Oscillation Theorems Concerning Damped Nonlinear Differential Equations Of Second Order

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Abstract:

In this paper, some oscillation criteria for solutions of second order damped nonlinear differential equations of the form

$$\left(r(t)\Psi(x(t))\dot{x}(t)\right)^{\bullet} + h(t)\dot{x}(t) + q(t)\Phi\left(g(x(t)), r(t)\Psi(x(t))\dot{x}(t)\right) = H\left(t, x(t), \dot{x}(t)\right) \qquad (E)$$

are obtained. Our results improve and extend some existing results in the literature. Some examples are given with its numerical solutions, which are computed using Runge Kutta method of fourth order to illustrate our results.

الملخص:

في هذا البحث، نقدم بعض معايير التذبذب لحلول المعادلات التفاضلية الغير خطية من الدرجة الثانية ذات الصورة العامة:

$$\left(r(t)\Psi(x(t))\dot{x}(t)\right)^{\bullet} + h(t)\dot{x}(t) + q(t)\Phi\left(g(x(t)), r(t)\Psi(x(t))\dot{x}(t)\right) = H\left(t, x(t), \dot{x}(t)\right) \qquad (E)$$

نتائجنا تحسن وتوسع بعض النتائج الموجودة في الدراسات السابقة. تم إعطاء بعض الأمثلة التوضيحية والتي تم حلها عدديا" باستخدام طريقة رونج– كوتا من الرتبة الرابعة لتوضيح صحة النتائج النظرية المتحصل عليها في هذا البحث.

Keywords: Damping terms, Nonlinear differential equations, Oscillatory solutions, Runge Kutta method.

AMO (MOS) Subject Classification : 34C 10, 34C15.

1. Introduction

This paper deals with the problem of oscillation of the solutions of the second order damped nonlinear differential equations of the form

$$\left(r(t)\Psi(x(t))\dot{x}(t)\right)^{\bullet} + h(t)\dot{x}(t) + q(t)\Phi\left(g(x(t)), r(t)\Psi(x(t))\dot{x}(t)\right) = H(t, x(t), \dot{x}(t)) \qquad (E)$$

where r, Ψ, h and q are continuous functions and $r(t) > 0$ *for* $t \geq t_0 > 0$. g is a continuous function for $x \in (-\infty, \infty)$, continuously differentiable and satisfies

$$xg(x) > 0 \text{ and } g'(x) \geq k > 0 \text{ for all } x \neq 0. \qquad (1)$$

The function Φ is continuous on RxR with

$$u\Phi(u,v) > 0 \text{ for all } u \neq 0 \text{ and } \Phi(\lambda u, \lambda v) = \lambda\Phi(u,v) \text{ for any } (\lambda, u, v) \in \mathrm{R}^3. \qquad (2)$$

and H is a continuous function on $[t_0, \infty) \times \mathrm{R} \times \mathrm{R}$ with

$$\frac{H(t, x(t), \dot{x}(t))}{g(x(t))} \leq p(t) \text{ for all } x \neq 0 \text{ and } t \geq t_0. \qquad (3)$$

Equation (E) is said to be superlinear if

$$0 < \int_{\pm\varepsilon}^{\pm\infty} \frac{du}{g(u)} < \infty \text{ for every all } \varepsilon > 0.$$

Throughout this study, our attention is restricted only to the solutions of the equation (E) which exist on some ray $[t_0, \infty)$. Such solution of the equation (E) is said to be oscillatory if it has an infinite number zeros, and otherwise it is said to be non-oscillatory. Equation (E) is called oscillatory if all its solutions are oscillatory, otherwise it is called non-oscillatory. The problem of determining oscillation criteria for second order nonlinear ordinary differential equations has received a great deal of attention of many authors.

Among numerous papers dealing with this subject we refer, for example to [1-26] and the references cited therein.

The equation (E) includes Emden-Fowler equation :

$$\ddot{x}(t)+q(t)|x(t)|^{\alpha}\operatorname{sgn} x(t)=0,\alpha>0 \qquad (E_1)$$

The conditions for oscillatory soltions of the equation (E_1) are studied by many authors (see [1], [26], etc).

The equation (E) also includes the following equation:

$$(r(t)\dot{x}(t))^{\cdot}+p(t)\dot{x}(t)+q(t)f(x(t))=0 \qquad (E_2)$$

which have been considered by X. Beqiri and E. Koci [1]. They established some sufficient conditions for the oscillation of the equation(E_2).

2. MAIN RESULTS

We state and prove here our oscillation theorems

Theorem ٢.1: Suppose that conditions (1), (2) and (3) hold and

(4) $a_1 \le \Psi(x) \le a_2$, $a_1, a_2 > 0$ and for $x \in R$.

(5) $h(t) \le 0$ $for\, t \ge t_0$.

(6) $q(t) > 0$, $for\, all\ t > 0$.

Furthermore, suppose that there exists a positive continuous differentiable function ρ on the interval $[t_0, \infty)$ with $\dot{\rho}(t) \ge 0$, $(\dot{\rho}(t) r(t))^{\bullet} \le 0$, $(\rho(t)h(t))^{\bullet} \le 0$, and such that

(7) $\displaystyle\int_{t_0}^{\infty} \frac{ds}{r(s)\rho(s)} = \infty$, $forevery\ t \ge t_0$.

(8) $\int_{t_0}^{\infty} \Omega(s)\,ds = \infty$,

where $\Omega(t) = \rho(t)\big(C_0 q(t) - p(t)\big) - \frac{a_2\,\dot{\rho}^2(t) r(t)}{4k\rho(t)}$.

Then, every solution of super-linear equation (E) is oscillatory.

Proof: Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (E) such that $x(t)>0$ on $[T,\infty)$, for some $T \geq t_0 \geq 0$. Define

$$\omega(t) = \frac{r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))} \quad , t \geq T$$

This, conditions (1), (2), (3), (6) and (E) imply

$$\left(\frac{r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))}\right)^{\bullet} \leq -[C_0 q(t) - p(t)] - \frac{h(t)\dot{x}(t)}{g(x(t))} - \frac{kr(t)\Psi(x(t))\dot{x}^2(t)}{g^2(x(t))} \quad , t \geq T \qquad (1\text{-}1)$$

We multiply the inequality (1-1) by $\rho(t)$ and integrate form T to t we have

$$\frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))}$$

$$\leq C_1 - \int_T^t \rho(s)[C_0 q(s) - p(s)]ds - \int_T^t \frac{\rho(s)h(s)\dot{x}(s)}{g\big(x(s)\big)} ds$$

$$+ \int_T^t \left[\dot{\rho}(s)\omega(s) - \frac{k}{a_2}\frac{\rho(s)}{r(s)}\omega^2(s)\right] ds$$

Where $C_1 = \frac{\rho(T)r(T)\Psi(x(T))\dot{x}(T)}{g(x(T))}$.

Thus

$$\frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))}$$
$$\leq C_1 - \int_T^t \rho(s)[C_0 q(s) - p(s)]ds - \int_T^t \frac{\rho(s)h(s)\dot{x}(s)}{g\big(x(s)\big)}ds$$
$$- \int_T^t \frac{k}{a_2}\frac{\rho(s)}{r(s)}\left[\eta^2(s) - \left(\frac{a_2\dot{\rho}(s)r(s)}{2k\rho(s)}\right)^2\right]ds$$

where $\eta(t) = \omega(t) - \frac{a_2\dot{\rho}(s)r(s)}{2k\rho(s)}$

Thus, for $t \geq T$, we have

$$\frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))}$$
$$\leq C_1 - \int_T^t \rho(s)\left[[C_0 q(s) - p(s)] - \frac{a_2\dot{\rho}^2(s)r(s)}{4k\rho(s)}\right]ds$$
$$- \int_T^t \frac{\rho(s)h(s)\dot{x}(s)}{g\big(x(s)\big)}ds \qquad (1-2)$$

The second integral in R. H. S. of the inequality (1-2) is bounded from above. This can be by using the Bonnet theorem, for all $t \geq T$, there exists $a_t \in [T, t]$ such that

$$- \int_T^t \frac{\rho(s)h(s)\dot{x}(s)}{g(x(s))}ds = -\rho(T)h(T)\int_T^{a_t} \frac{\dot{x}(s)}{g(x(s))}ds$$
$$= -\rho(T)h(T)\int_{x(T)}^{x(a_t)} \frac{du}{g(u)}$$

Since $(-\rho(t)h(t)) \geq 0$ and the equation (E) is super-linear, we have

$$-\infty < \int_T^t -\rho(s)h(s)\frac{\dot{x}(s)}{g(x(s))}ds \leq B_1, \qquad (1-3)$$

where $B_1 = -\rho(T)h(T)\int_{x(T)}^{\infty}\frac{du}{g(u)}$.

By (1-3) and the condition (8), (1-2) becomes

$$\frac{\rho(t)r(t)\Psi(x(s))\dot{x}(t)}{g(x(t))} \le C_1 + B_1 - \int_T^t \Omega(s)\,ds$$

By the condition (8), we have

$$\lim_{t\to\infty}\frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))} = -\infty.$$

Thus, there exists $T_1 \ge T$ such that $\dot{x}(t) < 0$ for $t \ge T_1$. The condition (8) also implies that there exists $T_2 \ge T_1$ such that

$$\int_{T_1}^{T_2}\rho(s)\left(C_0 q(s) - p(s)\right)ds = 0 \quad and \quad \int_{T_2}^{t}\rho(s)\left(C_0 q(s) - p(s)\right)ds \ge 0 \qquad for\ t \ge T_2$$

Multiplying the equation (E) by $\rho(t)$, from the conditions (2), (3), (5) and (6), we have

$$\rho(t)\left(r(t)\Psi(x(t))\dot{x}(t)\right)^{\bullet} + C_0\rho(t)g(x(t))q(t) \le \rho(t)g(x(t))p(t) \ ,t \ge T_2$$

where $0 < C_0 = \min_{\omega(t)\in R}\Phi(1,\omega(t))$.

Integrate the last inequality from T_2 to t, we obtain

$$\rho(t)r(t)\Psi(x(t))\dot{x}(t) \le \rho(T_2)r(T_2)\Psi(x(T_2))\dot{x}(T_2) + \int_{T_2}^{t}\dot{\rho}(s)r(s)\Psi(x(s))\dot{x}(s)ds$$
$$- g(x(t))\int_{T_2}^{t}\rho(s)\left(C_0 q(s) - p(s)\right)ds + \int_{T_2}^{t} g'(x(s))\dot{x}(s)\int_{T_2}^{s}\rho(u)\left(C_0 q(u) - p(u)\right)du\,ds$$

By the condition (4) and the Bonnet's theorem, for $t \geq T_2$ there exists $\gamma_t \in [T_2, t]$ such that

$$a_2 \rho(t) r(t) \dot{x}(t) \leq \rho(T_2) r(T_2) \Psi(x(T_2)) \dot{x}(T_2) + a_1 \dot{\rho}(T_2) r(T_2) [x(\gamma_t) - x(T_2)] - g(x(t)) \int_{T_2}^{t} \rho(s)(Cq(s) - p(s)) ds$$
$$+ \int_{T_2}^{t} g'(x(s)) \dot{x}(s) \int_{T_2}^{s} \rho(u)(Cq(u) - p(u)) du\, ds \;, t \geq T_2$$

Thus

$$a_2 \rho(t) r(t) \dot{x}(t) \leq \rho(T_2) r(T_2) \Psi(x(T_2)) \dot{x}(T_2) \;,\; t \geq T_2$$

Dividing the last inequality by $\rho(t) r(t)$, integrate from T_2 to t and the condition (7), we obtain

$$a_2 x(t) \leq a_2 x(T_2) + \rho(T_2) r(T_2) \Psi(x(T_2)) \dot{x}(T_2) \int_{T_2}^{t} \frac{ds}{\rho(s) r(s)} \to -\infty, as\, t \to \infty,$$

which is a contradiction to the fact that $x(t) > 0$ for $t \geq T$. Hence, the proof is completed.

Example2.1

Consider the following differential equation

$$\left(\frac{x^6(t) + 2}{t^3 (x^6(t) + 1)} \dot{x}(t) \right)^{\bullet} - t^2 \dot{x}(t) + t^3 \Big(x^3(t) + \frac{x^{15}(t)}{5x^{12}(t) + 8 \left(\frac{x^6(t) + 2}{t^3 (x^6(t)+1)} \dot{x}(t) \right)^4} \Big) = \frac{x^3(t) \sin(x(t) \dot{x}(t))}{t^5} \quad , t > 0$$

We note that $r(t) = \frac{1}{t^3}$, $h(t) = -t^2$, $q(t) = t^3$, $g(x) = x^3$, $\Phi(u, v) = u + \frac{u^5}{5u^4 + 8v^4}$,

$H(t, x(t), \dot{x}(t)) = \frac{x^3(t) \sin(x(t) \dot{x}(t))}{t^5}$, $\frac{H(t, x(t), \dot{x}(t))}{g(x(t))} = \frac{\sin(x(t) \dot{x}(t))}{t^5} \leq \frac{1}{t^5} = p(t)$ and for all $x \neq 0$ and $t > 0$. $\Psi(x) = \frac{x^6 + 2}{x^6 + 1}$ and $1 \leq \Psi(x) \leq 2$ for all $x \in \mathrm{R}$

Taking $\rho(t)=t$, $\dot{\rho}(t)r(t)=\frac{1}{t^3}>0$, $(\rho(t)h(t))^{\bullet}=-3t^2<0$ and $\left(\dot{\rho}(t)r(t)\right)^{\bullet}=\left(\frac{1}{t^3}\right)^{\bullet}=\frac{-3}{t^4}<0$ for all $t>0$ and $\int_{t_0}^{\infty}\frac{ds}{\rho(s)r(s)}=\int_{t_0}^{\infty}s^2ds=\infty.$

$$(1)\quad \int_{t_0}^{\infty}\Omega(s)ds=\int_{t_0}^{\infty}\left[\rho(s)(C_0q(s)-p(s))-\frac{a_1\rho^{\bullet 2}(s)r(s)}{4k\rho(s)}\right]ds$$

$$=\int_{t_0}^{\infty}\left(s\left[C_0s^3-\frac{1}{s^5}\right]-\frac{1}{4ks^4}\right)ds$$

$$=\left[\frac{C_0s^5}{5}+\frac{1}{3s^3}+\frac{1}{12ks^3}\right]_{t_0}^{\infty}=\infty.$$

All conditions of Theorem 2.1 are satisfied, thus, the given equation is oscillatory. We also compute the numerical solutions of the given differential equation using the Runge Kutta method of fourth order (RK4). We have

$$\ddot{x}(t)=f(t,x(t),\dot{x}(t))=x^3(t)\sin(x(t)\dot{x}(t))-\left(x^3(t)+\frac{x^{15}(t)}{5^{12}(t)+8\dot{x}^4(t)}\right)$$

with initial conditions $x(1)=-0.5$, $\dot{x}(1)=1$ on the chosen interval $[1,100]$, the functions $\Psi(x)\equiv 1$ and $h(t)\equiv 0$, finding values of the function r, q and f where we consider $H(t,x,\dot{x})=f(t)l(x,\dot{x})$ at t=1, n=500 and h=0.198.

k	t_k	$x(t_k)$

1	1	-0.5
2	1.198	-0.2996
3	1.396	-0.0975
4	1.594	0.1048
5	1.792	0.3069
6	1.99	0.5082
.	.	.
.	.	.
17	4.168	-0.0385
18	4.366	-0.4831
19	4.564	-0.9274
.	.	.
.	.	.
28	6.346	0.0846
29	6.544	0.7061
30	6.742	0.3129

Table 1: numerical solution of ODE1

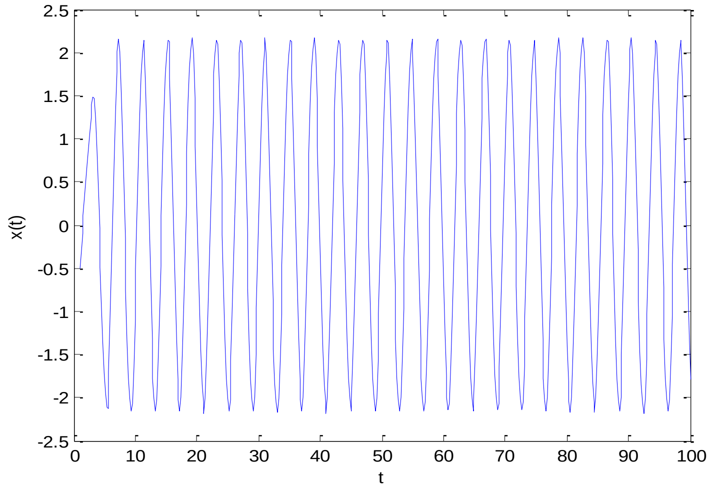

Figure1: solution curve of ODE 1

Remark2.1

Theorem2.1 is extension of Theorem 1 of Greaf, Rankin and Spikes [11], Theorem 1 of Grace and Lalli [10], Theorem 1 of Moussadek Remail [16] and results of Saad [18, 20-22]. All these results of them cannot be applied to the given equation in example2.1

Theorem2.2: Suppose, in addition to the conditions (1), (2), (3) and (4) hold that

(9) $\int_T^{\infty} \frac{ds}{r(s)} \le k_1 \,,\; k_1 > 0$

(10) $\int_{\pm\varepsilon}^{\pm\infty} \frac{\Psi(u)du}{g(u)} < \infty \quad for\ all\ \varepsilon > 0.$

(11) There exists a constant B^* such that

$$G(m) = \int_0^m \frac{ds}{\Phi(1,s)} > B^* m,\; B^* < 0,\; for\ every\ m \in \mathrm{R}^+.$$

Furthermore, suppose that there exists a positive continuous differentiable function ρ on the interval $[t_0,\infty)$ with $\rho(t)$ is a non-decreasing function on the interval $[t_0,\infty)$ such that

(12) $$\limsup_{t\to\infty}\int_T^t \frac{1}{r(s)\rho(s)}\int_T^s \rho(u)\left[C_0 q(u) - p(u) - \frac{h^2(u)}{4a^* r(u)}\right]du\, ds = \infty,$$

where $p:[t_0,\infty)\to(0,\infty)$.

Then, every solution of super-linear equation (E) is oscillatory.

Proof: Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (E) such that $x(t)>0$ *on* $[T,\infty)$ *for some* $T\ge t_0\ge 0$. Define

$$\omega(t) = \frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))}\ ,t\ge T$$

This and by conditions (1), (2), (3), (4) and (E), we have

$$\dot{\omega}(t) \le \rho(t)p(t) - \frac{\rho(t)h(t)\dot{x}(t)}{g(x(t))} - \rho(t)q(t)\Phi(1,\omega(t)/\rho(t)) + \frac{\dot{\rho}(t)}{\rho(t)}\omega(t) - \frac{a_1 k\rho(t)r(t)\dot{x}^2(t)}{g^2(x(t))}\ ,t\ge T$$

Thus for $t\ge T$, we have

$$\rho(t)\left(\frac{\omega(t)}{\rho(t)}\right)^{\bullet} \le \rho(t)p(t) - \rho(t)q(t)\Phi(1,\omega(t)/\rho(t)) - \frac{\rho(t)h(t)\dot{x}(t)}{g(x(t))} - \frac{a_1 k\rho(t)r(t)\dot{x}^2(t)}{g^2(x(t))},\ t\ge T$$

Dividing the last inequality by $\Phi(1,\omega(t)/\rho(t))>0$, by condition (2), there exists a positive constant C_0 such that $\Phi(1,\omega(t)/\rho(t))>C_0$ then, $0<\frac{1}{\Phi(1,\omega(t)/\rho(t))}<\frac{1}{C_0}$.

Thus, for $t\ge T$, we obtain

$$\rho(t)\left(C_0q(t)-p(t)\right)\le-\frac{C_0\rho(t)\left(\omega(t)/\rho(t)\right)^{\bullet}}{\Phi\left(1,\omega(t)/\rho(t)\right)}-\frac{C_0\rho(t)h(t)\dot{x}(t)}{\Phi\left(1,\omega(t)/\rho(t)\right)g(x(t))}-\frac{C_0a_1k\rho(t)r(t)\dot{x}^2(t)}{\Phi\left(1,\omega(t)/\rho(t)\right)g^2(x(t))},\ t\ge T$$

Integrate from T to t, we obtain

$$\begin{aligned}\int_T^t \rho(s)[C_0q(s)-p(s)]ds \le -C_0\int_T^t \frac{\rho(s)(\omega(s)/\rho(s))^{\cdot}}{\Phi(1,\omega(s)/\rho(s))}ds\\ -C_0\int_T^t\left[\frac{\rho(s)h(s)}{\Phi(1,\omega(s)/\rho(s))}\frac{\dot{x}(s)}{g(x(s))}\right.\\ \left.+\frac{a_1k\rho(s)r(s)}{\Phi(1,\omega(s)/\rho(s))}\frac{\dot{x}^2(s)}{g^2(x(s))}\right]ds \qquad (1-4)\end{aligned}$$

From the second integral in R. H. S. of (1-4), we have

$$\begin{aligned}-C_0\int_T^t\left[\frac{\rho(s)h(s)}{\Phi(1,\omega(s)/\rho(s))}\frac{\dot{x}(s)}{g(x(s))}+\frac{a_1k\rho(s)r(s)}{\Phi(1,\omega(s)/\rho(s))}\frac{\dot{x}^2(s)}{g^2(x(s))}\right]ds\\ = -C_0\int_T^t\left[\sqrt{\frac{a_1k\rho(s)r(s)}{\Phi(1,\omega(s)/\rho(s))}}\frac{\dot{x}(s)}{g(x(s))}\right.\\ \left.+\frac{1}{2}\sqrt{\frac{\rho(s)}{a_1\mathrm{kr}(s)}}h(s)\right]^2 ds+\frac{C_0}{4a_1k}\int_T^t\frac{\rho(s)h^2(s)}{\Phi(1,\omega(s)/\rho(s))r(s)}ds\\ \le\frac{1}{4a^*}\int_T^t\frac{\rho(s)h^2(s)}{r(s)}ds \qquad (1-5)\end{aligned}$$

where $a^*=a_1k$.

By the Bonnet's theorem, since $\rho(t)$ is a non-decreasing function on the interval $[t_0,\infty)$, there exists $T_1\in[T,t]$ such that

$$\int_T^t\frac{\rho(s)\left(\omega(s)/\rho(s)\right)^{\bullet}}{\Phi\left(1,\omega(s)/\rho(s)\right)}ds=\rho(t)\int_{T_1}^t\frac{\left(\omega(s)/\rho(s)\right)^{\bullet}}{\Phi\left(1,\omega(s)/\rho(s)\right)}ds \qquad (1\text{-}6)$$

From the inequalities (1-6) and (1-5) in the inequality (1-4), we have

$$\int_T^t \rho(s)\left[C_0 q(s) - p(s) - \frac{h^2(s)}{4a^* r(s)}\right]ds \le -C_0\rho(t)\int_{T_1}^t \frac{(\omega(s)/\rho(s))^{\bullet}}{\Phi(1,\omega(s)/\rho(s))}ds = -C_0\rho(t)\int_{\omega(T_1)/\rho(T_1)}^{\omega(t)/\rho(t)} \frac{du}{\Phi(1,u)}$$

$$\le -C_0\rho(t)\left[-\int_0^{\omega(T_1)/\rho(T_1)} \frac{du}{\Phi(1,u)} + \int_0^{\omega(t)/\rho(t)} \frac{du}{\Phi(1,u)}\right]$$

$$\le C_0\rho(t)G\left(\frac{\omega(T_1)}{\rho(T_1)}\right) - C_0\rho(t)G\left(\frac{\omega(t)}{\rho(t)}\right)$$

By the condition (11), we obtain

$$\int_T^t \rho(s)\left[C_0 q(s) - p(s) - \frac{h^2(s)}{4a^* r(s)}\right]ds \le C_0\rho(t)G\left(\frac{\omega(T_1)}{\rho(T_1)}\right) - C_0 B^* \omega(t)$$

Integrating the last inequality divided by $\rho(t)r(t)$ from T to t, taking the limit superior on both sides and by conditions (9) and (10), we have

$$\limsup_{t\to\infty}\int_T^t \frac{1}{r(s)\rho(s)}\int_T^s \rho(u)\left[C_0 q(u) - p(u) - \frac{h^2(s)}{4a^* r(s)}\right]du\,ds \le \limsup_{t\to\infty}\left\{C_0 G\left(\frac{\omega(T_1)}{\rho(T_1)}\right)\int_T^t \frac{ds}{r(s)} - C_0 B^* \int_{x(T)}^{x(t)} \frac{\Psi(u)du}{g(u)}\right\}$$
$$< \infty$$

as $t\to\infty$, which contradicts to the condition (12). Hence, the proof is completed.

Example2.2: Consider the following differential equation

$$\left(\frac{(x^4(t)+2)}{t^2(x^4(t)+1)}\dot{x}(t)\right)^{\bullet} + \frac{\dot{x}(t)}{t^3} + t^6\left(\frac{x^9(t)}{x^6(t) + \left((x^4(t)+2)\dot{x}(t)/t^2(x^4(t)+1)\right)^2}\right) = \frac{x^3(t)\cos(x(t))}{t^7} \quad ,t>0$$

We note that $r(t) = \frac{1}{t^2}, h(t) = \frac{1}{t^3}, q(t) = t^6, g(x) = x^3,$

$\Psi(x) = \frac{x^4(t)+2}{x^4(t)+1} > 0\ and\ 1 \le \Psi(x) \le 2\ \ for\ all\ x \in R$ and $\Phi(u,v) = \frac{u^3}{u^2+v^2}$ such that

(1) $G(m)=\int_0^m \frac{ds}{\Phi(1,s)}=\int_0^m (1+s^2)ds=\left[s+\frac{1}{3}s^3\right]_0^m=m+\frac{1}{3}m^3>-m$, $B=-1$, $B\in R^-$ *and for all* $m\in R^+$.

(2) $\frac{H(t,x(t),\dot{x}(t))}{g(x(t))}=\frac{\cos(x(t))}{t^7}\le\frac{1}{t^7}=p(t)$ *for all* $t>0$ *and* $x\neq 0$. Let $\rho(t)=t^4$ such that

$$(3)\ \limsup_{t\to\infty}\int_T^t \frac{1}{r(s)\rho(s)}\int_T^s \rho(u)\left[C_0 q(u)-p(u)-\frac{h^2(u)}{4a^* r(u)}\right]du\,ds$$
$$=\limsup_{t\to\infty}\int_T^t \frac{1}{s^2}\int_T^s u^4\left[C_0 u^6-\frac{1}{u^7}-\frac{1}{4a^* u^6}\right]du\,ds$$
$$=\limsup_{t\to\infty}\left[C_0\frac{s^{10}}{110}-\frac{1}{6s^3}-\frac{1}{8a^* s^2}+(C_0\frac{T^{11}}{11}+\frac{1}{2T^2}+\frac{1}{4a^* T})\Big/ s\right]_T^t=\infty.$$

All conditions of Theorem2.2 are satisfied and hence every solution of the given equation is oscillatory. The numerical solutions of the given differential equation are found out using the Runge Kutta method of fourth order (RK4).

We have

$$\ddot{x}(t)=f(t,x(t),\dot{x}(t))=x^3(t)\cos(x(t))-\frac{x^9(t)}{x^6(t)+\dot{x}^2(t)}$$

with initial conditions $x(1)=0.5, \dot{x}(1)=1$ on the chosen interval $[1,100]$, $\Psi(x)\equiv 1, h(t)\equiv 0$, finding values the function *r*, *q*, *f* where we consider $H(t,x,\dot{x})=f(t)l(x,\dot{x})$ at *t*=1 n=500 and h=0.198.

k	t_k	$x(t_k)$

1	1	0.5
2	1.98	0.7009
3	1.396	0.9104
.	.	.
.	.	.
16	3.97	-0.1655
17	4.168	-0.3608
18	4.366	-0.5581
.	.	.
.	.	.
31	7.138	0.0267
32	7.336	0.2217
33	7.534	0.4173
.	.	.
.	.	.
49	10.504	-0.082
50	10.702	-0.277
51	10.9	-0.473

Table 2: numerical solution of ODE 2

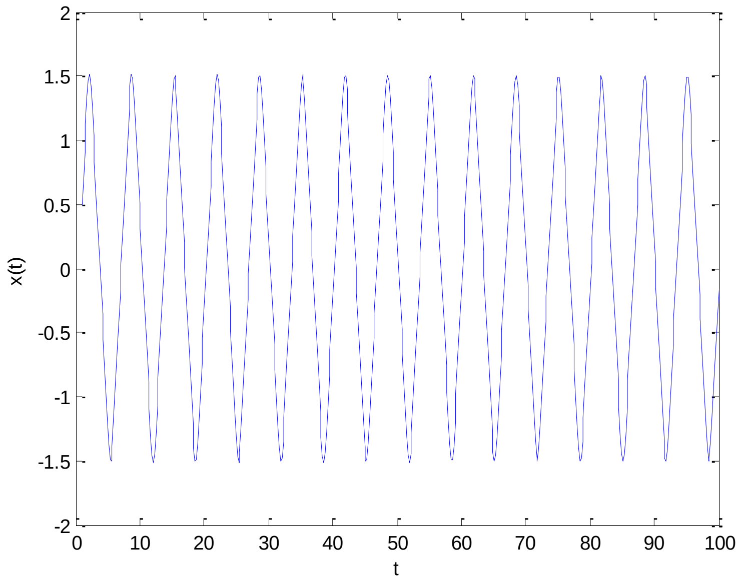

Figure2: solution curve of ODE 2

Remark2.2: Theorem2.2 extends results of Bihari [5] and Kartsatos [13] and results of Saad [18, 20-22]. Results of Bihari [5], Kartsatos [13] and Saad[18, 20-22] cannot be applied to the given equation in example2.2.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this manuscript.

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