ A+(0,-1) (the intersection of some translation of A) according to equation ❷

Dilation 4.2.6.2

The dual operation to erosion is dilation. Dilation of an input image A by a structuring element B , is defined as follows:

$$A \oplus B = \bigcup \{ B + a : a \in A\} \quad \rightarrow ❸$$

This means that in order to perform the dilation of A by B we first translate B by all points of A. The union of these translations constitutes $A \oplus B$.

Figure (4.2.6.3) illustrates the dilation of a triangle by a disk.

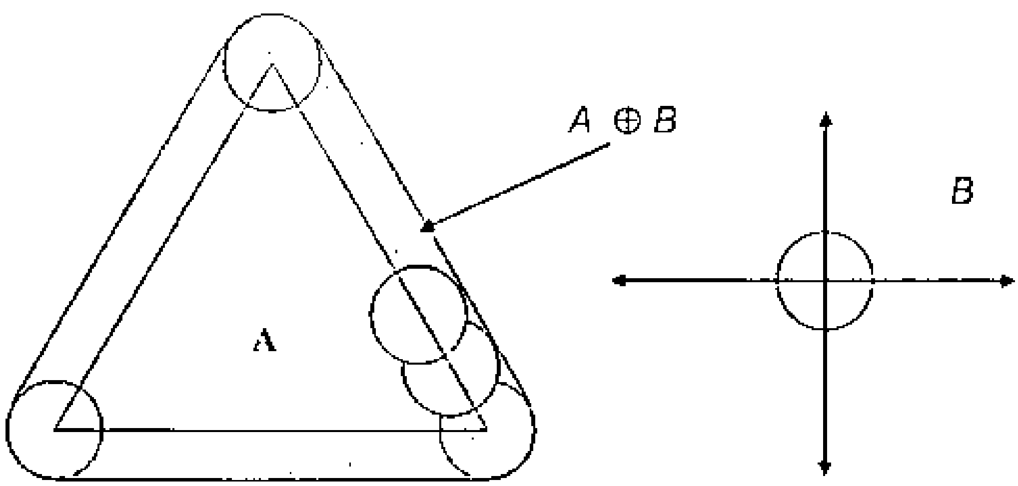


Figure (4.2.6.3) A ⊕ B is the external triangle with rounded corners , according to equation ❸

Dilation is both commutative and associative

$A \oplus B = B \oplus A$ and $(A \oplus B) \oplus C = A \oplus (B \oplus C) \rightarrow$ ❹

By commutatively $A \oplus B = \bigcup\{ A + B : b \in B \} \rightarrow$ ❺

This means that dilation can also be formed by translating the input image by all points in the structuring element and then taking the union. An application of equation (5) is illustrated in example of figure (4.2.6.4)

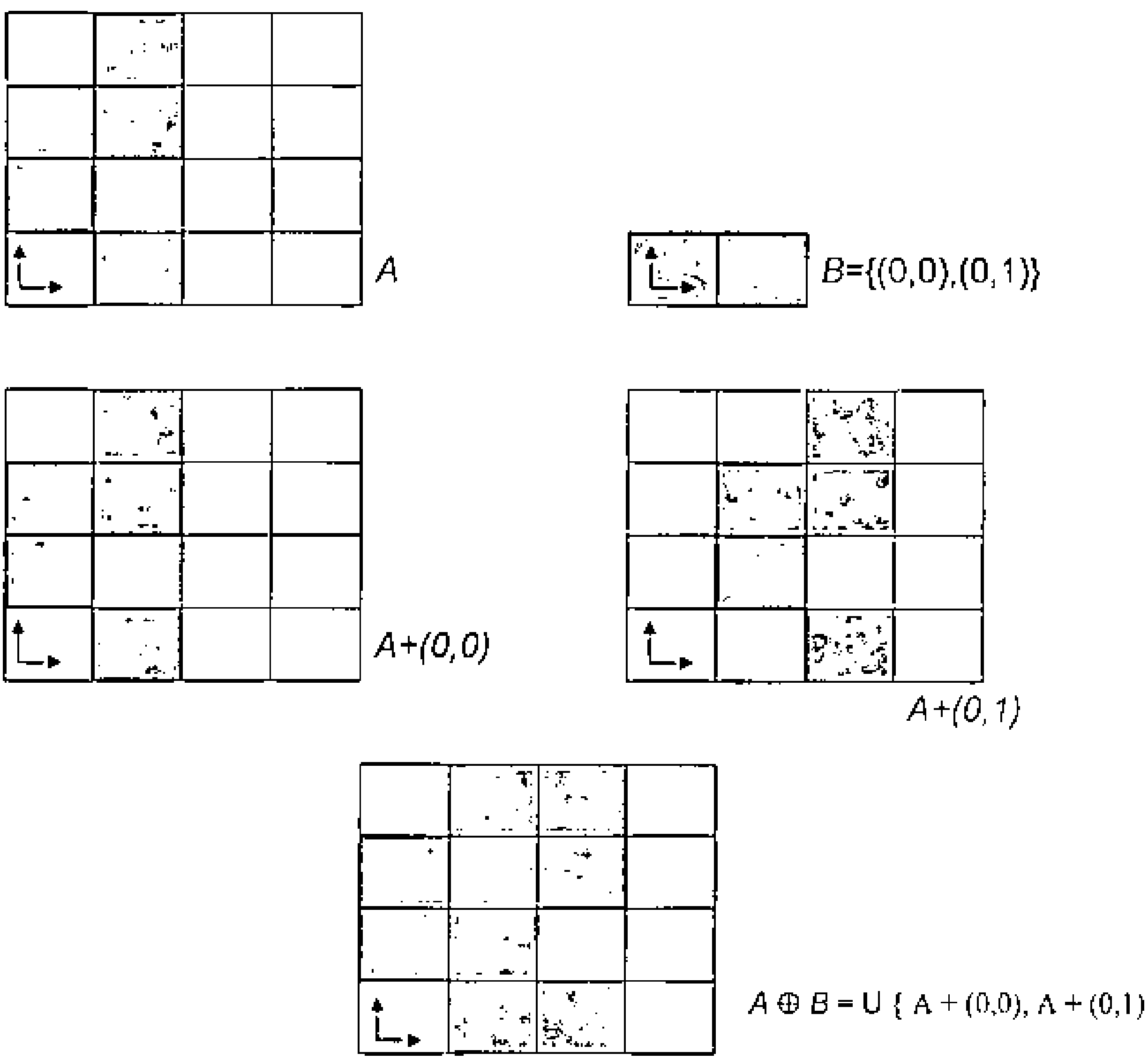


Figure (4.2.6.4) $A \oplus B$ results from A + (0,0) UA + (0,1) according to equation ❺

The body of operation produced from erosions and dilation is the morphological algebra of operations.

In particular, the composition : the erosion followed by the dilation is the opening , and the composition. The dilation followed by erosion is the closing.

Opening 4.2.6.3

A secondary operation of great importance in mathematical morphology is the opening operation. Opening of an input image A by a structuring element B is defined as follows:

$$o(A,B) = A \circ B = (A \ominus B) \oplus B \rightarrow ❻$$

An equivalent definition for opening is :

$$o(A, B) = A \circ B = \cup \{B + \chi : B + \chi \subseteq A\} \rightarrow ❼$$

This means that in order to open *A* by *B* we first translate *B* by χ so that this lies inside *A*. the union of these translations constitutes *A* o *B* for instance , the opening of a triangle *A* by a disk *B* (the origin coincides with the centre of the disk) is the triangle *A* with rounded corners. In general opening by disk round or eliminates all peaks extending into the image background.

Closing 4.2.6.4

The other important secondary operation is closing. Closing of an input image *A* by a structuring element *B* is defined as follows:

$$C(A,B) = A \bullet B = (A \oplus\ )\ \theta\ B \rightarrow ❽$$

For instance , closing a triangle *A* by a disk *B* (the origin is on the centre of the disk) yields the same triangle *A*. in general, closing by a disk rounded or eliminates all cavities extending into image foreground.

Proposition 4.2.6.1

$Z \in O(A, B)$ if and only if $[(-B) + Z\} \cap [A\ \theta\ (-B)] \neq \Phi$

Proof

$Z \in O(A, B)$ if and only if $Z \in \bigcup_{y \in B} [A \ominus (-B)] + y$

If and only if there exists $b \in B$ with $Z \in [A \ominus (-B)] + b$

If and only if there exists $b \in B$ and $w \in [A \ominus (-B)]$

Where become $Z = w + b$, or equivalently , with $w = -b + Z$. But this means precisely that ,$w \in [(-B) + Z] \cap [A\theta\ (-B)]$

Which means that intersection in the proposition is nonempty.

The binary case 4.3

Let us assume that B is a given subset of E called a structuring object ,we denote by + the vector addition in E . Morphological operations are defined by means of two set operations on subset of E called the Minkowski sum and the Minkowski difference. For simplicity sake, in presenting Morphological operations we assume from now on that the set B is symmetric i.e. $B = -B$

The Minkowski sum $\oplus$ is defined for subset A, B $\subseteq$ E by

I. $A \oplus B = \{ \chi + y : \chi \in A \text{ and } y \in B \}$

And the Minkowski difference 0 is defined for A, $B \subseteq$ E by

II. $A \ominus B = \{ \chi \in E : \{\chi\} \oplus B \subseteq A \}$.

The Minkowski sum and the Minkowski difference operations are employed in the definition of two basic morphological operations vis versa . The dilation by B and the erosion by B. the dilation of $X \subseteq E$ by B , denoted by $d_B(X)$ is defined by

III. $d_B(X) = X \oplus B$

The erosion of X by B , denoted by $e_B(X)$ is defined by

IV. $e_B(X) = X \ominus B$

Morphological operations can be composed and new operations can be generated by means of set – theoretical operations. The body of operations produced from erosions and dilation is the morphological algebra of operations.

In particular, the composition : the erosion by *B* followed by the dilation by B is the opening by *B*, denoted by O_B , and , the composition : the dilation by B followed by the erosion by B , is the closing by B denoted by C_B .

Formally

V. $O_B(x) = d_B \circ e_B(x)$

VI. $C_B(x) = e_B \circ d_B(x)$ for any $x \subseteq E$

The meaning of the opening by B and the closing by *B* can be best understood from the following characterization.

Proposition 4.3.1

a. $O_B(X) = \{ \chi \in E : \exists y (\chi \in \{ y \} \oplus B \subseteq X \}$.

b. $C_B(X) = \{ \chi \in E : \forall y (\chi \in \{y\} \oplus B \Rightarrow (\{y\} \oplus B) \cap X \neq \emptyset) \}$

proof (a)

$O_B(X) = \{ \chi \in E : \exists y (\chi \in \{ y \} \oplus B \subseteq X \}$

This formula is similar to $O(A,B) = \cup \{ B+y : B+y \subset A \}$

That we will to proof $O(A,B)$

By proposition (4.2.6.1) , $Z \in O(A,B)$ if and only if there exists a point y such that $y \in [(-B) + Z] \cap [A \,\theta\, (-B)]$, that is , if and only if there is a point y such that $Z \in B+y$ and $B+y \subset A$.

But this last statement means precisely that Z is in the union specified in the statement of the proposition.

proof (b)

$C_B(X) = \{ \chi \in E : \forall y\ (\chi \in \{y\} \oplus B \Rightarrow (\{y\} \oplus B) \cap X \neq \emptyset) \}$ This formulation similar to $C(A,B) = \cap \{ (B+y)^c : B+y \subset A^c \}$

Then we will to proof $C(A,B)$,We apply duality to the proposition (a)

$C(A,B)^c = O(A^c, B) = \cup \{ B + y : B + y \subset A^c \}$

Application of De Morgans's law gives:

$C(A, B) = \cap \{ (B + y)^c : B + y \subset A^c \}$.

Let us observe the parallelism between the description of X provided by the opening of X by B and the closing of X by B and the description of X provided by the B-lower and the B-upper approximation of X in the rough set theory.

The operations in the binary case based on the additive structure of E and set theoretical notation of inclusion and intersection have been extended for needs of image and signal processing to grayscale, respectively, we present below the most essential points of grayscale morphology.

The grayscale case 4.4

The objects of grayscale morphology are functions, representing e.g. grayscale visual objects or signals.

An object F is a function $F{:}A \rightarrow E$ where $A \subseteq E^n$ for some n.

the operation of mathematical morphology are performed on functions indirectly by means of associated objects called umbrae.

A subset $X \subseteq E^n{}_x E$ is called an umbrae if the following holds

(i) $(\chi,y) \in X \wedge z \leq y \Rightarrow (\chi,z) \in X$.

for a function $F{:}A \rightarrow E$, the umbrae of F, denoted by $U[F]$ is the set $\{ (x,y) \in A_x E : y \leq F(x) \}$.

The objects are recovered from operations on umbrae by means of the envelop operation. Suppose the X is an umbrae. The envelop of X, denoted by E (X), is defined as follows

(ii) $E(X)(\chi) = \sup \{ y \in E: (\chi,y) \in x\}$ for $\chi \in E^n$

Let $F{:}A \rightarrow E$ be an object and $K{:}C \rightarrow E$ be a fixed object called a structuring object. We denote again by $\oplus$, $\ominus$, the Minkowski sum, difference operators, respectively in the space E^{n+1} induced by the vector addition +.

The operation of dilation by K, denoted by d_k and of erosion by K, denoted by e_k are defined F by

(iii) $d_k(F) = E(U[F] \oplus U[K])$.

(iv) $e_k(F) = E(U[F] \ominus U[K])$.

The morphological algebra is defined in the grayscale case in the manner analogous to the case of binary morphology. in particular, the opening by K, denoted by O_k, and the closing by K, denoted by C_k are defined as

(v) $O_k(F) = E(O_{U[k]}(U[F])$

(vi) $C_k(F) = E(C_{U[k]}(U[F])$

It will be convenient for our purposes to present the above operations in binary as well as grayscale in an analytical form.

Mathematical morphology in analytical form 4.5

we will present morphological operations in an analytical form i.e as operations on vector representations of objects. we will restrict our selfes in the binary case .

The binary case 4.5.1

we will represent the binary objects $X \subseteq E^2$ as binary vectors of the form

$$V = < V_x : \chi \in E > \ (V_x = 1 \text{ iff } \chi \in X).$$

The function M, N, ρ and $\sqcup$ on Z^2 into Z^2 which represent, respectively, the dilation by *B*, the erosion by *B*, the opening by *B*, and closing by B are defined as follows

(i) $M(V) = V'$ where $V'_x = \max\{V_y : y \in \{x\} \oplus B\}$.

(ii) $N(V) = V'$ where $V'_x = \min\{V_y : x \in \{y\} \oplus B\}$.

(iii) ρ (V) = V' where V'_x = max { min {V_z : z∈ {y} ⊕ B}; x ∈ {y} ⊕ B.

(iv) ⊔ (V) = V' where V'_x = min { max {V_z : y∈ {z} ⊕ B}; y ∈{x} ⊕ B }.

We call X a centre of {x} ⊕ B and we call the set {x} ⊕ B the influence set of x . Let us observe the following:

(a) The operations M and *N* can be regarded as analytical representation of some strategies for negotiating among conflicting influences on X of elements of the influence set of X and among conflicting influences on X of centers of influence sets containing X, respectively.

(b) The operations ρ and ⊔ can be similarly regarded as analytical representations of some strategies for expressing a common influence on an element X of influence sets containing X (first we erode {x}⊕ B to y for y∈ {x} ⊕ B next we dilate these y's to x) and of influence sets intersecting the influence set of X, respectively (first we dilate centers of influence sets

$\{z\} \oplus B$ containing y to y for $y \in \{x\} \oplus B$ next $\{x\} \oplus B$ to x).

Analytical morphology 4.6

The scheme of mathematical morphology presented in the last has been based on two assumptions.

(a) Morphological objects are subset of a space E with the underlying linear structure, therefore elements at E can be translated one into another by means of this structure.

(b) Objects are represented as vectors in a space of states with a given algebraic structure. e.g. $< Z^2$, max, min $>$ in the binary case and $< E , + >$ in the grayscale case, therefore a new states of $X \in E$ can be found as a function of states of some other elements of E .

Analytical morphology as formulated below can be regarded as an attempt to develop a version of mathematical morphology suitable for contexts in which either assumption (a) or (b) may not be fulfilled.

(1) The general scheme 4.6.1

(a) objects, structuring objects 4.6.1.1

we assume that objects of analytical morphology from a set $o\ell$. A structuring object is a collection y of objects $y \subseteq o\ell$ together with an action $\Im_{\checkmark y}$: $o\ell \rightarrow p(o\ell)$ the function $\Im_{\checkmark y}$ can be regarded as a generalization of the translation in binary or grayscale morphology.

(b) influence sets, reciprocal influence sets 4.6.1.2

the set $\Im_{\checkmark y}(\chi)$ will be denoted by $S(\chi)$. for an object χ, we will say that χ is a centre of $S(x)$, let us observe that a set of the form $S(\chi)$ may have more than one centre. the set $S(\chi)$ will be called the influence set of χ. the set $R(\chi) = \{y \in o\ell : \chi \in s(y)\}$ will be called the reciprocal influence set of χ.

(c) dilations, erosions 4.6.1.3

We assume that for any $\chi \in o\ell$ a set V_χ of states over X is given.

for each $\chi \in o\ell$, the dilation at χ, denoted by d_x, is mapping

$$d_\chi : \Pi \{V_y : y \in R(\chi)\} \rightarrow V_x$$

and the erosion at x , denoted by e_χ is a mapping

$$e_\chi : \Pi \{V_y : y \in S(\chi)\} \rightarrow V_x$$

(d) openings and closing 4.6.1.4

for $\chi \in o\ell$, we lat $N(\chi) = \cup \{S(y) : y \in R(\chi)\}$; the opening at x, denoted by O_χ, is a mapping

$$O\chi : \Pi\{V_z : Z \in N(\chi)\} \to V_\chi$$

given by

$$O_\chi (< V_z : Z \in N(x) >) = d_x (< e_y (< V_z : Z \in S(y) > : y \in R(x) >)$$

Similarly

$$M(\chi) = \cup \{R(y) : y \in S(\chi)\}$$

and

$$C_x : \Pi \{ V_z : Z \in M(x)\} \to V_x$$

defined by

$$C_\chi \quad (< V_z : Z \in M(\chi) = e_\chi (< d_y (< V_z : Z \in R(y) > : y \in S(\chi) >)$$

Define the closing C_χ at χ.

Now we present an interpretation of these general operations in the case of case of decision tables.

(2) analytical morphology for decision tables 4.6.2

(a) objects, structuring objects 4.6.2.1

Let $/A = (U, A \cup \{d\})$ be a decision table. Objects of analytical morphology for $/A$ are non-empty subsets of $INF(C, V)$ Where $C \subseteq A$. A structuring object Ψ_Γ is defined by a given family $\mathbb{F}$ of approximation function in the following way:

$$\Psi_{\mathbb{F}} = \{ S_f(B, \tau, K) : F (B, \tau, K) \in \mathbb{F} \}$$

Where $S_F (B, \tau, K) = INF (/ \Lambda_\tau \mid BUC$ and

$$F(B, \tau, K) = INF (\Lambda_\tau) | B \rightarrow INF (C,V)$$

The influence set $S(\chi)$ for a given $\Phi \neq \chi \subseteq INF(C,V)$ is equal to

$\{S_f(B, \tau, K) : F (B, \tau, K) : INF (/ \Lambda_\tau) | B \rightarrow INF (C',V)$

where $C \cap C' \neq \Phi \}$

(b) dilatations, erosions 4.6.2.2

One can consider any approximation function

$F(B, \tau, K) : INF (/ \Lambda_\tau) | B \rightarrow INF (C,V)$ as a partial

function from INF(A,V) into INF(C,V) with the domain

$INF(/ \Lambda_\tau) | B$ on which the partial function equals to

$F(B, \tau, K)$. We assume that

$V_x = INF(C, V)$ if $\Phi \neq \chi \subseteq INF(C, V)$.

We define for the family $\mathbb{F}$ the sets N(C) and M (B) where

$N(C) = \{B \subseteq A : \exists F \in \mathbb{F} (F: INF(A) | B \rightarrow INF(C',V)$ and $C \cap C' \neq \Phi) \}$

$M(B) = \{C \subseteq A : \exists F \in \mathbb{F} (F: INF(B',V) \rightarrow INF(C',V)$, $C \cap C' \neq \Phi$ and

$B = B' \cup C') \}$.

If F : INF(A) / B → INF(C',V) then (B,C') (where C ∩ C' ≠ Φ is treated as a component of the influence set of X. the set N(c) corresponds to all such components. similarly , M (B) corresponds to the family of all objects S_F associated with approximation function influencing C and hence also X.

For each $x \in \mathcal{O}\ell$ the erosion at $X \subseteq V_\chi$ = INF(C, V), denoted by e_χ is a mapping.

$e_\chi : \Pi$ { INF(B,V): B ∈ N (C) } → INF(C,V)

Which can be defined by choosing a strategy for conflict resolving among different influences of approximation functions on χ.

The dilation of $X \subseteq V_\chi$ = INF (B, V), denoted by d_χ , is a mapping

d_χ: Π { INF(C,V): C ∈ M (B) } → INF(B,V)

Which can be also defined by choosing for conflict resolving among different influences of approximation functions on χ.

Conclusions

The method presented her is an approach to data filtering. The approach called analytical morphology combines the rough set theoretical ideas with the ideas of mathematical morphology. We expect that implementations based on ideas of analytical morphology will give effective tools for filtering of data encoded in decision tables especially when no inherent geometrical structure of the attribute set is assumed.

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الملخص

سوف ندرس طريقة تسمى بالتحليل الشكلي لتصفية البيانات وهذه الطريقة مبنية من بعض الافكار الاساسية لنظرية المجموعات الخشنة (التقريبية) والاشكال الرياضية.

الاشكال الرياضية معتمدة اصلا على البناء الهندسي للاشياء بينما الهدف من الطريقة المدروسة في هذا البحث هو تقديم الادوات المناسبة لتصفية البيانات وذلك عندما لايشير مباشرة شكل هندسي لمجموعة البيانات.

مقدمة

فى الكثير من التطبيقات توجد حاجة ملحة لطرق مناسبة لتنقية البيانات حتى لو لم توجد فرضيات بنية هندسية مسبقة لمجموعة(الشروط) الخصائص على الاساسيات التى على ضوئها توجد القرارات .

غالبا هذه الوضعية تظهر عندما تكون البيانات العملية مسجلة فى جداول قرارات. المشكلة تظهر فى ايجاد اطار للعمل لتصفية البيانات دون الاسناد على فكرة هندسية.

فى هذه الدراسة سوف نرى كيف نتحصل من علاقات محلية من خلال بيانات بعض العلاقات الدالية القريبة ومن هذه العلاقات سوف نتحصل على مايسمى بالدوال التقريبية.

فى هذه الدراسة سوف نرى الية خاصة لتطبيق هذه الدوال التقريبية على البيانات لانتاج (للحصول) على التنقيه المناسبة (المطلوبة) للبيانات

هذا البحث مقسم الى اربع فصول كالاتى

الفصل التمهيدى: العلاقات والمعرفة

الفصل الاول : نظرية المجموعات التقريبية

الفصل الثانى : منظومة المعلومات

الفصل الثالث : تصفية المعلومات

الفصل الرابع : التحليل الشكلى لتصفية المعلومات

كليـــة العلـــوم

قســـم الرياضيات

عنـــوان البحـــث

((دراسة حول الإشكال الرياضية لجداول القرارات – وذلك باستخدام مفهوم المجموعات التقريبية))

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