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((Fuzzy sets and its applications in artificial neural networks))

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INTRODUCTION

The fuzzy sets were introduced by zadeh (1965) as means of representing and Manipulating data that was not precise but rather fuzzy. Fuzzy logic provides an inference morphology that enables approximate human reasoning capabilities to be applied to knowledge-based systems. The theory of fuzzy logic provides a mathematical strength to capture the uncertainties associated with human cognitive processes such as thinking and reasoning. The conventional approaches to knowledge representation lack the means for representing the meaning of fuzzy concepts. As a consequence, the approaches based on first order logic and classical probability theory do not provide an appropriate conceptual framework for dealing with the representation of common sense knowledge, since such knowledge is by it's nature both lexically imprecise and non categorical.

The development of fuzzy logic was motivated in large measure by the need for a conceptual framework which can address the issue of uncertainty and lexical imprecision. Some of the essential characteristics of fuzzy logic relate to the following:-

- In fuzzy logic, exact reasoning is viewed as a limiting case of approximate reasoning.
- In fuzzy logic, every thing is matter of degree.
- In fuzzy logic, knowledge is interpreted a collection of elastic or, equivalently, fuzzy constraint on a collection of variables.
- Inference is viewed as a process of propagation of elastic constraints.
- Any logical systems can be fuzzified.

There are two main characteristics of fuzzy systems that give them better performance for specific applications:-

- Fuzzy systems are suitable for uncertain or approximate reasoning, especially for the systems with a mathematical model that is difficult to derive.
- Fuzzy logic allows decision making with estimated values under incomplete or uncertain information.

Artificial neural systems can be considered as simplified mathematical models of brain-like systems and they function as parallel distributed computing networks. However, in contrast to conventional computers which are programmed to perform specific task, most neural networks must be taught or trained. The most important advantage of neural network is their adaptively. Neural networks can automatically adjust their weights to optimize their behavior as pattern recognizes, decision makers, system controllers, etc.

To enable a system to deal with cognitive uncertainties in a manner more like humans, one may incorporate the concept of fuzzy logic into the neural networks, the resulting hybrid system is called fuzzy neural, neural fuzzy.

This work is divided and organizes into four chapters:

Chapter one: contains some basic concepts of fuzziness, and in this chapter we give definition of fuzzy set, representation of fuzzy set, some methods for determining membership functions. Manual and automatic methods consider as two options of determining membership functions. Also in this chapter we study the common mathematical operations which dealing with fuzzy sets such as union, intersection, etc.

Chapter two: in this chapter we present new concept of basic tool for doing operations in fuzzy sets which is call the extension principle and gives its definition, operations of applied it.

In this chapter we also discuss the level set which is very important to determining the fuzzy set. In this chapter we also discuss the convex fuzzy sets which is important to study the behaves of continuity. In this chapter we explain the important idea of fuzzy sets; the fuzzy quantities which is divided into fuzzy number and fuzzy interval. We give the definitions and operations on it, we study the fuzzy arithmetics which is give the better way to add, subtract, multiplied, division and max -min of fuzzy numbers. We have present some examples for doing it.

Chapter three: in chapter three we discuss the fuzzy logic which is known as a part of fuzzy set theory. Fuzzy logic is regard as generalising of ordinary logic. We discuss in this chapter fuzzy logic and it's connective. Also we discuss the fuzzy relations and its definitions; operations: representation and classifications.

In this chapter also we explain the concept of fuzzy partition and we have explain that it can be obtain by two way; the first one called equivalence relation and the second one called alpha - cuts.

Chapter four: in this chapter we study the concept of approximate reasoning and its division (linguistic variable), its definitions and operations on linguistic variables

Finally we give Application of fuzzy set in (Artificial neuron networks) where we study the simple neural and how to combined fuzzy logic operation with neural networks to introduce the hybrid neural networks or fuzzy neural.

ABSTRACT

The purpose of this Dissertation is to study and discuss the theory of fuzzy set and how we can apply it in artificial neuron networks. Where we combined the fuzzy logic and neuron networks to get a hybrid system that has two characteristics. The first one is that the ability of fuzzy logic to deal with uncertainties, and the second one it that ability of neural networks to learn.

CHAPTER 1

1.1 The Concept of Fuzziness:-

In this chapter, we will discuss the intrinsic notion of fuzziness in natural language. Following (Lotfi Zadeh) fuzzy concepts will be modeled as fuzzy sets, which are generalizations of (crisp) sets. In using our every day natural language to impart knowledge and information, there is a great deal of imprecision and vagueness, or fuzziness such statements as "Ali is tall " and "Salem is young" are simple examples

The materials in this chapter taken from the following references [1],[31],[16],[28],[22],[12],[19],[14],[21],[7],[17],[4],[30]

Now We begin with some examples.

Example 1.1.1:-

The description of a human characteristic such as healthy;

Example 1.1.2:-

The classification of patients as depressed;

Example 1.1.3:-

The classification of certain objects as large;

Example 1.1.4:-

The classification of people by age such as old;

Example 1.1.5:-

A rule for driving such as "if an obstacle is close, then brake immediately". In the examples above, terms such as depressed and old are fuzzy in the sense that they cannot be sharply defined. However, as humans, we do make sense out of this kind of information, and use it in decision making.

These "fuzzy notions" are in sharp contrast to such terms ' married over 39 year's old, or under 6 feet tall. in ordinary mathematics we are used to dealing with collections of objects, say certain subsets of a given set such as the subset of even integers in the set of all integers .but when we speak of the subset of depressed people in a given set of people, it may be impossible to decide whether a person is in that subset or not. Forcing a yes-or-no answer is possible and is usually done, but there may be information lost in doing so because no account is taken of the degree of depression. Although this situation has existed from time immemorial, the dominant context in which science is applied is that in which statements are precise (say either is true or false), no imprecision is present. But in this time of rapidly advancing technology, the dream of producing machines that mimic human reasoning which is usually based on uncertain and imprecise information, has captured the attention of many scientists .the theory and application of fuzzy concepts are central in this endeavor but remain to a large extent in the domain of engineering and applied sciences. With the success of automatic control and expert systems, we are now witnessing an endorsement of fuzzy concepts in technology. The mathematical elements that from the basis of fuzzy concepts have existed for along time but the emergence of applications has provided a motivation for anew focus for the underlying mathematics. Until the emergence of fuzzy set theory as an important tool in practical applications, there was no compelling reason to study its mathematics .but because of the practical significance of these developments; it has become important to study the mathematical basis of this theory.

The primitive notion of fuzziness as illustrated in the examples above needs to be represented in a mathematical way. This is a necessary step in getting to the heart of the notion, in manipulating fuzzy statements. And in applying them. This is a familiar situation in science. A good example is that of "chance". The outcome produced by many physical systems may be "random". And to deal with such phenomena, the theory of probability came into being and has been highly developed and widely used. The mathematical modeling of fuzzy concepts was presented by Zadeh in 1965 and we will now describe his approach. His contention is that meaning in natural language is a matter of degree. If we have a proposition such as "Ali is young". Then it is not always possible to assert that it is either true or false. When we know that "Ali's age is x then the "truth", or more correctly, the "compatibility" of x with "is young ", is a matter of degree. It depends on our understanding of concept "young". If the proposition is "Ali is under 22 years old" And we know Ali's age. then we can give a yes or no answer to whether the proposition is true or not .this can be formalized a bit by considering possible ages to the interval $[0,\infty)$.letting F be the subset $\{x : x \in [0,\infty) : x < 22\}$. and then determining whether or not (Ali age) is in F .But "young" cannot be defined as an ordinary subset of $[0,\infty)$ .Zadeh was led to the notion of fuzzy subsets. Clearly, 18 and 20 years old are young. but with different degrees. 18 is younger then 20. this suggests that membership in a fuzzy subset should not be on a "0 or 1" basis, but rather on "a 0 to 1 scale". that is. the membership should be an element of the interval $[0,1]$ .This is handled as follows. An ordinary subset A of a set X is determined by, its indicator function or characteristic function χA defined by

$$\chi A : X \to \{0,1\}$$

$$\chi A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

The indicator function of a subset A of a set X specifies whether or not an element is in A. There are only two possible values the indicator function can take. This notion is generalized by allowing images of elements to be in the interval $[0,1]$ rather a being restricted to the two element set $\{0,1\}$. From the above introduction we will introduce some important definitions of fuzzy set.

Definition 1.1.1:-

Let X be the (universe of discourse)

1. We shall call X to be the (space of objects).

2. If $p_1, p_2, \ldots, p_n$ are n mutually unrelated properties of a generic element x of X, then we defined the property vector as the n- tuple vector ($p_1, p_2, \ldots, p_n$) and the property space denoted by p as the set of all possible values which the vector ($p_1, p_2, \ldots, p_n$) can assume. We denote each point in the property space by $x = (p_1, p_2, \ldots, p_n)$.

Definition 1.1.2:-

Let X be a space of objects and x be generic element of X Thus $X = \{x\}$. A fuzzy subset F of X is characterized by a membership (characteristic) function with respect to certain properties of x of interest, $p_1, p_2, \ldots, p_n$, denoted by μ_F ($x = (p_1, p_2, \ldots, p_n)$), which is a functional mapping from the property space defined by the object space X into the interval $[0,1]$. The value of μ_F ($x = (p_1, p_2, \ldots, p_n)$) at x represents the grade of membership of x in F. For simplicity, we shall sometimes write $\mu_F(x)$ instead of μ_F ($x = (p_1, p_2, \ldots, p_n)$). However, it is assumed to be understood that when we are concerned with the n properties $p_1, p_2, \ldots, p_n$ of x in F, x will be considered as an n-tuple vector in p whose components are $p_1, p_2, \ldots, p_n$ and $\mu_F(x)$ is a function of variables $p_1, p_2, \ldots, p_n$. Thus we can repeat the above definition of a fuzzy set as Following

Definition 1.1.3:-

Let X be a space of objects and x be generic element of X

Thus $X = \{x\}$. A fuzzy set F of X can be defined by its membership function $\mu_F(x)$ as follow: $\mu_F(x): X \longrightarrow [0,1]$,where

$$\mu_F(x) = \begin{cases} 1 & \text{if } x \in F \\ 0 & \text{if } x \notin F \\ 0 < \mu_{F(x)} < 1 & \text{if } x \text{ is partly in } F \end{cases}$$

So every element x of X has a membership degree

(or grade of membership) $\mu_F(x) \in [0, 1]$ and F is completely determined by the set of tuples $F = \{ (x, \mu_F(x)) / x \in X, \mu_F(x) \in [0,1] \}$.

So we can redefine a fuzzy set as follows:-

Definition 1.1.4:-

Let X be universe of discourse. And let F be subset of X then F is said to be Fuzzy subset of X if and only if $F = \{(x, \mu_F(x)) / x \in X, \mu_F(x): X \rightarrow [0,1]\}$.

Remark 1.1.1:-

- $\mu_F(x): X \longrightarrow [0,1]$ is called membership function of a fuzzy subset F of X and denoted by (MF).
- The value $\mu_F(x)$ for $x \in X$ is a number from the real closed interval $[0, 1]$ and it's called (grade of membership) of x (or membership degree). or (membership value) of element x in F.
- A special case of fuzzy set is an (crisp) set, where we use

$\mu_F : X \rightarrow \{0,1\}$ Instead of $\mu_F : X \rightarrow [0,1]$.

- The universal set always has [$\mu_X (x) = 1$] for all x in X .
- The empty set is described by its membership function always zero [$\mu_\phi (x) = 0$ for all x in X].
- In fuzzy sets an element whose grade of membership is 1 will be said to have (full membership). An element whose grade of membership is 0 will be said to have (non-membership).
- The crisp set has a unique membership function .where a fuzzy set can have an infinite number of membership function to represent it.
- We usually denote to any fuzzy set by (F). Where F (x) the membership function values of F .
- We usually denote the set of all fuzzy sets of X as $F(X)$. or (the family of all fuzzy subsets in X).
- Fuzzy sets are always (and only) functions from "a universe of objects," say X, into [0, 1] as defined every function $\mu_F : X \rightarrow [0,1]$ is a fuzzy set.

While this is true in formal mathematical sense, many functions that qualify on this ground cannot be suitably interpreted as realizations of a conceptual fuzzy set. In other words, functions that map X into [0, 1] may be fuzzy set, but become fuzzy sets, when and only when they match some intuitively plausible semantic description of imprecise properties of the objects in X .

1.2 Representation of fuzzy set:-

Let X be the universe of discourse, of five elements such that $X = \{x_1, x_2, x_3, x_4, x_5\}$ and let A be crisp subset of X .

And assume that A consists of only two elements, $A=\{x_2,x_3\}$. Now by using the characteristic function of crisp set A as $\chi_A(x): X \to \{0,1\}$

We find that $A=\{(x_1,0),(x_2,1),(x_3,1),(x_4,0),(x_5,0)\}$ now the question is how to represent fuzzy set by using membership function. We begin with the discussion of concept of [universe of discourse] as follows: we usually, denote X as the (universe of discourse) as (zadeh suggested), It's the world which talking about .or it's simply the universe, and it may consist of discrete (ordered or non-ordered) objects or it can be a continuous space or (finite or non-finite) objects. Now let $F = \{x_1,x_2,\dots,x_n\}$ be a finite fuzzy subset of universe X and let $\mu_F(x)$ be the membership function ,then we can write F as follows:-

- $F=\{(x_1,\mu_F(x_1)),(x_2,\mu_F(x_2)),\dots\dots\dots,(x_n,\mu_F(x_n))\}$
- or $F=\{(x_i,\mu_F(xi)\}$ where i =1 to n

  or
- $F= \{(\mu_F(x_1)/x_1),(\mu_F(x_2)/x_2),\dots\dots\dots,(\mu_F(x_n)/x_n)\}$.

Where we use the separating symbol / to associated the membership value with it's coordinate on the horizontal axis. Also we described $F=\{(x,\mu_F(x)):x\in X,\mu_F(x)\in[0,1]\}$ by

- $F=\mu_F(x_1)/x_1+\dots\dots\dots+\mu_F(x_n)/x_n= \sum_{i=1}^{n}\mu_F(x_i)/x_i$

Where (+) satisfies $a/u+b/u=max(a,b)/u$ and the summation symbol is not for algebraic summation but rather denotes the collection or aggregation of each element.

- For any countable or discrete universe X allows annotation $F=\sum_{x\in X}\mu_F(x)/x$.
- if X is uncountable or continuous, we will write $F=\int_X \mu_F(x)/x$

Hence the $\int$-sign denotes an uncountable enumeration.
Now we give some example to illustrate those ideas.

Example 1.2.1:-

Fuzzy sets with a discrete non-ordered universe.
Let X= {Tripoli, Sebha, Bengazi, ALzawya, Sirt, Darna} some cities in Libya one may choose to live in. the fuzzy set F (best city to live in) ,May be described as follows:-
F= {(Tripoli, 0.9), (Sabha, 0.4), (Bengazi, 0.6), (Sirt, 0.8), (ALzawya, 0.5), (Darna, 0.7)}. The universe of discourse X contains non-ordered objects. And the membership grades listed above are quite subjective; Any one can come up with different grades membership. but legitimate values to reflect his or her preference.

Example 1.2.2:-

Fuzzy sets with discrete ordered universe. Let X= {0, 1, 2, 3, 4, 5, 6} be the set of numbers of children in a family may choose to have. Then the fuzzy set F desirable number of children in a family may be described as follows: $F=\{(0,0.1),(1,0.3),(2,1.0),(3,0.8),(4,0.7),(5,0.3),(6,0.1)\}$. Here we have discrete ordered universe X. Again the membership grades of this fuzzy set are obviously subjective measures.

Example 1.2.3:-

Fuzzy sets with a continuous universe :Let $X = R^{+}$
(The set of positive real numbers) be the set of possible ages for human beings. Then the fuzzy set F = {about 50 years old} may be expressed as

$$F = \{(x,\mu_F(x))/x \in X \ \mu_F(x) = \frac{1}{1+(\frac{x-50}{10})^4},\}$$

Remark 1.2.1:-

Sometime the universe of discourse containing which we call (linguistic variable) as (Zadeh) say " by a linguistic variable we mean variable whose values are words or sentences in natural or artificial language, for example age is a linguistic variable if its value is linguistic rather than numerical. i.e., young ,not young , very young , quite young ,old ,not very old , and not very young , etc, rather than 20,21,22,23,..."

So like an algebraic variable takes numbers as values, a linguistic variable takes words or sentences as values.
For example "the statement {"Ali is tall man"} implies that the linguistic variable Ali take the linguistic value or term {tall}.
The range of possible values of linguistic variable represents the universe of discourse. For examples:

Example 1.2.4:-

The universe of discourse of the linguistic variable (speed) might have the range between 0 and 220 km per hour. So the universe X = {0,...., 120} and may include such fuzzy subsets as: very slow, medium fast, and very fast. and we can rewrite X as X = {very slow, medium fast, very fast,..., etc}. And each fuzzy subset also represents a linguistic value of the corresponding linguistic variable.

Example 1.2.5:-

the universe of discourse of the linguistic variable {body temperature} its might have temperature in Fahrenheit degrees form 94 to 106.And also the linguistic variable takes the linguistic values or terms. such as {low, normal, subfebrile, high and very high ...., etc}.

Remark 1.2.2:-

In general, fuzzy sets cannot be represented by Venn diagrams. Sometimes we can represent fuzzy subset of universe X on X and Y axis We suppose that allows that the first element on the order pairs of the set $F=\{(x,\mu_F(x))/x\in X\}$ take the coordinate on the horizontal axis,

Figure 1

And the second element takes the coordinate of the vertical axis. So we can sometimes draw fuzzy subset on X and Y axis as follow. To illustrate those ideas we give some example

Example 1.2.6:-

Suppose some one want to describe the class of cars having the property of being expensive be considering cars such as Bmw, Buick, Ferrari, Fiat, Lada, and Mercedes, Rolls-Royce. Some cars, like Ferrari or Rolls-Royce definitely belong to this class, which other cars, like Fiat or Lade do not belong to it. But there is a third group of cars. Where it is difficult to state whether they are expensive or not using a fuzzy set, the fuzzy set of expensive cars is.

F ={(Ferrari,1),(rollsRoyce,1),(Mercedes,0.8),(bmw,0.7),(buick,0.4)}

I.e. Mercedes belong to degree 0.8, Bmw to 0.7 and Buick to 0.4 to the class of expensive cars. So the set of expensive cars can be described by F = 1/FERRARI+1/ROLLSROYCE+0.8/MERCEDES+0.7/BMW+0.4/BUICK

Example 1.2.7:-

Suppose we want to model the notion of "high income" with a fuzzy set. Let the set X be the positive real numbers representing the totality of possible incomes. We survey a large number of people and find out that no one thought that an income x between $ 20.000 and $75.000 was high was approximately $p=\frac{x-20}{55}$ of course, every one thought that an income over $75.000 was high. Measuring in thousands of dollars, one reasonable model of the fuzzy set "high income" would be.

$$\mu_F(x)=\begin{cases}0 & \text{if } x<20\\ \frac{x-20}{55} & \text{if } 20\le x\le 75\\ 1 & \text{if } 75<x\end{cases}$$

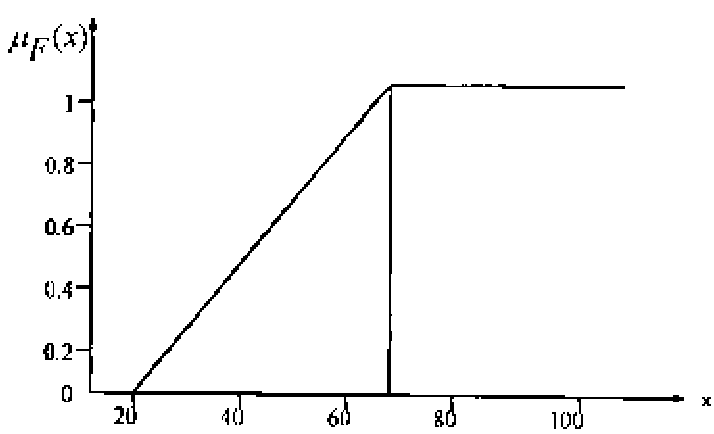

Figure 2

Example 1.2.8:-

The class of all real numbers which are much great then 1. We usually write this set as $F = \{ x / x \text{ is real number and } x > 1 \}$. Such a set may be defined subjectively by membership function such as:

$$\mu_F(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ \dfrac{1}{1+(x-1)^{-1}} & \text{if } x > 1 \end{cases}$$

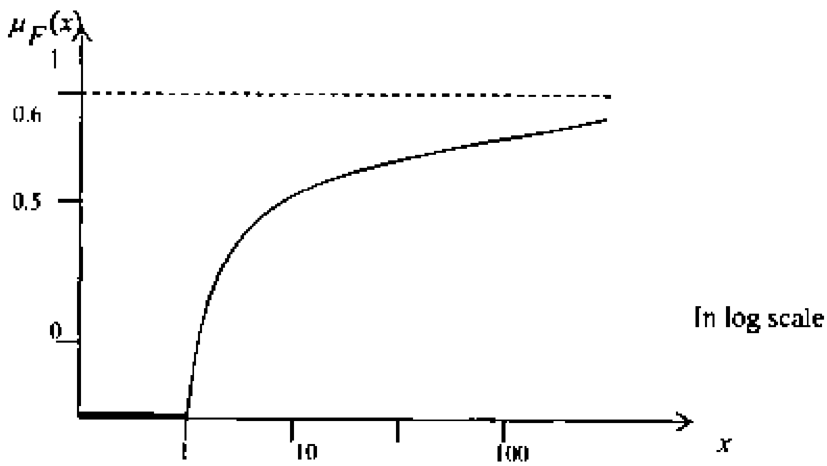

Figure 3

Example 1.2.9:-

Suppose we want to define the set of natural numbers "close to 1". This can be expressed by $F = \{(-2,0.0),(-1,0.3),(0,0.6),(1,1.0),(2,0.6),(3,0.3),(4,0.0)\}$.

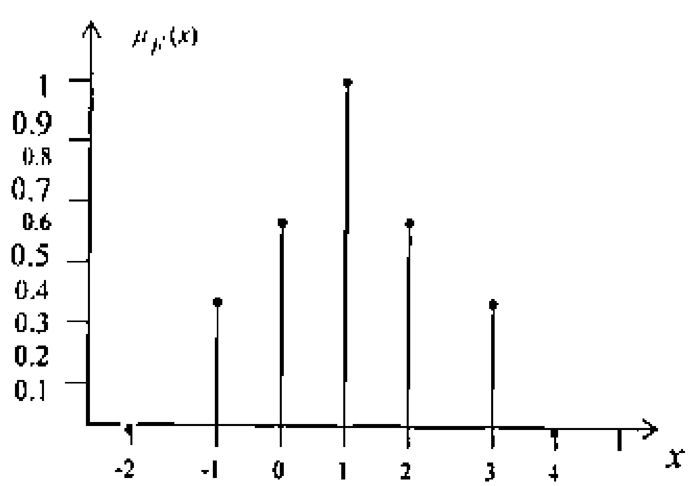

Figure 4

Example 1.2.10:-

Suppose some one want to define the set of integer "Close to 6". This can be expressed by. $F=\{(3,0.1),(4,0.3),(5,0.6),(6,1.0),(7,0.6),(8,0.3),(9,0.1)\}$

Or $F=\{0.1/3, 0.3/4, 0.6/5, 1.0/6, 0.6/7, 0.3/8, 0.1/9\}$

Now let X=R, and we fined that membership function is

$$\mu_F(x)=\begin{cases}1-\sqrt{\dfrac{|x-6|}{3}} & for\ \ 3\le x\le 9\\ 0 & \text{otherwise}\end{cases}$$

Then we can write $F=\int_R \mu_F(x)/x$.

Is the fuzzy set representing the real nmbers approximately equal to 6.

Remark 1.2.3:-

In previous examples, we found that its some times that not easy to represent a fuzzy sets on coordinates axis. And sometimes is too difficult to find relation between memberships functions. And also sometimes we find difficulty to get the right formula for membership. Thus the construction of a fuzzy set depends on two things:

- The identification of a suitable universe of discourse
- And specification of an appropriate membership functions.

The specification of membership function is subjective, which means that the membership functions specified for the same concept by different persons may vary considerably. This subjectivity comes from individual differences in perceiving or expressing an abstract concept and has little to do with randomness. Therefore, the subjectivity and non-randomness of fuzzy sets in the primary difference between the study of fuzzy sets and probability theory.

1.3 Some Methods for Determining Membership Functions:-

In real world application of fuzzy sets, an important task is to determine membership function of the fuzzy sets in question. Like the estimation of probabilities in the probability theory, we can only obtain an approximate membership function of a fuzzy set because of our cognitive limitations .In This section we discuss some methods for determining a membership function of a fuzzy set.

1.3.1 Manual methods

These methods fall into two broad categories

(1) Use of frequencies

(2) Use of direct estimation

The frequencies method, obtains a membership function by measuring the percentage of people in a group (typically experts in a particular domain). Who answer yes to a question about whether an object belongs to a particular set.

Direct estimation methods take a different approach by asking experts to grade an event on a scale from 0 to 1, to determine the ratio of the membership in particular set. All of these ways, in manual methods suffer from the deficiency of relying on a very subjective interpretation of words.

1.3.2 Automatic Methods for Determining Membership:-

The automatic generation of membership functions covers a wide variety of different approaches. What makes automatic generation different from the Manual methods is that either the expert is completely removed from the process or the membership functions are "fine tuned" based on an initial guess by the expert. The emphasis is on the use of modern soft computing techniques (in particular genetic algorithms and neural networks).

Now we will discuss some of terms or factors which interfere in when we begin to determine the membership function for a particular fuzzy set. namely: we take the first factor.

1. **Surroundings circumstances :-**

Consider, for example, the decision taken when analyzing a particular problem about how to determining the membership functions of fuzzy sets (i.e. fast cars). Let universe of cars speed be from 0 to 220 km/h. Where the speed is linguistic variable and the linguistic value of a variable will take {slow, very slow, fast, medial ... etc.}. To determine whether the numbers of the described cars consider as fast cars or not. we find that the judgment on cars speed depends on the personal opinion. But no matter what the used methods to expression on membership degree. Then there are a lot of factors or terms must be considered. And these factors are completely controls of Car speed. Such as {the weather, the road, the distance, the driver psycho situation, and car realism speed} .For example if the car speed is 150km/h and in the same time the weather was so bad then its speed will be less than reality.

2. **Time and Space:-**

The other factor or term is that enters in this process .is the time and space, for example if we decided to determine the membership function. For a particular fuzzy set (tall man). And if tall for some one is 150cm and that person lives in some space (place or country. Philippines for example). This person will be described as a tall man in Philippines .But may be described as a short man in other country (like Germany). Thus the time and space will change the expert views about an issue or in membership grade determination in a particular fuzzy set.

3. Personal Opinion:-

Sometimes the classification of an element belonging, different from an expert opinion to another. For example, when we classify the best five football players in the world, the classification will be based on the personal opinion of the expert .

4. The Nature Of The Study:-

In the case of classification of the membership functions an example, is the body temperature. The classification will depend on the nature of the problem that is to be studied. The matter of the temperature is connected with human body. To understand the temperature, its variation and its reflection on body health. So when we measure this temperature we must calculate the difference in the measurement. Sometimes we measure the temperature in centigrade or Fahrenheit. And here, we should consider the classification of temperature from a medical point of view more than physical perspective. And the expert here is a doctor who is able to judge the kind of the temperature.

5. The Accuracy in Dealing With a Particular Problem:-

The accuracy is very important in determining the membership degree. Because the most used applications depend on fine results of classification of a particular fuzzy set. So we must observe the accuracy in dealing with membership functions.

1.4 Operations on Fuzzy Sets:-

Let X be a space of points (objects) and x be a generic element of X. Let $p_1, p_2, \ldots\ldots, p_n$ be n unrelated properties of x interest.

Definition 1.4.1:-

A fuzzy set is empty iff its membership function is identically zero on X. Thus if F is a fuzzy subset of X, then $F = \phi$ iff $\mu_F(x)=0, \forall\ x \in X$.

Definition 1.4.2:-

A fuzzy set is universal iff its membership function is identically unity on X. if F is a fuzzy subset of X then, F is universal iff $\mu_F(x)=1, \forall\ x \in X$.

Definition 1.4.3:-

Two sets F_1 and F_2 are equal written as $F_1 = F_2$ iff

$\mu_{F_1}(x)=\mu_{F_2}(x), \forall\ x \in X$ for simplicity, we shall abbreviate the statement

" $\mu_{F_1}(x)=\mu_{F_2}(x), \forall\ x \in X$ " by " $\mu_{F_1}=\mu_{F_2}$ "

Example 1.4.1:-

Let X be the set of all real numbers greater than 1 and let F be the set of all real numbers greater than 0. Then $\mu_F(x)=1$, $\forall\ x \in X$ and thus F is a universal set in X.

Example 1.4.2:-

Let X be the set of all real numbers great than 1. Let F be the set of all real number less than 1. Then $\mu_F(x)=0$ for all $x \in X$. Hence F is an empty set in X.

Definition 1.4.4:-

Let F_1 and F_2 be two sets. We say that F_1 is contained in F_2 written as $F_1 \subseteq F_2$ if $\mu_{F_1}(x) \leq \mu_{F_2}(x)$. And we say that F_1 is strictly contained in F_2, denoted by $F_1 \subset F_2$ iff $\mu_{F_1}(x) < \mu_{F_2}(x)$.

Example 1.4.3:-

Let X be the set of all real numbers and let F_1 be the set of all real numbers which are much greater than 1. And let F_2 be the set of all integers which are much greater than 1. And let membership function of F_1 to be

$$\mu_{F_1}(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ \dfrac{1}{1+(x-1)^{-1}} & \text{if } x > 1 \end{cases}$$

Membership function of F_2 be

$$\mu_{F_2}(x) = \begin{cases} 0 & \text{if } x \leq 1 \text{ or } x \text{ is not an integer} \\ \dfrac{1}{1+(x-1)^{-1}} & \text{if } x \text{ is an integer and } x > 1 \end{cases}$$

Then the set F_2 is contained in the set F_1 because $\mu_{F_1} \geq \mu_{F_2}$.

Now if we defined $\mu_{F_2}(x)$ as

$$\mu_{F_2}(x) = \begin{cases} 0 & \text{if } x \leq 1 \text{ or } x \text{ is not an integer} \\ \dfrac{0.9}{1+(x-1)^{-1}} & \text{if } x \text{ is an integer and } x > 1 \end{cases}$$

Than F_2 is strictly contained in F_1. Thus $F_2 \subset F_1$.

Definition 1.4.5:-

The absolute complement of the fuzzy set F is denoted by F^c.

And is defined F^c is $\mu_{F^c}(x) = 1 - \mu_F(x)$.

Example 1.4.4:-

Let X be the set of all real numbers and let F be the set of all real numbers which are much greater than 1 and the membership function of F

is $\mu_F(x)=\begin{cases} 0 & \text{if } x \le 1 \\ \frac{1}{1+(x-1)^{-1}} & \text{if } x > 1 \end{cases}$

The complement of F is

$$\mu F^{c}(x)= \begin{cases} 1 & \text{if } x \le 1 \\ 1-\frac{1}{1+(x-1)^{-1}} & \text{if } x > 1 \end{cases}$$

In other word the set F^c is the set of real numbers which are less than or not much greater than 1.

Definition 1.4.6:-

The union of two sets F_1 and F_2 with respective membership functions $\mu_{F_1}(x)$ and $\mu_{F_2}(x)$ is set F_3 written as $F_3=F_1 \cup F_2$,whose membership function is related to those of F_1 and F_2 by $\mu_{F_3}(x)$=max[$\mu_{F_1}(x),\mu_{F_2}(x)$]

$\forall$ $x \in X$ or in abbreviated form $\mu_{F_3}=\mu_{F_1}+\mu_{F_2}$.

(This is equivalent to the smallest set contains both F_1 and F_2)

Definition 1.4.7:-

The intersection of two sets F_1 and F_2 with respective membership functions $\mu_{F_1}(x)$ and $\mu_{F_2}(x)$ is a fuzzy set F_3 written as $F_3 = F_1 \cap F_2$ who membership function is related to those of F_1 and F_2 by $\mu_{F_3}(x)$= min [$\mu_{F_1}(x),\mu_{F_2}(x)$] ,$\forall$ $x \in X$ or in abbreviated form

$\mu_{F_3}=\mu_{F_1}\cdot\mu_{F_2}$. (This is the largest set contained in both F_1 and F_2).

Remark 1.4.1:-

We write $(F_1 \cup F_2) = (\mu_{F_1} \vee \mu_{F_2})(x) = \max[\mu_{F_1}(x), \mu_{F_2}(x)]$

$= \mu_{F_1}(x) \vee \mu_{F_2}(x), \forall\ x \in X$.

And

$(F_1 \cap F_2) = (\mu_{F_1} \wedge \mu_{F_2})(x) =$

$= \min[\mu_{F_1}(x), \mu_{F_2}(x)] = \mu_{F_1}(x) \wedge \mu_{F_2}(x), \forall\ x \in X$.

Example 1.4.5:-

Let X be the set of real numbers and let F_1 be the set of real numbers which are close to 1 and let the membership function of F_1 be defined by

$\mu_{F_1}(x) = \frac{1}{1+(x-1)^2}, \forall\ x \in X$.

Let F_2 be the set of real numbers which are close to 2 and let the membership function of F_2 be defined by $\mu_{F_2}(x) = \frac{1}{1+(x-2)^2}, x \in X$

The union of F_1 and F_2 is $(F_1 \cup F_2) = \mu_{F_1}(x) \vee \mu_{F_2}(x) =$

$$= \max\{\mu_{F_1}(x), \mu_{F_2}(x)\} = \begin{cases} \frac{1}{1+(x-1)^2} & x \le 1.5 \\ \frac{1}{1+(x-2)^2} & x \ge 1.5 \end{cases}$$

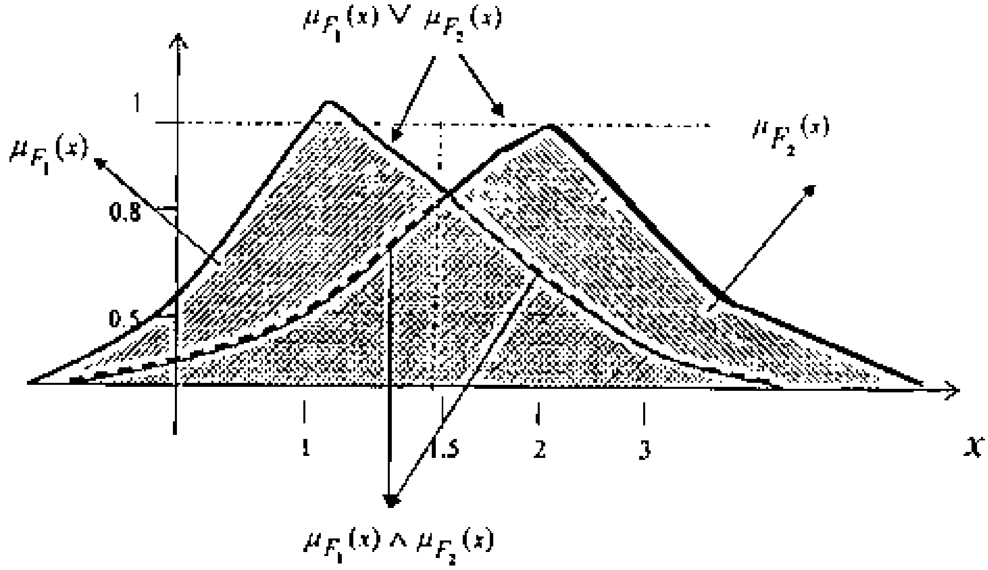

Figure 5

Since the two curves μ_{F_1} and μ_{F_2} intersect at $x=1.5$ then intersection of F_1 and F_2 is $\mu_{F_1}(x) \wedge \mu_{F_2}(x)=$

$$=\min\left[\mu_{F_1}(x), \mu_{F_2}(x)\right] = \begin{cases} \dfrac{1}{1+(x-2)^2} & x \le 1.5 \\ \dfrac{1}{1+(x-1)^2} & x \ge 1.5 \end{cases}$$

Definition 1.4.8:-

Let F_1 and F_2 be two sets, the relative complement of F_1 with respect to F_2 denoted by F_2 / F_1. Is defined by $\mu_{F_2/F_1} = \mu_{F_2} - \mu_{F_1}$.

Provided that $\mu_{F_2}(x) \ge \mu_{F_1}(x)$.

Example 1.4.6:-

In (Example 1.4.3), Since $F_2 \subseteq F_1$ then the relative complement of F_2 with respect to F_1 is defined as

$$\mu_{F_1} - \mu_{F_2} = \begin{cases} 0 & \text{for } x \le 1 \\ \dfrac{1}{1+(x-1)^{-1}} & \text{for } x \text{ is not an integer and } x > 1 \\ 0 & \text{for } x \text{ is an integer and } x > 1 \end{cases}$$

Definition 1.4.9:-

The symmetrical difference (or Boolean sum) of two sets F_1 and F_2 with membership functions μ_{F_1} and μ_{F_2} denoted by $F_1 \Delta F_2$ is a fuzzy set, whose membership function $\mu_{F_1} \Delta \mu_{F_2}$ is related to those of F_1 and F_2 by

$\mu_{F_1} \Delta \mu_{F_2} = \left|\mu_{F_1} - \mu_{F_2}\right|$.

Definition 1.4.10:-

The algebraic product of two sets whose membership functions μ_{F_1} and μ_{F_2}, denoted by $F_1 \bullet F_2$ is a fuzzy set whose membership function $\mu_{F_1 F_2}$ is related to those of F_1 and F_2 by $\mu_{F_1.F_2} = \mu_{F_1} \bullet \mu_{F_2}$.

Definition 1.4.11:-

The algebraic sum of two sets F_1 and F_2 with membership functions μ_{F_1} and μ_{F_2} denoted by $F_1 + F_2$ is a fuzzy set whose membership function $\mu_{F_1+F_2}$ is related to those of F_1 and F_2 by $\mu_{F_1+F_2} = \mu_{F_1} + \mu_{F_2}$.

Provided the sum $\mu_{F_1}(x)+\mu_{F_2}(x)\leq 1$ for all x.

Definition 1.4.12:-

The direct sum of two sets F_1 and F_2 with membership function μ_{F_1} and μ_{F_2} denoted by $F_1 \oplus F_2$, is a fuzzy set whose membership function $\mu_{F_1\oplus F_2}$ is related to those of F_1 and F_2 by $\mu_{F_1\oplus F_2}=\mu_{F_1+F_2}-\mu_{F_1F_2}$

$\mu_{F_1\oplus F_2}=\mu_{F_1+F_2}-\mu_{F_1F_2}=\mu_{F_1}+\mu_{F_2}-\mu_{F_1}\mu_{F_2}$.

Definition 1.4.12:-

The support of F is a set that contains all elements of F with non-zero membership grade supp (F) = $\{x\in X/\mu_F(x)>0\}$

Definition 1.4.13:-

A fuzzy set whose support is single point in X .or if support consists of only one point .it is called a fuzzy singleton.

Definition 1.4.14:-

If the membership grade of a fuzzy singleton is one, then F is called crisp singleton.

Definition 1.4.15:-

The core (nucleus. center) of a set F is defined by

Core (F) = $\{x\in X/\mu_F(x)=1\}$.

Definition 1.4.16:-

The height of a set F on X is defined by height (F) =sup $\{\mu_F(x), x\in X\}$

Definition 1.4.17:-

The set F is called normal if height (F) =1.

And it is called subnormal if height (F) <1.

In other word the set F is normal

If its core is non-empty. so we can always find at least a point $x \in X$ Such that $\mu_F(x)=1$.

Definition 1.4.18:-

A crossover point of a set F is a point $x \in X$ at which $\mu_F(x)=0.5$.

Theorem 1.4.1:-

The union and intersection operations of two fuzzy sets are idempotent, commutative and associative.

Let F_1, F_2, F_3 be three sets and $\mu_{F_1}, \mu_{F_2}, \mu_{F_3}$ be the membership functions of F_1, F_2 and F_3 respectively.

1. ($F_1=F_1 \cap F_1$, $F_2=F_2 \cup F_2$) (idempotent)
2. ($F_1 \cap F_2=F_2 \cap F_1$), ($F_1 \cup F_2=F_2 \cup F_1$) (commutative)
3. ($(F_1 \cup F_2) \cup F_3=F_1 \cup (F_2 \cup F_3)$) (associative union)
4. ($(F_1 \cap F_2) \cap F_3=F_1 \cap (F_2 \cap F_3)$) (associative intersection)

Proof:-

- To prove that $F_1=F_1 \cap F_1$ and $F_2=F_2 \cup F_2$

Since μ_{F_1} =min (μ_{F_1}, μ_{F_1}) than $F_1=F_1 \cap F_1$.

And since μ_{F_2} =max (μ_{F_2}, μ_{F_2}) then $F_2=F_2 \cup F_2$.

- We will prove that $F_1 \cap F_2=F_2 \cap F_1$ and $F_1 \cup F_2=F_2 \cup F_1$

Since min (μ_{F_1}, μ_{F_2}) =min (μ_{F_2}, μ_{F_1}).Thus $F_1 \cap F_2=F_2 \cap F_1$.

And since max (μ_{F_1}, μ_{F_2}) = max (μ_{F_2}, μ_{F_1}),Thus $F_1 \cup F_2=F_2 \cup F_1$.

- We will prove that

$F_1 \cup (F_2 \cup F_3) = (F_1 \cup F_2) \cup F_3$

Since $F_1 \cup (F_2 \cup F_3)$ =max $\{\mu_{F_1}, \max(\mu_{F_2}, \mu_{F_3})\}$ which is equal to

max $\{\max(\mu_{F_1}, \mu_{F_2}), \mu_{F_3}\} = (F_1 \cup F_2) \cup F_3$.

- We will prove that $F_1 \cap (F_2 \cap F_3) = (F_1 \cap F_2) \cap F_3$

$F_1 \cap (F_2 \cap F_3)$ =min $\{\mu_{F_1}, \min(\mu_{F_2}, \mu_{F_3})\}$ which is equal to

min $\{\min(\mu_{F_1}, \mu_{F_2}), \mu_{F_3}\} = (F_1 \cap F_2) \cap F_3$.

Theorem 1.4.2:-

Let F_1, F_2 and F_3 be sets then

(1) : $F_3 \cap (F_1 \cup F_2) = (F_3 \cap F_1) \cup (F_3 \cap F_2)$

(2): $F_3 \cup (F_1 \cap F_2) = (F_3 \cup F_1) \cap (F_3 \cup F_2)$

Proof:-

1. Let μ_{F_1}, μ_{F_2} and μ_{F_3} be membership functions of F_1, F_2 and F_3 , respectively. To prove (1) is simply to prove that

Min $\{\mu_{F_3}, \max(\mu_{F_1}, \mu_{F_2})\}$ =max $\{\min(\mu_{F_3}, \mu_{F_1}), \min(\mu_{F_3}, \mu_{F_2})\}$.

This can be verified to be an identity by considering the six cases.

a) $\mu_{F_1}(x) \ge \mu_{F_2}(x) \ge \mu_{F_3}(x)$.

b) $\mu_{F_1}(x) \ge \mu_{F_3}(x) \ge \mu_{F_2}(x)$.

c) $\mu_{F_2}(x) \ge \mu_{F_1}(x) \ge \mu_{F_3}(x)$.

d) $\mu_{F_2}(x) \ge \mu_{F_3}(x) \ge \mu_{F_1}(x)$.

e) $\mu_{F_3}(x) \ge \mu_{F_1}(x) \ge \mu_{F_2}(x)$.

f) $\mu_{F_3}(x) \geq \mu_{F_2}(x) \geq \mu_{F_1}(x)$

2. $F_3 \cup (F_1 \cap F_2) = (F_3 \cup F_1) \cap (F_3 \cup F_2)$

Let μ_{F_1}, μ_{F_2} and μ_{F_3} be membership functions of F_1, F_2 and F_3 ,respectively

to prove (2) is simply to prove that

Max $\{\mu_{F_3}, \min(\mu_{F_1}, \mu_{F_2})\}$ =min $\{\max(\mu_{F_3}, \mu_{F_1}), \max(\mu_{F_3}, \mu_{F_2})\}$

This can be verified to be an identity by considering the six cases.

a) $\mu_{F_1}(x) \leq \mu_{F_2}(x) \leq \mu_{F_3}(x)$

b) $\mu_{F_1}(x) \leq \mu_{F_3}(x) \leq \mu_{F_2}(x)$

c) $\mu_{F_2}(x) \leq \mu_{F_1}(x) \leq \mu_{F_3}(x)$

d) $\mu_{F_2}(x) \leq \mu_{F_3}(x) \leq \mu_{F_1}(x)$

e) $\mu_{F_3}(x) \leq \mu_{F_1}(x) \leq \mu_{F_2}(x)$

f) $\mu_{F_3}(x) \leq \mu_{F_2}(x) \leq \mu_{F_1}(x)$

Theorem 1.4.3:-

Let F_1 and F_2 be two set then

1- $(F_1 \cup F_2)^C = F^c{}_1 \cap F^c{}_2$

2-$(F_1 \cap F_2)^C = F^c{}_1 \cup F^c{}_2$

Proof:-

1. $(F_1 \cup F_2)^C$=1-max (μ_{F_1}, μ_{F_2})

And $F^c{}_1 \cap F^c{}_2$=min $[(1-\mu_{F_1}), (1-\mu_{F_2})]$

Thus it's similar to prove that

1-max (μ_{F_1}, μ_{F_2}) =min $[(1-\mu_{F_1}), (1-\mu_{F_2})]$ ....(*)

Where μ_{F_1} and μ_{F_2} are the membership functions of F_1 and F_2 respectively.

Now if $\mu_{F_1}(x) \geq \mu_{F_2}(x)$ both sides of equation (*)

become $1-\mu_{F_1}(x)=1-\mu_{F_1}(x)$ .than both sides are equal.

Now if $\mu_{F_2}(x) > \mu_{F_1}(x)$ then the both sides in equation (*)

$1-\mu_{F_2}(x)=1-\mu_{F_2}(x)$. Then the both sides are equal.

2. $(F_1 \cap F_2)^C = F^c{}_1 \cup F^c{}_2$

$(F_1 \cap F_2)^C = 1 - \min(\mu_{F_1}, \mu_{F_2})$

And $F^c{}_1 \cup F^c{}_2 = \max[(1-\mu_{F_1}), (1-\mu_{F_2})]$

Thus $1-\min(\mu_{F_1}, \mu_{F_2}) = \max[(1-\mu_{F_1}), (1-\mu_{F_2})]$ .... (**)

Now if $\mu_{F_1} < \mu_{F_2}$ then both sides of (**) are equal to $1-\mu_{F_1}$.

Now if $\mu_{F_2} < \mu_{F_1}$ then both sides of equation (**) are equal to $1-\mu_{F_2}$. In

Two cases the equation (**) are equal.

Theorem 1.4.4:-

Let F_1 and F_2 be sets than $F_1 \cdot F_2 \subseteq F_1 \cap F_2 \subseteq F_1 \cup F_2 \subseteq F_1 \oplus F_2 \subseteq F_1 + F_2$.

Remark 1.4.2:-

The law of contradiction and the law of excluded middle in ordinary set is not valid in fuzzy sets. Then in crisp set A we have.

(i) $A \cup A^c = X$

(ii) $A \cap A^c = \phi$

But in a fuzzy set F we may have $F \cup F^c \neq X$, $F \cap F^c \neq \phi$.

Example 1.4.6:-

Let μ_F be membership function of F and $\mu_F(x)=0.5 \; \forall x \in R$.
It is easy to see that $1-\mu_F(x)=\mu_{F^c}(x)=1-0.5=0.5$
And $F \cup F^c=\max\{0.5, 0.5\}=0.5 \neq 1$

$F \cap F^c=\min\{0.5, 0.5\}=0.5 \neq 0$. In general let $\mu_F(x)=\alpha \in R,\ 0<\alpha<1$
$\forall x \in R$, $F \cup F^c=\max\{\alpha, 1-\alpha\} \neq 1$ and $F \cap F^c=\min\{\alpha, 1-\alpha\} \neq 0$.

Remark 1.4.3:-

There are other operations represent the union and intersection

1. (Yeger union and intersection)

   $F_1 \cup F_2=\min(1, \mu_{F_1}(x)+\mu_{F_2}(x))$

   And $F_1 \cap F_2=(\mu_{F_1}(x)\cdot\mu_{F_2}(x))$

2. (Cheeseman 1986) says the intersection of two sets, F_1, F_2 over the universe X defined by $F_1 \cap F_2=\min\{\mu_{F_1}(x), \mu_{F_2}(x)\}$ is not always true specially when no sufficient information about F_1 and F_2 is provided ,the Only conclusion about $\min\{\mu_{F_1}(x), \mu_{F_2}(x)\}$ should be $0 \leq (\mu_{F_1} \wedge \mu_{F_2})(x) \leq \min\{\mu_{F_1}(x), \mu_{F_2}(x)\}$.

3. Other operators on union and intersection by (D.dubois and H.prade) which satisfy excluded middle-laws are defined by:

- $F_1 \cup F_2=\min\{1, \mu_{F_1}(x)+\mu_{F_2}(x)\}$
- $F_1 \cap F_2=\max\{0, \mu_{F_1}(x)+\mu_{F_2}(x)-1\}$
- F^c is $\mu_{F^c}(x)=1-\mu_F(x)$

4. (Drastic sum and Drastic product)

$$\forall x \in X, \mu_{F_1 \dot{\bigcup} F_2}(x) = \begin{cases} \mu_{F_1}(x), & when\ \mu_{F_2}(x) = 0 \\ \mu_{F_2}(x) & when\ \mu_{F_1}(x) = 0 \\ 1 & \text{others} \end{cases}$$

$$\forall x \in X, \mu_{F_1 \underset{\bullet}{\bigcap} F_2}(x) = \begin{cases} \mu_{F_1}(x), & when\ \mu_{F_2}(x) = 1 \\ \mu_{F_2}(x) & when\ \mu_{F_1}(x) = 1 \\ 0 & when\ \mu_{F_1}(x), \mu_{F_2}(x) < 1 \end{cases}$$

5. in general we use :-

$F_1 \bigcup F_2$=sup $\{ \mu_{F_1}(x), \mu_{F_2}(x) \}$, $F_1 \bigcap F_2$=inf $\{ \mu_{F_1}(x), \mu_{F_2}(x) \}$

CHAPTER 2

2.1 The Extension Principle:-

In order to use fuzzy numbers and interval in any intelligent systems when a fuzzy number is expressed in linguistic terms, we can compute words rather than numbers .the extension principle gives the basic way to define the operations on fuzzy quantities so we can add, subtract, multiply and divide with fuzzy quantities.

The materials in this chapter taken from the following references [16],[26],[20],[5],[25],[33],[27],[11],[28],[15],[10],[24],[19],[14],[21],[7],[8],[29]

Definition 2.1.1:-

Let R be a mapping from X to Y, $R: X \to Y$ such that $F_1 : X \to [0,1]$. And $F_1 \in F(X)$, then we have $R(F_1)$ is a fuzzy set in Y .defined by
$$R(F_1)(y)=\begin{cases}\sup\left\{\mu_{F_1}(x) : x \in R^{-1}(y)\right\} & \text{if} \quad R^{-1}(y) \neq \phi, \\ 0 & \text{if} \quad R^{-1}(y) = \phi\end{cases}$$

Remark 2.1.1:-

Let $F_1 \in F(X)$, $F_2 \in F(Y)$ such as $F_1 : X \to [0,1], F_2 : Y \to [0,1]$ then $R^{-1}(F_2)$ is a fuzzy set in X, defined by $R^{-1}(F_2)(x) = F_2(R(x))$, $x \in X$ .then we have $R^{-1}(F_2^c) = (R^{-1}(F_2))^c$. for any fuzzy set F_2 in Y.

- $R(R^{-1}(F_2)) \leq F_2$, for any fuzzy set F_2 in Y.
- $F_1 \leq R^{-1}(R(F_1))$, for any fuzzy set F_1 in X.

Definition 2.1.2:-

Let $F_1 \in F(X)$ and $F_2 \in F(Y)$. Then by $F_1 \times F_2$ we denote the fuzzy set in $X \times Y$ for which $(F_1 \times F_2)(x, y) = \min\{\mu_{F_1}(x), \mu_{F_2}(y)\}$, $\forall (x, y) \in X \times Y$.

Definition 2.1.3:-

Let $X_1, X_2, \ldots\ldots, X_n$ and Y be a family of sets and let R be a mapping from the Cartesian product $X_1 \times X_2 \times \ldots\ldots \times X_n$ into Y that is for each n-tuple $(x_1, x_2, \ldots\ldots, x_n)$ such that $x_i \in X_i$ we have $R(x_1, x_2, \ldots\ldots, x_n) = y \in Y$. Let $F_1, F_2, \ldots\ldots, F_n$ be fuzzy subset of $X_1, X_2, \ldots\ldots, X_n$ .Respectively the extension principle allows for the evaluation of :-

$$R(F_1, F_2, \ldots\ldots, F_n)(y) = \begin{cases} \sup\{\min(\mu_{F_1}(x_1), \ldots\ldots, \mu_{F_n}(x_n)) : x \in R^{-1}(y)\} & \text{if } R^{-1}(y) \neq \phi, \\ 0 & \text{if } R^{-1}(y) = \phi \end{cases}$$

.For n=2 the extension principle reads

$$R(F_1, F_2)(y) = \sup \{ \mu_{F_1}(x_1) \wedge \mu_{F_2}(x_2) : R(x_1, x_2) = y \}$$

or $R(F_1, F_2)(y) = \vee\{ \mu_{F_1}(x_1) \wedge \mu_{F_2}(x_2) : R(x_1, x_2) = y \}$

2.1.1 Operations By Using Extension Principle:-

Let R be a binary operation $R : X \times X \to X$ and $(a,b) \in X \times X$

Thus we can define:-

- (Extended addition) let $R : X \times X \to X$ be defined as $R(a,b) = a + b$ .i.e. R is the addition operation. Suppose that F_1, F_2 are fuzzy subset of X . Then using the extension principle we get:

  $$R(F_1, F_2)(x) = \sup_{a+b=x} \min\{ \mu_{F_1}(a), \mu_{F_2}(b) \}$$

  And we use the notation $R(F_1, F_2) = F_1 + F_2$. Thus we can say that

  $F_1 + F_2 = \vee(\wedge(F_1 \times F_2)) +^{-1}$ where $+^{-1} = \{(a,b) : a + b = x\}$

- (Extended subtraction) let $R : X \times X \to X$ be defined as $R(a,b) = a - b$ .i.e. R is the subtraction operator .then subtraction of two fuzzy sets given by :-

$$R(F_1,F_2)(x) = F_1 - F_2 = \sup_{a-b=x} \min\{\mu_{F_1}(a), \mu_{F_2}(b)\}$$

Remark 2.1.1.1:-

- $F_1 - F_2 = F_1 + (-F_2)$ holds. Since from definition of addition and subtract on two fuzzy sets we have
  $\sup_{a-b=x} \min\{\mu_{F_1}(a), \mu_{F_2}(b)\} = \sup_{a+b=x} \min\{\mu_{F_1}(a), \mu_{F_2}(-b)\}$.
- (Extended multiplication) let $R : X \times X \to X$

be defined as $R(a,b) = a \bullet b$ .i.e. R is the multiplication operation .suppose that F_1, F_2 are fuzzy subset of X. Then using the extension principle we get: $F_1 \bullet F_2 = R(F_1,F_2)(x) = \sup_{a \bullet b=x} \min\{\mu_{F_1}(a), \mu_{F_2}(b)\}$.

- (Extended division) let $R : X \times X \to X$

be defined as $R(a,b) = a/b$ .i.e. R is the division operation. Suppose that F_1, F_2 are fuzzy subset of X. Then using the extension principle we get: $F_1/F_2 = R(F_1,F_2)(x) = \sup_{a/b=x, b\neq 0} \min\{\mu_{F_1}(a), \mu_{F_2}(b)\}$.

2.2 The Alpha –Cuts or (Level Set):-

We give one of the important ideas in fuzzy sets theory. Since the notation of "belonging" which plays a fundamental role in the case of crisp sets, does not have the same role in the case of fuzzy sets. We now can introduce two levels α, β such that $(0 < \alpha < 1$, and $0 < \beta < 1, \alpha > \beta)$ and agree to say that:

1. "x belong to F " if $\mu_F(x) \geq \alpha$
2. "x dose not belong to F " if $\mu_F(x) \leq \beta$
3. " has an indeterminate status relative to F" if $\beta < \mu_F(x) < \alpha$

Definition 2.2.1:-

Let X be a set, and $[0,1]$ be a complete lattice and $\mu_F : X \to [0,1]$, the α-cut of F is given by: $F^{-1}(\uparrow \alpha) = F_\alpha = \{x \in X : \mu_F(x) \geq \alpha\}, \forall \alpha \in (0,1]$.

Definition 2.2.2:-

Let F be fuzzy subset of X then the strong α-cut of F is given by $F_{\overline{\alpha}} = \{x \in X : \mu_F(x) > \alpha\}, \forall \alpha \in [0,1)$.

Remark 2.2.1:-

- Note that α-cut in general is not fuzzy sets
- We can express the support and the core of fuzzy set by using the notation of α-cut and strong alpha cut as following:-

1. Supp(F) $= F_{\overline{\alpha}}$ when α=0. thus we have supp(F) $= F_{\overline{0}}$
2. Core (F) $= F_\alpha$ when $\alpha = 1$. Thus core (F) $= F_{\alpha=1}$.

Definition 2.2.3:-

Let $F_{\alpha=1}$ be the core of F and let $F_{\overline{0}}$ be the support of F then the boundary of F is difference set $F_{\overline{0}} / F_{\alpha=1}$

Remark 2.2.2:-

A fundamental fact about the α-cut F_α is that they determine F and this is easy to see it from equation $F^{-1}(\alpha) = F_\alpha \cap \left(\bigcup_{\beta > \alpha} F_\beta \right)^c$

this equation just says that the left sides $\{x : \mu_F(x) = \alpha\}$ namely the set of those elements that μ_F takes to α is the intersection of $\{x : \mu_F(x) \geq \alpha\}$ With the set $\{x : \mu_F(x) \not> \alpha\}$ .but these two sets are given strictly in terms of α-cut. So if we know all the α-cuts of F is the same as knowing F it self.

Theorem 2.2.1:-

Let F_1 and $F_2 \in F(X)$ be mapping from X into complete lattice [0,1] if $F_{1\alpha} = F_{2\alpha}, \forall\ \alpha \in (0,1]$, then $F_1 = F_2$.

Proof:-

Since ($F_{1\alpha} = F_{2\alpha}\ \forall\ \alpha \in (0,1]) \Rightarrow (\{x \in X : \mu_{F_1}(x) \geq \alpha\} = \{x \in X : \mu_{F_2}(x) \geq \alpha\}$. $\forall\ \alpha \in (0,1] \Rightarrow F_1 = F_2$).

Theorem 2.2.2:-

Let F_1, F_2 be fuzzy sets ,such that $F_1, F_2 \in F(X)$. And Let $\alpha, \beta, \gamma \in [0,1]$.

Then α -cut and strong α -cut has the following properties:-

1. $(F_1 \vee F_2)_\alpha = F_{1\alpha} \cup F_{2\alpha}$
2. $(F_1 \wedge F_2)_\alpha = F_{1\alpha} \cap F_{2\alpha}$
3. $(F_1 \vee F_2)_{\overline{\alpha}} = F_{1\overline{\alpha}} \cup F_{2\overline{\alpha}}$
4. $(F_1 \wedge F_2)_{\overline{\alpha}} = F_{1\overline{\alpha}} \cap F_{2\overline{\alpha}}$
5. $F_{\overline{\alpha}} \subset F_\alpha$
6. if $\gamma \leq \beta$ then $F_\gamma \subset F_\beta$ and $F_{\overline{\gamma}} \supset F_\beta \supset F_{\overline{\beta}}$
7. $(F^c)_\alpha = (F_{\overline{1-\alpha}})^c$
8. $(F^c)_{\overline{\alpha}} = (F_{1-\alpha})^c$
9. $F_{\alpha=0} = X,\ F_1 = \phi$
10. $(F^c)_\alpha \neq (F_\alpha)^c$

Proof:-

1. $(F_1 \vee F_2)_\alpha = \{x \in X : \mu(F_1 \vee F_2)(x) \geq \alpha\}, \forall\ \alpha \in (0,1] \Rightarrow$

$\{x \in X : (\mu_{F1}(x) \vee \mu_{F2}(x)) \geq \alpha\} \Rightarrow \{x \in X\ \mu_{F1}(x) \geq \alpha \vee \mu_{F2}(x) \geq \alpha\}$...(*)

And $F_{1\alpha} \cup F_{2\alpha} = \{x \in X : \mu_{F1}(x) \geq \alpha \text{ or } \mu_{F2}(x) \geq \alpha\}$...(**)

From (*) since for each x either $\mu_{F1}(x) \leq \mu_{F2}(x)$ or $\mu_{F2}(x) \leq \mu_{F1}(x)$ and this we have $\mu_{F1}(x) \geq \alpha$ or $\mu_{F2}(x) \geq \alpha$ thus (*)=(**).

2. $(F_1 \wedge F_2)_\alpha = \{x \in X : \mu(F_1 \wedge F_2)(x) \geq \alpha\} \Rightarrow$

$\{x \in X : \mu_{F_1}(x) \geq \alpha \wedge \mu_{F_2}(x) \geq \alpha\}$....(*)

$F_{1_\alpha} \cap F_{2_\alpha} \Longrightarrow \{x \in X : \mu_{F_1}(x) \geq \alpha$ *and* $\mu_{F_2}(x) \geq \alpha\}$..(**) from (*)

$\mu_{F_1}(x) \wedge \mu_{F_2}(x) \geq \alpha \Rightarrow \mu_{F_1}(x) \geq \alpha$ and $\mu_{F_2}(x) \geq \alpha \ \forall \alpha \in (0,1]$

Thus(*)=(**)

3. $(F_1 \vee F_2)_{\overline{\alpha}} = \{x \in X : \mu(F_1 \vee F_2)(x) > \alpha\} =$

$\{x \in X : \mu_{F_1}(x) > \alpha \vee \mu_{F_2}(x) > \alpha\})$ ...(*)

Since $\forall\ \alpha \in [0,1)$, $\mu_{F_1}(x) \leq \mu_{F_2}(x)$ or $\mu_{F_2}(x) \leq \mu_{F_1}(x)$ and

$\mu_{F_1}(x) \vee \mu_{F_2}(x) > \alpha \Rightarrow \mu_{F_1}(x) > \alpha$ or $\mu_{F_2}(x) > \alpha \Rightarrow F_{1\overline{\alpha}} \cup F_{2\overline{\alpha}}$

Thus $(F_1 \vee F_2)_{\overline{\alpha}} = F_{1\overline{\alpha}} \cup F_{2\overline{\alpha}}$.

4. Proof is similar of 3

5. obvious from (Definitions 2.2.1, 2.2.2).

6. since $\gamma \leq \beta$ then $\{x \in X : \mu_F(x) \geq \gamma \leq \beta\} \subset \{x \in X : \mu_F(x) \geq \beta\}$

$\therefore F_\gamma \subset F_\beta$.

Now we proof that $F_{\overline{\gamma}} \supset F_\beta \supset F_{\overline{\beta}}$ since $\gamma \leq \beta$ then

$F_{\overline{\gamma}} = \{x \in X : \mu_F(x) > \gamma \leq \beta\} \supset \{x \in X : \mu_F(x) \geq \beta\}$.

$\therefore \{x \in X : \mu_F(x) \geq \beta\} \subset \{x \in X : \mu_F(x) > \gamma \leq \beta\}$ thus $F_\beta \subset F_{\overline{\gamma}}$ or $F_{\overline{\gamma}} \supset F_\beta$.

and also $\{x \in X : \mu_F(x) > \beta\} \subset \{x \in X : \mu_F(x) \geq \beta\}$,

Thus $F_{\overline{\beta}} \subset F_\beta$ and $F_{\overline{\gamma}} \supset F_\beta \supset F_{\overline{\beta}}$.

7. $(F^c)_\alpha = \{x \in X : 1 - \mu_F(x) \geq \alpha\} = \{x \in X : \mu_F(x) \leq 1 - \alpha\}$ ....(*)

And $\left(F_{\overline{1-\alpha}}\right)^C = \{x \in X : \left(\mu_F(x) > 1 - \alpha\right)^c\} = \{x \in X : \mu_F(x) \leq 1 - \alpha\}$ ....(**)

$\therefore$ (*)=(**) thus $(F^c)_\alpha = (F_{\overline{1-\alpha}})^c$.

8. Proof is similar to 7.

9. $F_\alpha = \{x \in X : \mu_F(x) \geq \alpha\}$ since $\alpha = 0$ then $\{x \in X : \mu_F(x) \geq 0\}, \forall\, x \in X$

thus $\{x \in X : \mu_F(x) \geq 0\} = X$ (universal set)

and also $F_{\bar{1}} = \{x \in X : \mu_F(x) > 1\}$ since $\mu_F(x) \in [0,1]$.

then $F_{\bar{1}} = \phi$.

10. $(F^c)_\alpha = \{x \in X : 1 - \mu_F(x) \geq \alpha\}$ and $(F_\alpha)^c = \{x \in X : (\mu_F(x) \geq \alpha)^c\}$.

Thus $(F^c)_\alpha = \{x \in X : \mu_F(x) \leq 1 - \alpha\}$ and $(F_\alpha)^c = \{x \in X : \mu_F(x) < \alpha\}$.

$\therefore (F^c)_\alpha \neq (F_\alpha)^c$.

Theorem 2.2.3:-

Let F be fuzzy subset of a set X and let F_α be the α-cut of F. Then

for $x \in X$, $\mu_F(x) = \bigvee_{\alpha \in [0,1]} (\alpha \wedge \chi_{F_\alpha}(x))$

Where $\chi_{F_\alpha}(x) = \begin{cases} 1 & 0 \leq \alpha \leq \mu_F(x) \\ 0 & \mu_F(x) \leq \alpha \leq 1 \end{cases}$

Proof:-

We have

$$\mu_F(x) = \bigvee_{\alpha \in [0,1]} (\alpha \wedge \chi_{F_\alpha}(x)) = \left(\bigvee_{0 \leq \alpha \leq \mu F(x)} (\alpha \wedge \chi_{F_\alpha}(x))\right) \vee \left(\bigvee_{\mu F(x) \leq \alpha \leq 1} (\alpha \wedge \chi_{F_\alpha}(x))\right)$$

$$= \left(\bigvee_{0 \leq \alpha \leq \mu F(x)} (\alpha \wedge 1)\right) \vee \left(\bigvee_{\mu F(x) \leq \alpha \leq 1} (\alpha \wedge 0)\right) = \bigvee_{0 \leq \alpha \leq \mu F(x)} (\alpha) = \mu_F(x).$$

$\therefore$ thus we have $F = \bigcup_{\alpha \in [0,1]} \alpha F_\alpha$.

Remark 2.2.3:-

The above theorem is called the (first resolution theorem).

Theorem 2.2.4:-

Let $F \in F(X)$ and let $F_{\bar{\alpha}}$ be strong α-cut of F then for $x \in X$

$\mu_F(x) = \bigvee_{\alpha \in [0,1]} (\alpha \wedge \chi_{F_{\overline{\alpha}}}(x))$ thus $F = \bigcup_{\alpha \in [0,1]} \alpha F_{\overline{\alpha}}$

where $\chi_{F_{\overline{\alpha}}}(x) = \begin{cases} 1 & 0 \le \alpha \le \mu_F(x) \\ 0 & \mu_F(x) \le \alpha \le 1 \end{cases}$

Proof:-

Let $F \in F(X)$ we define the mapping $H : [0,1] \to P(X)$, $\alpha \to H(\alpha)$

such that $H(\alpha)$ satisfies $\forall \alpha \in [0,1]$ $(F_{\overline{\alpha}} \subseteq H(\alpha) \subset F_\alpha)$ so we can defined

F by $\{H(\alpha) : \alpha \in [0,1]\}$ then we can proof that $F = \bigcup_{\alpha \in [0,1]} \alpha H(\alpha)$.

By definition of $H(\alpha)$, we have

$(F_{\overline{\alpha}} \subset H(\alpha) \subset F_\alpha \Rightarrow \alpha F_{\overline{\alpha}} \subset \alpha H(\alpha) \subset \alpha F_\alpha)$

and from the (Theorem 2.2.3) we have $F = \bigcup_{\alpha \in [0,1]} \alpha F_\alpha$.

$\therefore F = \bigcup_{\alpha \in [0,1]} \alpha F_{\overline{\alpha}} \subset \bigcup_{\alpha \in [0,1]} \alpha H(\alpha) \subset \bigcup_{\alpha \in [0,1]} \alpha F_\alpha$.

$\therefore F = \bigcup_{\alpha \in [0,1]} \alpha H(\alpha)$ And $F = \bigcup_{\alpha \in [0,1]} \alpha F_{\overline{\alpha}}$.

2.3 The Images of Alpha –Cuts:-

Theorem 2.3.1:-

Let C be complete lattice, and $R : X \to Y$, $A : X \to C$ then

1. $R(A_\alpha) \subseteq (\vee AR^{-1})_\alpha$ for all $\alpha \in C$

2. $R(A_\alpha) = (\vee AR^{-1})_\alpha$ for $\alpha > 0$ if and only if for each member p of the partition induced by R, $\vee A(P) \ge \alpha$ implies $A(x) \ge \alpha$ for some $x \in P$.

Proof:-

1. Since $A_\alpha = \{x \in X : A(x) \ge \alpha\}$ then

$R(A_\alpha) = \{R(x) : A(x) \geq \alpha\} = \{y \in Y : A(x) \geq \alpha, R(x) = y\}$ ...(*)

and $(\vee AR^{-1})_\alpha = \{y \in Y : \vee AR^{-1}(x) \geq \alpha\} = \{y \in Y : \vee\{A(x) : R(x) = y\} \geq \alpha\}$

$= \{y \in Y : \vee\{A(x) \geq \alpha, R(x) = y\}$ ....(**).

$\therefore$ From (*) and (**) we have $R(A_\alpha) \subseteq (\vee AR^{-1})_\alpha \;\; \forall\; \alpha \in C$.

2. We prove that $(R(A_\alpha) = (\vee AR^{-1})_\alpha$ for $\alpha > 0) \Leftrightarrow$ (each member p of the partition induced by R, $\vee A(P) \geq \alpha$ implies $A(x) \geq \alpha$ for some $x \in p$).

$(\Rightarrow)$. let $R(A_\alpha) = (\vee AR^{-1})_\alpha$ for $\alpha > 0$ thus

$R(A_\alpha) = \{y \in Y : A(x) \geq \alpha, R(x) = y\} = (\vee AR^{-1})_\alpha = \{y \in Y : \vee\{A(x) : R(x) = y\} \geq \alpha\}$

thus $\{y : A(x) \geq \alpha\} = \{y : \vee\{A(x)\} \geq \alpha\} \, \forall \alpha$ then for each member p

We have $\vee\{A(P)\} \geq \alpha = A(P) \geq \alpha$ thus

$\vee\{A(P)\} \geq \alpha = A(x) \geq \alpha$ for some $x \in P$

$(\Leftarrow)$. for each member p of the partition induced by R, $\vee A(P) \geq \alpha$ implies $A(x) \geq \alpha$ for some $x \in P$. we will prove that

$R(A_\alpha) = (\vee AR^{-1})_\alpha$ for $\alpha > 0$ .now

$(\vee AR^{-1})_\alpha = \{y \in Y : \vee\{A(x) : R(x) = y\} \geq \alpha\}$ and

$\{y \in Y : \vee\{A(P)\} \geq \alpha \Rightarrow A(x) \geq \alpha\} \Rightarrow \{y \in Y : A(x) \geq \alpha, R(x) = y\} = R(A_\alpha)$

Thus $R(A_\alpha) = (\vee AR^{-1})_\alpha$.

2.4 Convex Fuzzy Sets:-

In this section we shall discuses some important notion in fuzzy set theory is that convex fuzzy set. This notion is very useful in neural networks and pattern classification, so we begin a convex fuzzy set as follows.

Definition 2.4.1:-

Let X be a real linear $\mathbb{R}^n$. A fuzzy set F is convex if $x_1, x_2 \in X$,for all $\lambda \in [0,1]$, $\mu_F[\lambda x_1 + (1-\lambda)x_2] \geq \min[\mu_F(x_1), \mu_F(x_2)]$.

or $F[\lambda x_1 + (1-\lambda)x_2] \geq F(x_1) \wedge F(x_2)$

Definition 2.4.2:-

A fuzzy set F is convex iff the sets $F_\alpha = \{x \in X : \mu_F(x) \geq \alpha\}$ are convex for all α in the interval $(0,1]$.thus

Theorem 2.4.1:-

F is convex iff all its α-cuts are convex.

Proof:-

Now we show that the both definitions above are equivalence.

Now if F is convex by the (definition 2.4.2) and

$\alpha = \mu_F(x_1) \leq \mu_F(x_2) \Rightarrow \mu_F(x_2) \geq \alpha$ then $x_2 \in F_\alpha$ and $\lambda x_1 + (1-\lambda)x_2 \in F_\alpha$,

by the convexity of F_α. Hence $\mu_F[\lambda x_1 + (1-\lambda)x_2] \geq \alpha = \mu_F(x_1) = \min[\mu_F(x_1), \mu_F(x_2)]$

$\therefore \mu_F[\lambda x_1 + (1-\lambda)x_2] \geq \min[\mu_F(x_1), \mu_F(x_2)]$.

If F is convex in the sense of the (definition 2.4.1)

$\mu_F[\lambda x_1 + (1-\lambda)x_2] \geq \min[\mu_F(x_1), \mu_F(x_2)]$, and let $\alpha = \mu_F(x_1)$ then F_α may be regarded as the set of all points x_2 for which $\mu_F(x_2) \geq \mu_F(x_1)$ every point of the form $\lambda x_1 + (1-\lambda)x_2$. $0 \leq \lambda \leq 1$ is also in F_α and hence F_α is convex set .

Remark 2.4.1:-

1. note that the definition of convexity does not imply that $\mu_F(x)$ must be a convex function of x .
2. The only set operation which preserves the convexity property is the intersection.

Theorem 2.4.2:-

Let F_1 and F_2 be two convex fuzzy sets in X then the intersection of F_1 and F_2 is convex.

Proof:-

Let μ_{F1} and μ_{F2} be the membership functions of F_1 and F_2

Since F_1 is convex then $\mu_{F_1}(\lambda x_1+(1-\lambda)x_2)\geq \min[\mu_{F1}(x_1),\mu_{F1}(x_2)]$ and

since F_2 is convex then $\mu_{F_2}(\lambda x_1+(1-\lambda)x_2)\geq \min[\mu_{F2}(x_1),\mu_{F2}(x_2)]$.

and the memberships function of the intersection F_3 of F_1 and F_2

evaluated at $x=\lambda x_1+(1-\lambda)x_2$

$\mu_{F_3}(\lambda x_1+(1-\lambda)x_2)=\min\{\mu_{F_1}[\lambda x_1+(1-\lambda)x_2],\mu_{F_2}[\lambda x_1+(1-\lambda)x_2]\}\geq$

$\min\{\min[\mu_{F_1}(x_1),\mu_{F_1}(x_2)],\min[\mu_{F_2}(x_1),\mu_{F_2}(x_2)]\}\geq$

$\min\{\min[\mu_{F_1}(x_1),\mu_{F_2}(x_1)],\min[\mu_{F_1}(x_2),\mu_{F_2}(x_2)]\}=\min[\mu_{F_3}(x_1),\mu_{F_3}(x_2)]$

$\therefore \mu_{F_3}(\lambda x_1+(1-\lambda)x_2)\geq \min[\mu_{F3}(x_1),\mu_{F3}(x_2)]$

Thus the intersection of two convex fuzzy set is convex.

Definition 2.4.3:-

A fuzzy set F is strictly convex if the sets F_α , $0<\alpha\leq 1$, are strictly convex (that is if the midpoint of any two distinct points in F_α lies in the interior of F_α).

Definition 2.4.4:-

A fuzzy set F is strongly convex if for any two distinct points x_1 and x_2 , $\lambda\in(0,1)$ then $\mu_F(\lambda x_1+(1-\lambda)x_2)>\min[\mu_F(x_1),\mu_F(x_2)]$.

Remark 2.4.2:-

- Strong convexity does not imply strict convexity or vice –versa.
- If F_1, F_2 are strictly (strongly) convex their intersection is strictly (strongly)convex.

Definition 2.4.5:-

A fuzzy set is bounded if and only if the set $F_\alpha=\{x\in X:\mu_F(x)\geq\alpha\}$ are bounded for all $\alpha>0$.

In other words, for every $\alpha > 0$ there exists a finite number $K(\alpha)$ such that the norm $\|x\|$ of every element x of F_α is less than or equal to $K(\alpha)$, thus $\|x\| \le K(\alpha), \forall x \in F_\alpha$.

2.5 Fuzzy Quantities:-

We study the class of fuzzy sets, namely those the real line R .Fuzzy quantities are fuzzy subset of R generalizing ordinary subsets of R.
In order to define operations ,among fuzzy quantities we will evoke the extension principle which was discussed. This principle provides a means for extending operations on R to these of $F(\mathrm{R})$.
We will look at special fuzzy quantities in particular fuzzy numbers and fuzzy intervals.

Definition 2.5.1:-

Let R denote the set of real numbers, the elements of $F(\mathrm{R})$ that is fuzzy subsets of R are called fuzzy quantities.

2.5.1 Operations on fuzzy quantities by using extension principle:-

1. (Addition) let $O: \mathrm{R} \times \mathrm{R} \to \mathrm{R}$,

be defined as $O(a,b) = a+b$ .i.e. O is the addition operation. Suppose that F_1, F_2 are fuzzy quantities of R. Then using the extension principle we get: $O(F_1,F_2)(x) = \sup\limits_{a+b=x} \min\{\mu_{F_1}(a), \mu_{F_2}(b)\}$

And we use the notation $O(F_1,F_2) = F_1 + F_2$. Thus we can say that

$(F_1+F_2)(x) = \vee(\wedge(F_1 \times F_2))+^{-1}$ where $+^{-1} = \{(a,b): a+b = x\}$

And $(a,b), \in \mathrm{R}\times\mathrm{R}$, $\quad x \in \mathrm{R}$

2. (Multiplication) let $O: \mathrm{R} \times \mathrm{R} \to \mathrm{R}$

Be defined as $O(a,b)=a\bullet b$ .i.e. O is the multiplication operation .suppose that F_1, F_2 are fuzzy quantities of X. Then using the extension principle we get: $F_1\bullet F_2=O(F_1,F_2)(x)=\sup\limits_{a\bullet b=x}\min\{\mu F_1(a),\mu F_2(b)\}$.

Thus $(F_1\bullet F_2)(x)=\vee(\wedge(F_1\times F_2))\bullet^{-1}$ where $\bullet^{-1}=\{(a,b):a\bullet b=x\}$

3. (Subtraction) The mapping $R\to R$ defined such that $(r\to -r)$ induces a mapping $F(R)\to F(R)$ and the image of F is denoted $-F$. For $x\in R$ $(-F)(x)=\vee_{x=-y}\{F(y)\}=F(-x)$. Now if we view (-) as a binary operation on R. Let O: $R\times R\to R$ be defined as $O(a,b)=a-b$ .i.e. O is the subtraction operator .then subtraction of two fuzzy quantities given by: - $O(F_1,F_2)(x)=F_1-F_2=\sup\limits_{a-b=x}\min\{\mu_{F1}(a),\mu_{F2}(b)\}$

$(F_1-F_2)(x)=\vee(\wedge(F_1\times F_2))-^{-1}$ where $-^{-1}=\{(a,b):a-b=x\}$.

4. (Division)

It is not a binary operation on R since it is not defined for pairs $(x,0)$ but it is the relation $\{((r,s),t)\in(R\times R)\times R: r=st\}$ by the extension principle this relation induces the binary operation on $F(R)$ given by:-

$\frac{F_1}{F_2}(x)=\vee\wedge(F_1\times F_2)\div^{-1}$, where $\div^{-1}=\{(a,b):a=xb\}$.

Remark 2.5.1:-

So division of any fuzzy quantity by any other fuzzy quantity is possible. In particular, a real number may be divided by 0(zero) in $F(R)$. Recall that R is viewed inside $F(R)$ as the characteristic functions $\chi(r)$ for elements r of R.

Theorem 2.5.1:-

For any fuzzy set F, $\frac{F}{\chi\{0\}}$ is the constant function whose value is $F(0)$. (or $\mu_{F(0)}$).

Proof:-

The function $\frac{F}{\chi\{0\}}$ is given by the formula

$$(\frac{F}{\chi\{0\}})(x) = \underset{a=b.x}{\vee}(F(a) \wedge \chi_{\{0\}}(b)) = \underset{a=0.x}{\vee}(F(a) \wedge \chi_{\{0\}}(0)) = F(0).$$

Thus $\frac{\chi\{r\}}{\chi\{0\}}$ is the constant function 0 (zero) $\frac{\chi\{r\}}{\chi\{0\}} = \begin{cases} 0 & \text{if } r \neq 0 \\ 1 & \text{if } r = 0 \end{cases}$

Remark 2.5.2:-

1. The performing operation on R is the same as performing the corresponding operation on R viewed as subset of $F(R)$ for binary operation O, this means that $\chi\{r\} o \chi\{s\} = \chi\{ros\}$.

Theorem 2.5.2:-

Let O be any binary operation on a set X and let $S, T \in X$ then

$\chi_S o \chi_T = \chi\{sot : s \in S, t \in T\}$

Proof:-

For $x \in X$

$(\chi_S O \chi_T)(x) = \underset{sot=x}{\vee}(\chi_S(s) \wedge \chi_T(t))$ The sup is either (zero) or 1 and is 1 exactly when there is an $s \in S$ and $t \in T$ with $sot = x$ the result follows.

Remark 2.5.3:-

1. Thus if X is a set with a binary operation O then $F(X)$ contains a copy of X with this binary operation in particular if $X =$ R then R with its various binary operation is contained in $F(R)$. We identify $r \in R$ with its corresponding element $\chi\{r\}$.

2. The characteristic function χ_{ϕ} has some special properties where ϕ denotes the empty set. From the last theorem

$\chi_{\phi} o \chi_{T} = \chi\{\phi o t : \phi \in \phi, t \in T\}$

$\chi_{\phi} o \chi_{T} = \chi_{\phi}$, but in fact $\chi_{\phi} o F = \chi_{\phi}$ for any fuzzy set F . ((It is simply the function that is zero every where))

3. A binary operations on a set induce binary operations on its set of subsets. For example if S and T are subsets of R then $S + T = \{s + t : s \in S, t \in T\}$ these operations on subsets S, T of R carry over exactly to operations on the corresponding characteristic sets χ_S, χ_T in $F(\mathrm{R})$.

Theorem 2.5.3:-

Let F_1, F_2 and F_3 be fuzzy quantities. The following hold.

1. $0 + F_1 = F_1$
2. $0 \bullet F_1 = 0$
3. $F_1 + F_2 = F_2 + F_1$
4. $F_1 + (F_2 + F_3) = (F_1 + F_2) + F_3$
5. $F_1 \bullet F_2 = F_2 \bullet F_1$
6. $1 \bullet F_1 = F_1$
7. $r(F_1 + F_2) = rF_1 + rF_2$
8. $F_1(F_2 + F_3) \leq F_1 F_2 + F_1 F_3$
9. $(F_1 \bullet F_2) \bullet F_3 = F_1 \bullet (F_2 \bullet F_3)$
10. $-(-F_1) = F_1$
11. $\frac{F_1}{F_2} = F_1 \frac{1}{F_2}$
12. $F_1 - F_1 \neq 0$
13. $\frac{F_1}{F_1} \neq 1$

Proof:-

1. $0+F_1(x)=\underset{y+z=x}{\vee}\chi_{\{0\}}(y)\wedge F_1(z)=\underset{0+x=x}{\vee}\chi_{\{0\}}(0)\wedge F_1(x)=\underset{x=x}{\vee}(1\wedge F_1(x))=F_1(x)$

2. since $0\bullet F_1=\underset{x=y.z}{\vee}\chi_{\{0\}}(y)\wedge F_1(z)=\underset{x=0.x}{\vee}\chi_{\{0\}}(0)\wedge F_1(x)=\underset{x=0}{\vee}(\chi_{\{0\}}(0)\wedge F_1(x))=0$

3. $(F_1+F_2)(x)=\underset{y+z=x}{\vee}(F_1(y)+F_2(z))=\underset{z+y=x}{\vee}(F_2(z)+F_1(y))=(F_2+F_1)(x)$

4. $F_1+(F_2+F_3)=\{F_1+(F_2+F_3)\}(v)=\underset{\substack{v=z+u\\ u=z+(x+y)}}{\vee}(F_1(z)\wedge(F_2+F_3)(u))=$

$\underset{\substack{v=z+u\\ u=x+y}}{\vee}(F_1(z)\wedge(F_2(x)\wedge F_3(y))=\underset{v=(z+x)+y}{\vee}((F_1(z)\wedge F_2(x))\wedge F_3(y))=(F_1+F_2)+F_3$

5. $F_1\bullet F_2=(F_1\bullet F_2)(z)=\underset{x.y=z}{\vee}\{F_1(x)\wedge F_2(y)\}=\underset{y.x=z}{\vee}\{F_2(y).F_1(x)\}=(F_2.F_1)(z)$

6. $1\bullet F_1=1\bullet F_1(x)=\underset{x.y=z}{\vee}\{\chi_{\{1\}}(x)\wedge F_1(y)\}=\underset{1.z=z}{\vee}\{\chi_{\{1\}}(1)\wedge F_1(y)\}=F_1(z)$

7. $r(F_1+F_2)=(\chi_{\{r\}}(F_1+F_2))(x)=\underset{u.v=x}{\vee}(\chi_{\{r\}}(u)\wedge(F_1+F_2)(v))=$

$=\underset{r.v=x}{\vee}(\chi_{\{r\}}(r)\wedge(F_1+F_2)(v))=\underset{\substack{r.v=x\\ s+t=v}}{\vee}(\chi_{\{r\}}(r)F_1(s)\wedge\chi_{\{r\}}(r)F_2(t))=(rF_1+rF_2)(x)$

8. If $(F_1(F_2+F_3))(x)>(F_1F_2+F_1F_3)(x)$ then there exist u,v,y with $y(u+v)=x$ such that $F_1(y)\wedge(F_2(u)\wedge F_3(v))>F_1(p)\wedge F_2(q)\wedge F_1(h)\wedge F_3(k)$, for all p,q,h,k with $pq+hk=x$ but this is not so for $p=h=y$, $q=u$ and $v=k$. Thus $(F_1(F_2+F_3))(x)\le(F_1F_2+F_1F_3)(x)$ for all x and $F_1(F_2+F_3)\le F_1F_2+F_1F_3$.

9. $(F_1\bullet F_2)\bullet F_3=\{(F_1\bullet F_2)\bullet F_3\}(v)=\underset{v=u.z}{\vee}((F_1\bullet F_2)(u)\wedge F_3(z))=$

$=\underset{\substack{v=u.z\\ u=x.y}}{\vee}((F_1(x)\wedge F_2(y))\wedge F_3(z))=\underset{v=x.(y.z)}{\vee}(F_1(x)\wedge(F_2(y)\wedge F_3(z))=$

$\underset{\substack{v=x.t\\ t=y.z}}{\vee}(F_1(x)\wedge(F_2\bullet F_3)(t))=((F_1\bullet(F_2\bullet F_3))(v).$

$\therefore(F_1\bullet F_2)\bullet F_3=F_1\bullet(F_2\bullet F_3)$

10. $-(-F_1)=-\{(-F_1)(x)\}=-\{\underset{x=-y}{\vee}F_1(x)\}=-\{\underset{x=-y}{\vee}F_1(y)\}=\{\underset{x=-y}{\vee}F_1(-y)\}=F_1(x)$

11. $\frac{F_1}{F_2}=\frac{F_1}{F_2}(x)=\underset{y=z.x}{\vee}(F_1(y)\wedge F_2(z))=\underset{y=z.x}{\vee}\{(F_1(y)\bullet(\chi_{\{1\}}(1)\wedge F_2(z)\}=$

$\underset{y \cdot z=x}{\vee}\{(F_1(y) \cdot (\chi_{\{1\}}(1) \wedge F_2(z))\} = \underset{y=x.z}{\vee}\{F_1(y) \cdot \{\underset{z=1.z}{\vee}\{\chi_{\{1\}}(1) \wedge F_2(z)\}\}\} = F_1 \cdot \frac{1}{F_1}$

12. $(F_1 - F_2)(x) = \underset{y-z=x}{\vee}(F_1(y) \wedge F_1(z))$

$(F_1 - F_1)(0) = \underset{x-y=0}{\vee}\{F_1(x) \wedge F_1(y)\} = \underset{x=y}{\vee}\{F_1(x) \wedge F_1(x)\} = \underset{x}{\vee}\{F_1(x)\}$

and the other side $0 = \chi_0(0) = 1$ thus $F_1 - F_1 \neq 0$.

13.

$\frac{F_1}{F_1}(x) = \underset{y=z.x}{\vee}(F_1(y) \wedge F_1(z)) = \frac{F_1}{F_1}(1) = \underset{y=z.1}{\vee}(F_1(y) \wedge F_1(z)) = \underset{y=z}{\vee}(F_1(y) \wedge F_1(z))$
$= \underset{y=z}{\vee}(F_1(z)) = F_1(z)$

and the other side $1 = \chi_1(1) = 1$. thus $\frac{F_1}{F_1} \neq 1$

Definition 2.5.2:-

A fuzzy quantity F is convex if its α -cuts are convex that is if its α -cuts are intervals.

Remark 2.5.4:-

Since fuzzy quantities are fuzzy numbers and intervals then a subset F of the plane that is of $R^2 = R \times R$ is convex if contains the straight line connecting any two of its points. This can be expressed by saying that for $t \in [0,1]$ $tx + (1-t)y$ is in F whenever, x, y are in F.

Theorem 2.5.4:-

A fuzzy quantity F is convex if and only if
$F(y) \geq (F(x) \wedge F(z))$, whenever $x \leq y \leq z$.

Proof:-

a) let F be convex, $x \leq y \leq z$. and let $\alpha = F(x) \wedge F(z)$ then $x, z \in F_\alpha$
And since F_α is an interval, $y \in F_\alpha$ there for $F(y) \geq (F(x) \wedge F(z))$.

b) Suppose that $F(y) \geq F(x) \wedge F(z)$, whenever $x \leq y \leq z$ to prove F is convex, Let $x < y < z$, with $x, z \in F_\alpha$. Then
$F(y) \geq (F(x) \wedge F(z)) \geq \alpha$, where $y \in F_\alpha$ and F_α is convex.

Theorem 2.5.5:-

If F_1, F_2 are convex fuzzy sets then so $F_1 + F_2$.

Proof:-

a) We show that $F_1 + F_2$ is convex.

Let $x < y < z$. we need that $(F_1 + F_2)(y) \geq [(F_1 + F_2)(x) \wedge (F_1 + F_2)(z)]$

Let $\in > 0$, there are numbers x_1, x_2, z_1, and z_2, with $x_1 + x_2 = x$

and $z_1 + z_2 = z$, satisfying $F_1(x_1) \wedge F_2(x_2) \geq (F_1 + F_2)(x) - \in$

$F_1(z_1) \wedge F_2(z_2) \geq (F_1 + F_2)(z) - \in$.

Now $y = \alpha x + (1-\alpha)z$ for some $\alpha \in [0,1]$. Let $x' = \alpha x_1 + (1-\alpha)z_1$ and

$z' = \alpha x_2 + (1-\alpha)z_2$ then $x' + z' = y$, x' lies between x_1, z_1 and z' lies

between x_2, z_2 thus we have $(F_1 + F_2)(y) \geq F(x') \wedge F(z') \geq$

$\geq F_1(x_1) \wedge F_1(z_1) \wedge F_2(x_2) \wedge F_2(z_2)$

$\geq [(F_1 + F_2)(x) - \in] \wedge [(F_1 + F_2)(z) - \in] \geq [(F_1 + F_2)(x) \wedge (F_1 + F_2)(z)] - \in$.

It follows that $F_1 + F_2$ is convex.

Definition 2.5.3:-

A function $f : \mathrm{R} \to \mathrm{R}$ is upper semi continuous if $\{x : f(x) \geq \alpha\}$ is closed. The following definition is consistent with this terminology.

Definition 2.5.4:-

A fuzzy quantity is upper semi continuous if its α -cuts are closed.

Theorem 2.5.6:-

A fuzzy quantity F is upper semi continuous in R if and only if whenever $x, y \in \mathrm{R}$ and $\in > 0$, there is a $\delta > 0$ such that $|x - y| < \delta$ implies that $F(y) < F(x) + \in$.

Proof:-

a) Suppose that F_α is close for all α. Let $x \in \mathrm{R}$ and $\in > 0$. if

$F(x) + \in > 1$, Then $F(y) < F(x) + \in$ for any y if $F(x) + \in \leq 1$ then for $\alpha = F(x)+\in, x \notin F_\alpha$. And so there is $\delta > 0$ such that

$(x-\delta, x+\delta) \cap F_\alpha = \phi$ thus $F(y) < \alpha = F(x)+\in$, for all y with $|x-y| < \delta$

b) Let $\in > 0$ $x \in R$ $\exists$ $\delta > 0 : |x-y| < \delta$, $F(y) < F(x)+\in$ we show that F is upper semi continuous (thus we will show that α -cuts are closed)

take $\alpha \in [0,1]$, $x \notin F_\alpha$ and $\in = \frac{\alpha - F(x)}{2}$, there is $\delta > 0$ such that

$|x-y| < \delta \Rightarrow F(y) < F(x) + \frac{\alpha - F(x)}{2} < \alpha$ and so $(x-\delta, x+\delta) \cap F_\alpha = \phi$.

Thus F_α is closed.

The following theorem is the crucial fact that enables us to use α -cuts in computing with fuzzy quantities.

Theorem 2.5.7:-

Let $O: \mathrm{R} \times \mathrm{R} \to \mathrm{R}$ be a continuous binary operation on R and let F_1, F_2 be fuzzy quantities with closed α -cuts and bounded supports. Then for each $u \in \mathrm{R}$, $(F_1 \circ F_2)(u) = F_1(x) \wedge F_2(y)$ for some x, y with $u = x \circ y$

Proof:-

By definition $(F_1 \circ F_2)(u) = \vee_{x \circ y = u} \{F_1(x) \wedge F_2(y)\}$

the equality certainly holds if $(F_1 \circ F_2)(u) = 0$, suppose

$\alpha = (F_1 \circ F_2)(u) > 0$ and $F_1(x) \wedge F_2(y) < \alpha$ for all x and y such that $x \circ y = u$

then there is a sequence $\{F_1(x_i) \wedge F_2(y_i)\}_{i=1}^{\infty}$ in the set $\{F_1(x) \wedge F_2(y) : x \circ y = u\}$ having the following properties.

1) $\{F_1(x_i) \wedge F_2(y_i)\}_{i=1}^{\infty}$ converges to α

2) Either $\{F_1(x_i)\}_{i \geq 1}$ or $\{F_2(y_i)\}_{i \geq 1}$ converges to α

3) each x_i is in the support of F_1 and each y_i is in the support of F_2.

Suppose that it is $\{F_1(x_i)\}_{i \geq 1}$ that converges to α since the support of F_1

is bounded. The set $\{x_i\}$ has a limit point x and hence a subsequence converging to x .

Since the support of F_2 is bounded the corresponding subsequence of y_i has a limit point y and hence a subsequence converging to y. the corresponding subsequence of x_i converging to x .

thus we have a sequence $\{F_1(x_i) \wedge F_2(y_i)\}_{i=1}^{\infty}$ satisfying the three properties above and with $\{x_i\}_{i\geq 1}$ converging to x and $\{y_i\}_{i\geq 1}$ converging to

y, If $F_1(x) = \gamma < \alpha$, then for $\delta = \frac{\alpha + \gamma}{2}$ and for sufficiently large

$i, x_i \in F_\delta$, x is a limit point of those x_i and since all cuts are close , $x \in F_\delta$. But it is not, so $F_1(x) = \alpha$ in a similar $F_1(y) \geq \alpha$ and we have $(F_1 \circ F_2)(u) = F_1(x) \wedge F_2(y)$. Finally $u = x \circ y$ since $u = x_i \circ y_i$ for all i.
And $\circ$ is continuous.

Corollary 2.5.1:-

If F_1 and F_2 are fuzzy quantities with bounded support all α -cuts are closed and $\circ$ is continuous binary operation on R then
$(F_1 \circ F_2)_\alpha = F_{1\alpha} \circ F_{2\alpha}$

Proof:-

1) Applying the (Theorem 2.5.7), for $u \in (F_1 \circ F_2)_\alpha$
$(F_1 \circ F_2)(u) = F_1(x) \wedge F_2(y)$ for some x and y with $u = x \circ y$ thus $x \in F_{1\alpha}$
$y \in F_2\alpha$, and therefore $(F_1 \circ F_2)_\alpha \subseteq F_1\alpha \circ F_2\alpha$.

2) Let $u \in F_1\alpha \circ F_2\alpha \Rightarrow x \in F_1\alpha$ and $y \in F_2\alpha$ for some x, y and $u = x \circ y$
thus $u \in (F_1\alpha(x) \wedge F_2\alpha(y))$, $u = x \circ y$ for some x, y
$u \in (F_1 \circ F_2)_\alpha(u) \Rightarrow F_1\alpha \circ F_2\alpha \subseteq (F_1 \circ F_2)(u)$,
Form (1) and (2) $(F_1 \circ F_2)_\alpha = F_{1\alpha} \circ F_{2\alpha}$

Corollary 2.5.2:-

If F_1 and F_2 are fuzzy quantities with bounded support and α -cuts are closed then

1. $(F_1+F_2)_\alpha = F_{1\alpha}+F_{2\alpha}$
2. $(F_1-F_2)_\alpha = F_{1\alpha}-F_{2\alpha}$
3. $(F_1\bullet F_2)_\alpha = F_{1\alpha}\bullet F_{2\alpha}$

Remark 2.5.5:-

About division of set of real numbers, we have no obvious way to divide a set S by a set T we cannot take $\frac{S}{T}=\{\frac{s}{t}:s\in S,t\in T\}$

(Since t may be 0.)But we can perform the operation $\frac{\chi_S}{\chi_T}$ .therefore fuzzy arithmetic gives a natural way to divide sets of real one by the other an in particular to divide intervals. And note that if S and T are closed and bounded then $(\chi_S/\chi_T)(u)=\chi_S(ux)\wedge\chi_T(x)$ for a suitable x

2.6 fuzzy numbers:-

in this section we disuses a special case of fuzzy quantities which is known as fuzzy numbers and we shall show that because of its important of convexity and normality in definition of fuzzy number and also we defined the .operations on fuzzy quantities, type of fuzzy numbers also because fuzzy number is expressed in linguistic terms we can compete words rather than numbers. Which is very important in artificial intelligent.

Definition 2.6.1:-

A fuzzy numbers is a fuzzy quantity F that satisfies the following conditions:-

1. $F(x)=1$ for exactly one x (F is normal fuzzy quantity)
2. the support of F is bounded ,

3. The α -cuts of F are closed intervals, For every $\alpha \in (0,1]$.

Remark 2.6.1:-

1. Let I be a fuzzy number then I_{α} (α -cuts of I) is a closed convex (subset of R) for all $\alpha \in (0,1]$, and I_{α} from type $I_{\alpha} = \left[a_{\alpha}^{L}, a_{\alpha}^{R}\right]$ closed and bounded interval (L-left, R-right)
2. We shall note to fuzzy number as I and I_r (family of fuzzy number). We know that the only convex sets in R are interval ,and when the membership function of the convex quantities is upper Semi-continuous ,Then all these α -cuts I_{α} ,for $\alpha \in (0,1]$,are closed intervals. Since the basic requirement to define a fuzzy number is that all it's α -cuts are closed and bounded intervals, we may have the following

Proposition 2.6.1:-

The following hold:-

1. real numbers are fuzzy numbers ;
2. a fuzzy number is convex fuzzy quantity ;
3. a fuzzy number is upper semi continuous ;
4. if I is a fuzzy number with $I(r)=1$ where $r \in R$ then I is

a non-decreasing on $(-\infty, r]$ and non-increasing on $[r, \infty)$.

Proof:-

1. It should be clear that real numbers are fuzzy numbers.
2. A fuzzy number is convex since its α -cuts are intervals.
3. A fuzzy number is upper semi continuous since its α -cuts are closed.
4. If I is fuzzy number with $I(r)=1$ and $r \in R$ $x < y < r$ since I is convex. And $I(x) \leq I(y)$, so I is monotone increasing on $(-\infty, r]$

Similarly I is monotone decreasing on $[r, \infty)$.

2.6.1 Some Types of Fuzzy Numbers:-

2.6.1.1 Triangular Fuzzy Number:-

Among the various shapes of fuzzy number triangular fuzzy number (TFN) is the most popular one such this type have shape like triangular which can determined by a triple (a,b,c) of fuzzy numbers with $a \le b \le c$.

Definition 2.6.1.1.1:-

A triangular fuzzy number I is a fuzzy quantity such that

$$I = (a1, a2, a3)\ , \ \mu_I(x) = \begin{cases} 0 & \text{if} \quad x < a_1 \\ \dfrac{x-a_1}{a_2-a_1} & \text{if} \quad a_1 \le x \le a_2 \\ \dfrac{a_3-x}{a_3-a_2} & \text{if} \quad a_2 \le x \le a_3 \\ 0 & \text{if} \quad x > a_3 \end{cases}$$

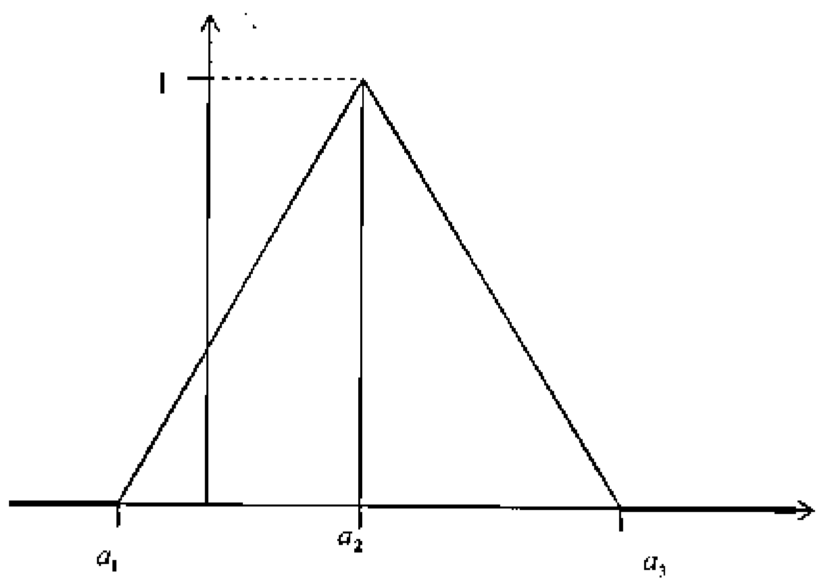

Figure 6

So we use the notation $I = (a1, a2, a3)$ used for a triangular fuzzy number with center $a2$ is a fuzzy quantity which x is a proximately equal to $a2$ Where the support of I is $(a1,, a3)$.

2.7 Fuzzy Intervals:-

A subset S of R is identified with χ_s and in particular, intervals $[a,b]$ are identified with their characteristic functions, namely the fuzzy quantities $\mu[a,b]$. The use of intervals with their arithmetic is appropriate in some situations involving impreciseness. When the intervals themselves are not sharply defined, we are driven to the concept of fuzzy interval. Thus we want to generalize intervals to fuzzy intervals, and certainly a fuzzy quantity generalizing the interval $[a,b]$.
should have value 1 on $[a,b]$.

Definition 2.7.1:-

A fuzzy interval is fuzzy quantity F satisfying the following:

1. F is normal
2. the support $\{x \in X : \mu_{F(x)} \geq 0\}$ of F is bounded
3. The α -cuts of F are closed intervals.

Remark 2.7.1:-

1. In fact fuzzy numbers are fuzzy intervals the only difference is that a fuzzy number can attain the value 1 at only one place while a fuzzy interval can have an interval of such places.
2. We denote to fuzzy intervals by I^*.

2.7.1 Some Types of Fuzzy Interval:-

Definition 2.7.1.1:-

A fuzzy quantity is called trapezoidal fuzzy interval such that

$I^* = (a_1, a_2, a_3, a_4)$ and its membership function is:-

$$\mu_I^*(x) = \begin{cases} 0 & \text{if} \quad x < a_1 \\ \dfrac{x-a_1}{a_2-a_1} & \text{if} \quad a_1 \le x \le a_2 \\ 1 & \text{if} \quad a_2 \le x \le a_3 \\ \dfrac{a_4-x}{a_4-a_3} & \text{if} \quad a_3 \le x \le a_4 \\ 0 & \text{if} \quad x > a_4 \end{cases}$$

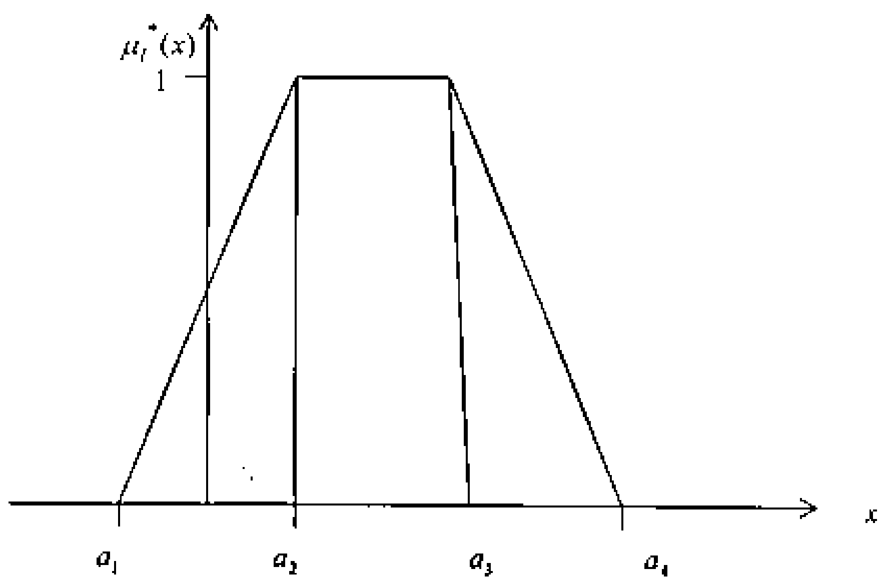

Figure 7

Remark 2.7.1.1:-

A trapezoidal fuzzy interval may be seen as a fuzzy quantity, x is approximately in the interval $[a_2, a_3]$.

2.8 L-R representation of fuzzy number:-

Definition 2.8.1:-

An L-R fuzzy number I denoted by (m, α, β) is a fuzzy set which has membership function defined for all $x \in R$ by

$$\mu_l(x) = \begin{cases} L(\frac{m-x}{\alpha}) & \text{if} \quad x \le m \\ R(\frac{x-m}{\beta}) & \text{if} \quad x \ge m \end{cases}$$

With $\alpha > 0, \beta > 0$,respect to left and right spreads.

Thus , L is monotonically increasing toward 1and R is monotonically decreasing from 1 with $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$, and the highest membership value 1 is at $x = m$ as shown in the next figure

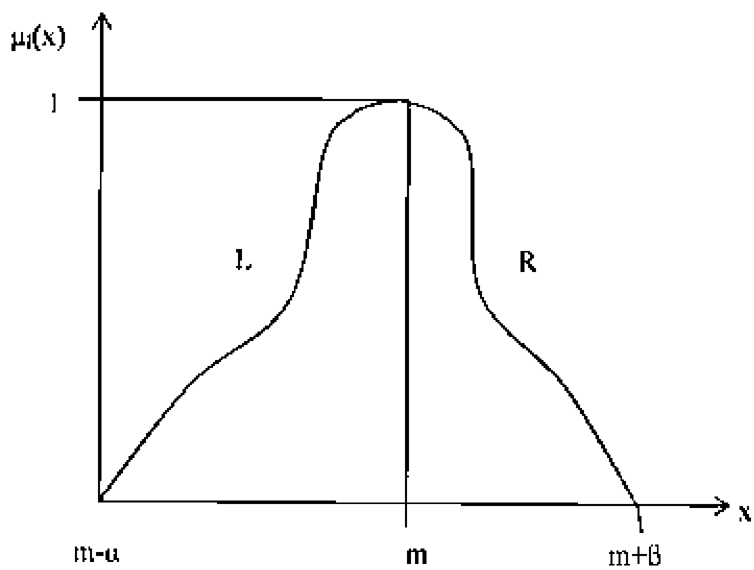

Figure 8

If $m<0$, we have a left translation, and for $m>0$ we have a right translation. .

Remark 2.8.1:-

L-R fuzzy numbers allow one to tune the shape of membership function.

1. if $\alpha<1$ and $\beta<1$,we have contraction ;
2. if $\alpha>1$ and $\beta>1$,we have a dilation ;
3. and for $m_1<x<m_2$ we have a flatting as shown in the next figure

$$\mu_m(x)=\begin{cases} L(\frac{m_1-x}{\alpha}) & \text{if} \quad x\le m_1 \\ 1 & \text{if} \quad m_1\le x\le m_2 \\ R(\frac{x-m_2}{\beta}) & \text{if} \quad x\ge m_2 \end{cases}$$

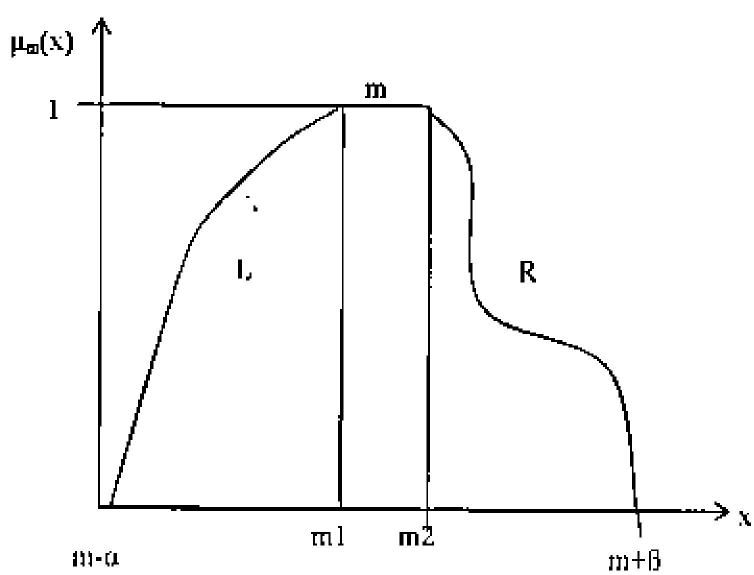

Figure 9

Dilation (L), contraction(R), and flatting (m) if both L and R are linear ,we have a triangular fuzzy number

4. if a linear L-R fuzzy number has $m_{I_1} \le m_I \le m_{I_2}$ then it become a trapezoidal fuzzy numbers.

2.9 Operations On Fuzzy Number:-

If I_1, I_2 are two fuzzy numbers we need to add, subtract, multiply and divide them. There are two basic methods to do their operations:-

1. the extension principle;
2. α -cuts as interval arithmetic;

on fuzzy quantities in the section **2.5.1** by using the extension principle. And because the α -cuts of any fuzzy number is closed interval and we have bounded support ,thus we can use the notion of interval arithmetic as operations on fuzzy number .

2.9.1 Operations on Fuzzy Number by using alpha Cuts:-

Let I_1, I_2 be two fuzzy numbers and let

$I_{1\alpha} = \left[a_\alpha^L, a_\alpha^R\right]$, $I_{2\alpha} = \left[b_\alpha^L, b_\alpha^R\right]$ be α -cuts, $\alpha \in (0,1]$ of

I_1, I_2 respectively. Let $\circ$ denote any of the arithmetic operations

(+), (-) ,(.),(/),($\wedge$),($\vee$) on fuzzy numbers. Then by using

Theorem 2.5.7, corollary 2.5.2 and Theorem 2.2.3 to compute

$I_1 \circ I_2 = \bigcup_\alpha \alpha(I_1 \circ I_2)_\alpha, (I_1 \circ I_2)_\alpha = I_{1\alpha} \circ I_{2\alpha}, \alpha \in (0,1]$,Thus

$(I_1 \circ I_2)_\alpha$ is closed interval and we can compute it by applying the interval arithmetic as follow:-

1. <u>addition</u> $I_{1\alpha} + I_{2\alpha} = \left[a_\alpha^{\ L} + b_\alpha^{\ L}, a_\alpha^{\ R} + b_\alpha^{\ R}\right]$
2. <u>subtraction</u> $I_{1\alpha} - I_{2\alpha} = \left[a_\alpha^{\ L} - b_\alpha^{\ R}, a_\alpha^{\ R} - b_\alpha^{\ L}\right]$

3. Multiplication. of two closed intervals $I_{1\alpha}, I_{2\alpha}$ denoted by

$$I_{1\alpha} \bullet I_{2\alpha} = \left[a_\alpha^L, a_\alpha^R\right] \bullet \left[b_\alpha^L, b_\alpha^R\right] =$$

=

$$\left[\min(a_\alpha^L b_\alpha^L, a_\alpha^L b_\alpha^R, a_\alpha^R b_\alpha^L, a_\alpha^R b_\alpha^R), \max(a_\alpha^L b_\alpha^L, a_\alpha^L b_\alpha^R, a_\alpha^R b_\alpha^L, a_\alpha^R b_\alpha^R)\right]$$

.

In case these intervals are in R^+, the multiplication formula gets simplified to $I_{1\alpha} \bullet I_{2\alpha} = \left[a_\alpha^L, a_\alpha^R\right] \bullet \left[b_\alpha^L, b_\alpha^R\right] = \left[a_\alpha^L b_\alpha^L, a_\alpha^R b_\alpha^R\right]$

4. scalar multiplication

Let $I_{1\alpha} = \left[a_\alpha^L, a_\alpha^R\right]$ be closed interval in R^+ and $k \in R^+$, identifying the scalar k as the closed interval $[k,k]$, the scalar multiplication $k \bullet I_{1\alpha}$ is defined as $(k \bullet I_1)_\alpha = k \bullet I_{1\alpha} = \left[Ka_\alpha^L, Ka_\alpha^R\right]$

5. division :-

The division of two closed intervals $I_{1\alpha}$ and $I_{2\alpha}$ of R, denoted by $\frac{I_{1\alpha}}{I_{2\alpha}}$, is defined as the multiplication of $[a^L_\alpha, a^R_\alpha]$ and $\left[\frac{1}{b_\alpha^R}, \frac{1}{b_\alpha^L}\right]$ Provided $0 \notin \left[b_\alpha^L, b_\alpha^R\right]$. Therefore $\frac{I_{1\alpha}}{I_{2\alpha}} =$

$$\left[a_\alpha^L, a_\alpha^R\right] \bullet \left[\frac{1}{b_\alpha^R}, \frac{1}{b_\alpha^L}\right] = \left[\min\left(\frac{a_\alpha^L}{b_\alpha^R}, \frac{a_\alpha^L}{b_\alpha^L}, \frac{a_\alpha^R}{b_\alpha^R}, \frac{a_\alpha^R}{b_\alpha^L}\right), \max\left(\frac{a_\alpha^L}{b_\alpha^R}, \frac{a_\alpha^L}{b_\alpha^L}, \frac{a_\alpha^R}{b_\alpha^R}, \frac{a_\alpha^R}{b_\alpha^L}\right)\right]$$

In case these intervals are in R^+ and as before $0 \notin \left[b_\alpha^L, b_\alpha^R\right]$, this formula for the division gets simplified to $\frac{I_{1\alpha}}{I_{2\alpha}} = \left[\frac{a_\alpha^L}{b_\alpha^R}, \frac{a_\alpha^R}{b_\alpha^L}\right]$

Remark 2.9.1.1:-

As in intervals arithmetic ,it can be verified that (+) and multiplication (.) operations on closed intervals as defined above are commutative and associative but subtraction (-) and division (/) are neither commutative nor associative. Also image of a closed interval $\left[a_\alpha^L, a_\alpha^R\right]$ is $\left[-a_\alpha^R, -a_\alpha^L\right]$ thus

$\left[a_\alpha^L, a_\alpha^R\right] + \left[-a_\alpha^R, -a_\alpha^L\right] \neq [0,0]$. In case $\left[a_\alpha^L, a_\alpha^R\right]$ is in R^+ and $0 \notin \left[a_\alpha^L, a_\alpha^R\right]$, thus

$$\left[a_\alpha^L, a_\alpha^R\right] \cdot \left[a_\alpha^L, a_\alpha^R\right]^{-1} = \left[a_\alpha^L, a_\alpha^R\right]^{-1} \cdot \left[a_\alpha^L, a_\alpha^R\right] \neq [1,1]$$

6. max-min operations

Let $I_{1\alpha}$ and $I_{2\alpha}$ be two closed interval in R ,then the max$(\vee)$ and min$(\wedge)$ operations on $I_{1\alpha}$ and $I_{2\alpha}$ are defined as:-

$$I_{1\alpha} \vee I_{2\alpha} = \left[a_\alpha^L, a_\alpha^R\right] \vee \left[b_\alpha^L, b_\alpha^R\right] = \left[a_\alpha^L \vee b_\alpha^L, a_\alpha^R \vee b_\alpha^R\right]$$

$$I_{1\alpha} \wedge I_{2\alpha} = \left[a_\alpha^L, a_\alpha^R\right] \wedge \left[b_\alpha^L, b_\alpha^R\right] = \left[a_\alpha^L \wedge b_\alpha^L, a_\alpha^R \wedge b_\alpha^R\right]$$

2.10 Fuzzy arithmetic:-

In this section we use the both operations the extension principle and α -cuts as interval arithmetic. And since fuzzy number have closed α -cuts and bounded interval ,thus we have.

2.10.1 Fuzzy addition:-

From extension principle and corollary 2.5.2 we have that

Definition 2.10.1.1:-

Fuzzy addition of two fuzzy numbers I_1, I_2 is defined by $I_1 + I_2 = I_3$

Where $\mu_{I_3}(z) = \sup_{x+y=z}\{\min(\mu_{I1}(x), \mu_{I2}(y)\}$

If and only if $I_{3_\alpha} = I_{1_\alpha} + I_{2_\alpha} = \left[a_\alpha{}^L + b_\alpha{}^L, a_\alpha{}^R + b_\alpha{}^R\right], \forall \alpha \in (0,1]$.

Theorem 2.10.1.1:-

Addition of two fuzzy numbers is a fuzzy number.

Proof:-

Let I_1, I_2 be two fuzzy numbers we need to proof that $I_1 + I_2 = I_3$ is fuzzy number.

Thus we proof that $I_1 + I_2$

1. is normal ;
2. α -cuts are closed interval;
3. and have bounded support;

1. Assume $I_1 + I_2$ have the value 1 in exactly one place.

2, 3 from the corollary 2.5.2 and Theorem 2.5.7;

Corollary 2.10.1.1:-

Let I_1, I_2, I_3 be fuzzy numbers. The addition of fuzzy numbers satisfies the following:-

1. (+) is commutative and associative

$I_1 + I_2 = I_2 + I_1$

$(I_1 + I_2) + I_3 = I_1 + (I_2 + I_3)$

2. Nonsymmetric on image :

$I_1(+)((-)I_1) = ((-)I_1)(+)I_1 \neq 0$

3. Identity for fuzzy addition :

$I_1 + 0 = 0 + I_1 = I_1$

Definition 2.10.1.2:-

Given a fuzzy number I_1 and a crisp number K, a hybrid addition of two types of numbers is defined by $I_2 = I_1 + K$ if and only if $\mu_{I_2}(x) = \mu_{I_1}(x - K)$ .therefore, this is simply a translation and $I_1 = I_1 + 0$

Definition 2.10.1.3:-

Fuzzy addition for triangular fuzzy numbers:

$I_1 = (a_1, a_2, a_3)$ and $I_2 = (b_1, b_2, b_3)$ defined by

$I_1 + I_2 = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

Definition 2.10.1.4:-

Fuzzy addition for trapezoidal fuzzy numbers. $I_1 = (a_1, a_2, a_3, a_4)$

$I_2 = (b1, b2, b3, b4)$ defined by $I_1 + I_2 = (a1 + b1, a2 + b2, a3 + b3, a4 + b4)$

Definition 2.10.1.5:-

Fuzzy addition for L-R fuzzy numbers. $I_1 = (x, \alpha, \beta)$

$I_2 = (y, \gamma, \delta)$ defined by $I_1 + I_2 = (x + y, \alpha + \gamma, \beta + \delta)$

Definition 2.10.1.6:-

Giving a fuzzy number I, $\tilde{n}$ is called fuzzy integer defined by

$\tilde{n} = \tilde{1} + (\tilde{n} - \tilde{1})$

2.10.2 Fuzzy Subtraction:-

Definition 2.10.2.1:-

Let I_1, I_2 be fuzzy numbers. Then fuzzy subtraction of I_1, I_2 is defined by $I_1(-)I_2 = I_3$ Where $\mu_{I_3}(z) = \sup_{x-y=z}\{\min(\mu_{I_1}(x), \mu_{I_2}(y)\}$

If and only if $I_{3\alpha} = I_{1\alpha} - I_{2\alpha} = \left[a_\alpha^L - b_\alpha^R, a_\alpha^R - b_\alpha^L\right], \forall \alpha \in (0,1]$

Theorem 2.10.2.1:-

Subtraction of two fuzzy numbers is a fuzzy number.

Proof:-

Is similar to proof of Theorem 2.10.1.1

Definition 2.10.2.2:-

Fuzzy subtraction for two triangular fuzzy numbers.

$I_1 = (a_1, a_2, a_3)$ And $I_2 = (b_1, b_2, b_3)$ defined by

$I_1 - I_2 = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$

Remark 2.10.2.1:-

The results from addition or subtraction between triangular fuzzy numbers are also triangular fuzzy numbers.

If we have two triangular fuzzy numbers I_1, I_2 then

$I_1 - I_2 = I_1(+)((-)I_2)$ and if $I_1 = I_2$ then $I_1 - I_1 \neq 0$ if $I_1 \neq 0$

Definition 2.10.2.3:-

Fuzzy subtraction for two trapezoidal fuzzy numbers.

$I_1 = (a_1, a_2, a_3, a_4)$ And $I_2 = (b_1, b_2, b_3, b_4)$ defined by

$I_1 - I_2 = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1)$

Remark 2.10.2.2:-

The results from addition or subtraction between trapezoidal fuzzy numbers are also trapezoidal fuzzy numbers.

Corollary 2.10.2.1:-

The subtraction of two fuzzy numbers in general is neither commutative nor associative; it is defined in integer Z and real numbers R but not in N or R^+.

2.10.3 Fuzzy Multiplication:-

Definition 2.10.3.1:-

Let I be a fuzzy number and K is a scalar in R, then

1. $K = 0$, $K \bullet I = 0$
2. $K \neq 0$, $K \bullet I_1 = I_2$ If and only if $\mu_{I_2}(z) = \mu_{I_1}(\frac{z}{k})$, $\forall z \in X$.

Definition 2.10.3.2:-

Let I_1, I_2 be two positive fuzzy numbers. Then fuzzy multiplication of I_1, I_2 is defined by $I_1(\bullet)I_2 = I_3$ where

$\mu_{I_3}(z) = \sup_{x \bullet y = z}\{\min(\mu_{I_1}(x), \mu_{I_2}(y)\}$

If and only if $I_{3_\alpha} = I_{1_\alpha} \bullet I_{2_\alpha} = \left[a_\alpha^L \bullet b_\alpha^L, a_\alpha^R \bullet b_\alpha^R\right], \forall \alpha \in (0,1]$

is called fuzzy multiplication in R^+ or N .

Definition 2.10.3.3:-

Let I_1, I_2 be two negative fuzzy numbers. Then fuzzy multiplication of I_1, I_2 is defined by $I_1(\bullet)I_2 = I_3$ If and only if

$I_{3_\alpha} = I_{1_\alpha} \bullet I_{2_\alpha} = \left[a_\alpha^R \bullet b_\alpha^R, a_\alpha^L \bullet b_\alpha^L\right] \forall \alpha \in (0,1]$

(Is called fuzzy multiplication in R^- or N^-)

Definition 2.10.3.4:-

1. Let I_1 be positive and I_2 is negative fuzzy numbers. Then fuzzy multiplication of I_1, I_2 is Defined by $I_1(\bullet)I_2 = I_3$ If and only if

   $I_{3_\alpha} = I_{1_\alpha} \bullet I_{2_\alpha} = \left[a_\alpha^L \bullet b_\alpha^R, a_\alpha^R \bullet b_\alpha^L\right] \forall \alpha \in (0,1]$

2. let I_1 is negative and I_2 is positive fuzzy numbers , then

   $I_{3_\alpha} = I_{1_\alpha} \bullet I_{2_\alpha} = \left[a_\alpha^L \bullet b_\alpha^R, a_\alpha^R \bullet b_\alpha^L\right] \forall \alpha \in (0,1]$

(1, 2 is called fuzzy multiplication in R and Z)

Corollary 2.10.3.1:-

1. (.) is commutative and associative.
2. there is an identity "1" for fuzzy multiplication :
   $I \bullet 1 = I = 1 \bullet I, \forall I \in I_r$.
3. $((-)I_1)(\bullet)I_2 = (-)(I_1(\bullet)I_2) = I_1(\bullet)((-)I_2)$.
4. inverse ,in general , does not exist :
   $I \bullet I^{-1} \neq 1$.
5. $(\bullet)$ is distributive if I_2, I_3 with same sign. both positive , or negative as below: $I_1(\bullet)(I_2(+)I_3) = (I_1(\bullet)I_2)(+)(I_1(\bullet)I_3)$

2.10.4 Fuzzy Division:-

Definition 2.10.4.1:-

Let I_1 and I_2 be two fuzzy numbers either have the same sign, both negative, positive or have different sign one positive and the other negative. Fuzzy division is defined by: $I_3 = I_1(/)I_2$ Where

$$\mu_{I_3}(z) = \sup_{x/y=z}\{\min(\mu_{I_1}(x), \mu_{I_2}(y))\}$$

$$= \sup_{x \bullet y = z}\left\{\min(\mu_{I_1}(x), \mu_{I_2}(\frac{1}{y}))\right\} = \sup_{x \bullet y = z}\{\min(\mu_{I_1}(x), \mu_{I_2}{}^{-1}(y))\}$$

If and only if $I_{3\alpha} = I_{1\alpha} / I_{2\alpha} = \left[a_\alpha{}^L / b_\alpha{}^R, a_\alpha{}^R / b_\alpha{}^L\right] \forall \alpha \in (0,1]$

2.10.5 Fuzzy Max and Fuzzy Min:-

Definition 2.10.5.1:-

Let I_1 and I_2 be two fuzzy numbers.

with $I_{1\alpha} = \left[a_\alpha{}^L, a_\alpha{}^R\right], I_{2\alpha} = \left[b_\alpha{}^L, b_\alpha{}^R\right]$ then fuzzy max is defined by: $I_3 = \max\{I_1, I_2\} = I_1 \vee I_2$ Where

$$\mu_{I_3}(z) = \sup_{x \vee y = z} \{\min(\mu_{I_1}(x), \mu_{I_2}(y))\}$$

If and only if $I_{3_\alpha} = \left[a_\alpha^{\ L} \vee b_\alpha^{\ L}, a_\alpha^{\ R} \vee b_\alpha^{\ R}\right]$.

And for fuzzy min:

$I_3 = \min\{I_1, I_2\} = I_1 \wedge I_2$, where

$$\mu_{I_3}(z) = \sup_{x \wedge y = z} \{\min(\mu_{I_1}(x), \mu_{I_2}(y))\}$$

If and only if $I_{3_\alpha} = \left[a_\alpha^{\ L} \wedge b_\alpha^{\ L}, a_\alpha^{\ R} \wedge b_\alpha^{\ R}\right]$.

Corollary 2.10.5.1:-

1. $(\vee)$ and $(\wedge)$ are commutative and associative.
2. distributive : $I_1(\wedge)[I_2(\vee)I_3] = (I_1(\wedge)I_2)(\vee)(I_1(\wedge)I_3)$

   $I_1(\vee)[I_2(\wedge)I_3] = (I_1(\vee)I_2)(\wedge)(I_1(\vee)I_3)$.
3. Absorption $I_1(\vee)[I_1(\wedge)I_2] = I_1$, $I_1(\wedge)[I_1(\vee)I_2] = I_1$.
4. de Morgan's law $1-[I_1(\wedge)I_2] = [1(-)I_1](\vee)[1(-)I_2]$

   $1-[I_1(\vee)I_2] = [1(-)I_1](\wedge)[1(-)I_2]$
5. Idempotence $I_1(\vee)I_1 = I_1 = I_1(\wedge)I_1$
6. $I_1(+)[I_2(\vee)I_3] = (I_1(+)I_2)(\vee)(I_1(+)I_3)$
7. $[I_1(\vee)I_2](+)[I_1(\wedge)I_2] = (I_1(+)I_2)$

Remark 2.10.5.1:-

The results from multiplication or division of triangular fuzzy numbers are not necessarily triangular fuzzy numbers. And also multiplication or divisions for trapezoidal fuzzy numbers are not necessarily trapezoidal fuzzy numbers.

2.11 Examples on chapter 2:-

2.11.1 Example on The extension principle

Let $f(x)=x^2$ and let I be a triangular fuzzy number with membership function $I(x)=\begin{cases}1-|a-x|/\alpha & \text{if } |a-x|\le\alpha \\ 0 & \text{otherwise}\end{cases}$

Then using the extension principle we get

$$f(I)(y)=\begin{cases}I(\sqrt{y}) & \text{if } \; y\ge 0 \\ 0 & \text{otherwise}\end{cases}$$

That is $f(I)(y)=\begin{cases}1-\left|a-\sqrt{y}\right|/\alpha & \text{if } \; \left|a-\sqrt{y}\right|\le\alpha \text{ and } y\ge 0 \\ 0 & \text{otherwise}\end{cases}$

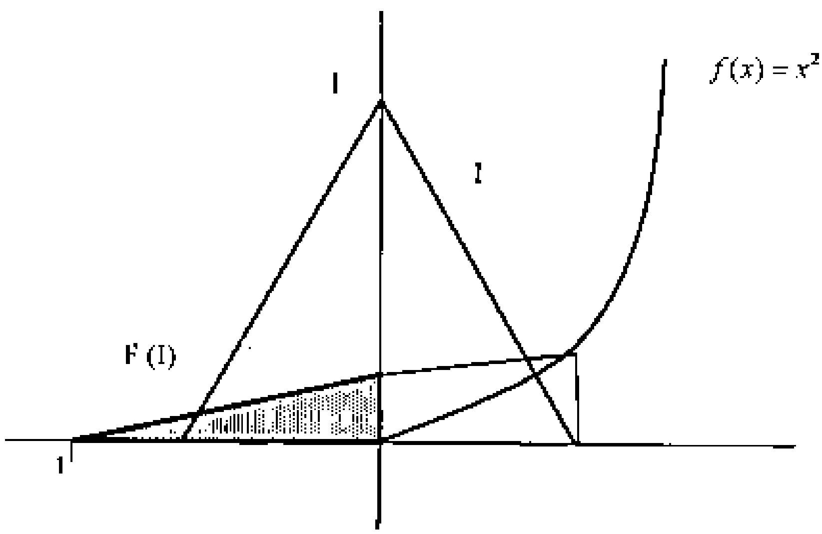

Figure 10

2.11.2 Example on convex fuzzy sets:-

Let F be a fuzzy set whose membership function is defined by

$$\mu_F(x)=\begin{cases}0 & , \text{ if } \; x<0 \\ e^{-x} & , \text{ if } \; x\ge 0\end{cases}$$

Assume $x_2 \geq x_1$.

Three possible cases are considered below:

1. for $x_1, x_2 < 0$, trivially the condition of equation

$\mu_F[\lambda x_1 + (1-\lambda)x_2] \geq \min[\mu_F(x_1), \mu_F(x_2)]$ is satisfied.

2. for $x_1 < 0$ but $x_2 > 0$ two sub cases arise:

i) $x_1 \leq \lambda x_1 + (1-\lambda)x_2 < 0$

ii) $0 \leq \lambda x_1 + (1-\lambda)x_2 \leq x_2$

Since $x_1 \leq \lambda x_1 + (1-\lambda)x_2 \leq x_2$, for the (i) case equation $\mu_F[\lambda x_1 + (1-\lambda)x_2] \geq \min[\mu_F(x_1), \mu_F(x_2)]$ is satisfied as for the (ii) is also satisfied ,since

$\mu_F[\lambda x_1 + (1-\lambda)x_2] \geq e^{-x_2} > 0 = \min[\mu_F(x_1), \mu_F(x_2)]$

3. for $x_1, x_2 \geq 0$

$\mu_F[\lambda x_1 + (1-\lambda)x_2] = \exp[-[\lambda x_1 + (1-\lambda)x_2]] \geq \exp[-[\lambda x_2 + (1-\lambda)x_2]] =$

$\exp[-[x_2]] = \min[\mu_F(x_1), \mu_F(x_2)]$

since $\mu_F(x_1) \geq \mu_F(x_2)$. Hence F is convex fuzzy set.

2.11.2 Example on fuzzy number:-

Let $I = (a1, a2, a3)$ be triangular fuzzy number where the membership function given as following :-

$$\mu_I(x) = \begin{cases} 0 & \text{if} \quad x < a1 \\ \dfrac{x - a1}{a2 - a1} & \text{if} \quad a1 \leq x \leq a2 \\ \dfrac{a3 - x}{a3 - a2} & \text{if} \quad a2 \leq x \leq a3 \\ 0 & \text{if} \quad x > a3 \end{cases}$$

We can obtained I_α as follows : $\forall \alpha \in (0,1]$, from

$\frac{a_1^L - a_1}{a_2 - a_1} = \alpha$, $\frac{a_3 - a_3^R}{a_3 - a_2} = \alpha$ we get $a_1^L = (a_2 - a_1)\alpha + a_1$,

$a_1^R = -(a_3 - a_2)\alpha + a_3$ thus

$$I_\alpha = \left[a_1^L, a_3^R\right] = \left[(a_2 - a_1)\alpha + a_1, -(a_3 - a_2)\alpha + a_3\right]$$

In case of the triangular fuzzy number $I_1 = (-5, -1, 1)$ the membership function value will be

$$\mu_{I_1}(x) = \begin{cases} 0 & \text{if} \quad x < -5 \\ \frac{x+5}{4} & \text{if} \quad -5 \le x \le -1 \\ \frac{1-x}{2} & \text{if} \quad -1 \le x \le 1 \\ 0 & \text{if} \quad x > 1 \end{cases}$$

α-cut interval from this fuzzy number is

$(\frac{x+5}{4} = \alpha) \Rightarrow (x = 4\alpha - 5)$ $(\frac{1-x}{2} = \alpha) \Rightarrow (x = -2\alpha + 1)$.

$I_{1\alpha} = \left[a_1^L, a_3^R\right] = [4\alpha - 5, -2\alpha + 1]$ if $\alpha = 0.5$, substituting 0.5, we get $I_{1(0.5)} = [-3, 0]$.

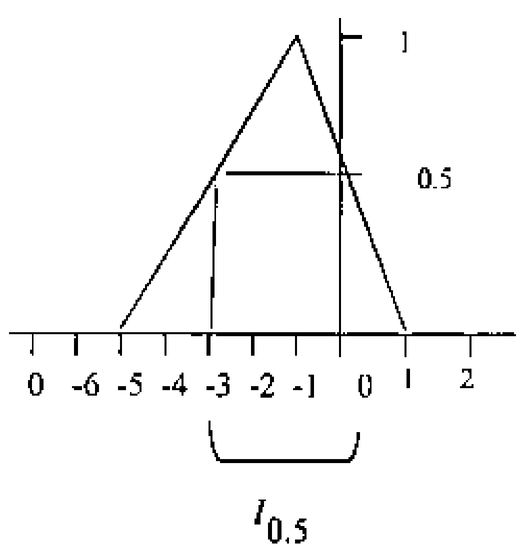

Figure 11

2.11.3 Example on fuzzy arithmetic:-

1 .addition

Let $I_1 \approx 2$ and $I_2 \approx 8$ be fuzzy numbers where

$$\mu_2(x)=\begin{cases}0 & \text{if } x\le 0 \text{ or } x>4\\ \frac{x}{2} & \text{if } 0<x\le 2\\ \frac{4-x}{2} & \text{if } 2<x\le 4\end{cases}$$

$$\mu_8(y)=\begin{cases}0 & \text{if } y\le 3 \text{ or } y>11\\ \frac{y-3}{5} & \text{if } 3<y\le 8\\ \frac{11-y}{3} & \text{if } 8<y\le 11\end{cases}$$

Using $\alpha-cuts$, for $\alpha\in(0,1]$

$\mu_2(x)\ge\alpha \Rightarrow \frac{x}{2}\ge\alpha$ And $\frac{4-x}{2}\ge\alpha \Rightarrow 2\alpha\le x\le 4-2\alpha$.

Therefore $2_\alpha=[2\alpha,4-2\alpha]$.

$\mu_8(y)\ge\alpha$

$\Rightarrow \frac{y-3}{5}\ge\alpha$ and $\frac{11-y}{3}\ge\alpha \Rightarrow 3+5\alpha\le y\le 11-3\alpha$.

Therefore $8_\alpha=[3+5\alpha,11-3\alpha]$ then, we have $Z=I_1+I_2$ with

$Z_\alpha=[3+7\alpha,15-5\alpha]\equiv[Z^L{}_\alpha,Z^R{}_\alpha]$,$\forall\alpha\in(0,1]$

$$3+7\alpha=Z^L{}_\alpha \Rightarrow \frac{Z^L{}_\alpha-3}{7}=\alpha$$

$$15-5\alpha=Z^R{}_\alpha \Rightarrow \frac{15-Z^R{}_\alpha}{5}=\alpha$$

Therefore

$$\mu_z(x) = \begin{cases} 0 & \text{if} \quad z \le 3 \text{ or } z > 15 \\ \dfrac{z-3}{7} & \text{if} \quad 3 < z \le 10 \\ \dfrac{15-z}{5} & \text{if} \quad 10 < z \le 15 \end{cases}$$

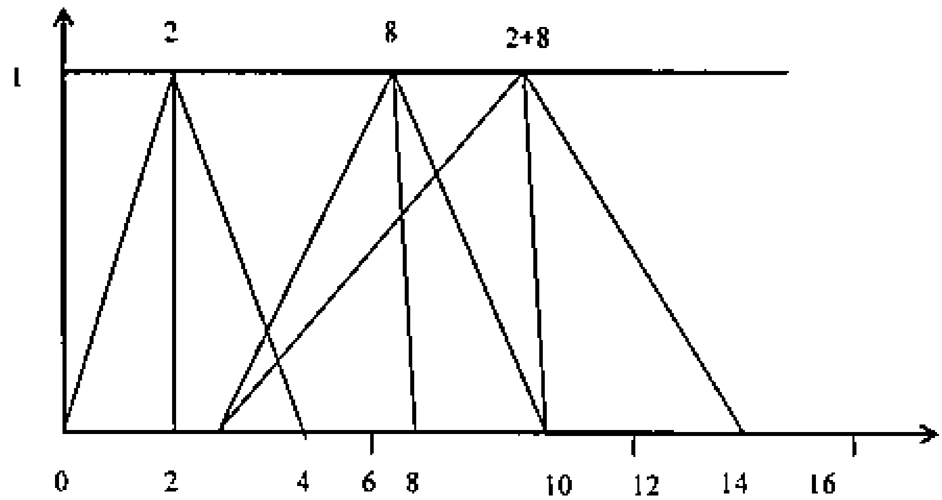

Figure 12

2 .subtraction :

For each $\alpha \in (0,1]$ we have $2_\alpha = [2\alpha, 4-2\alpha]$, $8_\alpha = [3+5\alpha, 11-3\alpha]$.

Therefore, the α-level after subtraction is

$Z_\alpha = [-11+5\alpha, 1-7\alpha] = [Z^L{}_\alpha, Z^R{}_\alpha]$, $\forall \alpha \in (0,1]$.

$$5\alpha - 11 = Z^L{}_\alpha \Rightarrow \alpha = \frac{Z_\alpha{}^L + 11}{5}$$

because $1 - 7\alpha = Z_\alpha{}^R \Rightarrow \alpha = \dfrac{1 - Z_\alpha{}^R}{7}$ the membership function of Z is

$$\mu_z(z) = \begin{cases} 0 \quad \text{if} & z \le -11 \text{ or } z > 1 \\ \dfrac{z+11}{5} & \text{if } -11 < z \le -6 \\ \dfrac{1-z}{7} & \text{if} \quad -6 < z \le 1 \end{cases}$$

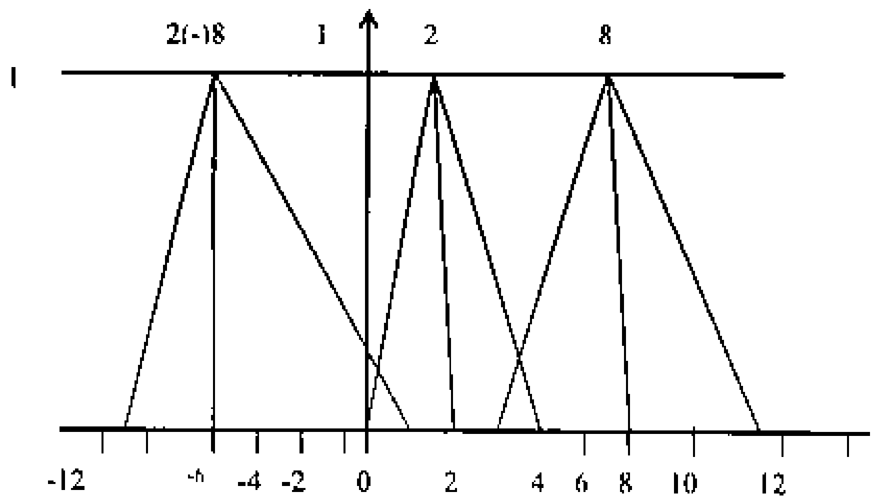

Figure 13

3 .multiplication

Again, we first have $2_\alpha = [2\alpha, 4-2\alpha]$, $8_\alpha = [3+5\alpha, 11-3\alpha]$.

Then because $Z_\alpha = 2_\alpha \cdot 8_\alpha = [10\alpha^2 + 6\alpha, 6\alpha^2 - 34\alpha + 44]$.

We have $z^L{}_\alpha = 10\alpha^2 + 6\alpha \Rightarrow \alpha = \dfrac{-6+\sqrt{36+40Z^L{}_\alpha}}{20}$.

And $Z^R{}_\alpha = 6\alpha^2 - 34\alpha + 44 \Rightarrow \alpha = \dfrac{34-\sqrt{100+24Z_\alpha{}^R}}{12}$.

Therefore $\mu_z(z) = \begin{cases} 0 & \text{if } \quad z \le 0 \text{ or } z > 44 \\ \dfrac{-6+\sqrt{36+40z^L{}_\alpha}}{20} & \text{if } \quad 0 < z \le 16 \\ \dfrac{34-\sqrt{100+24Z_\alpha{}^R}}{12} & \text{if } \quad 16 < z \le 44 \end{cases}$

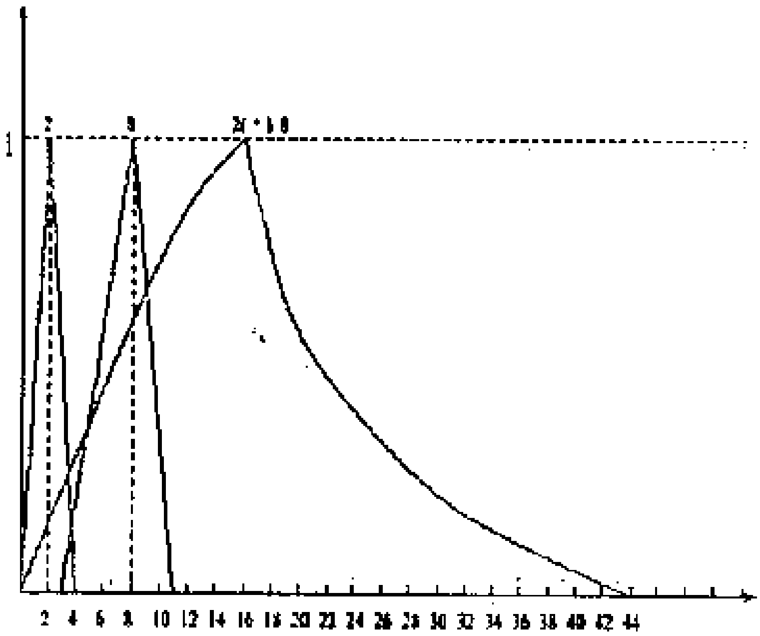

Figure14

4 .division :

For $2_\alpha = [2\alpha, 4-2\alpha]$ and $8_\alpha = [3+5\alpha, 11-3\alpha]$, we have

$$Z_\alpha = 2_\alpha / 8_\alpha = \left[\frac{2\alpha}{11-3\alpha}, \frac{4-2\alpha}{3+5\alpha}\right] \text{ where } Z^L{}_\alpha = \frac{2\alpha}{11-3\alpha} \Rightarrow \alpha = \frac{11Z_\alpha{}^L}{3Z^L{}_\alpha + 2}$$

$$Z_\alpha{}^R = \frac{4-2\alpha}{3+5\alpha} \Rightarrow \alpha = \frac{4-3Z^R{}_\alpha}{2+5Z^R{}_\alpha}$$ therefore ,we have

$$\mu_Z(z) = \begin{cases} 0 & \text{if} \quad z \le 0 \text{ or } z > \frac{4}{3} \\ \frac{11z}{3z+2} & \text{if} \quad 0 < z \le \frac{1}{4} \\ \frac{4-3z}{2+5z} & \text{if} \quad \frac{1}{4} < z \le \frac{4}{3} \end{cases}$$

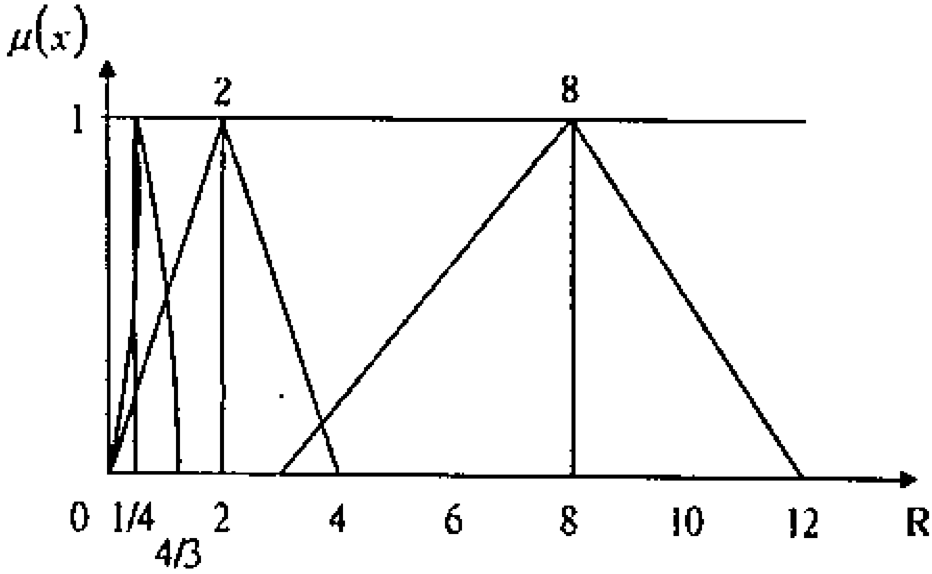

Figure 15

CHAPTER 3

3.1 Fuzzy logic:-

Formal language is a language in which the syntax is precisely given and thus is different from informal language like English and French. The study of the formal languages is the content of mathematics known as mathematical logic. The mathematical logic is called classical logic in this chapter. The classical logic considers the binary logic which consists of truth and false. The fuzzy logic is generalization of the classical logic and deals with the ambiguity in the logic.

The material in this chapter taken from the following references [31],[16],[20],[33],[11],[13],[1],[10],[24],[22],[19],[14],[21],[7],[23],[17],[6] ,[29],[4],[5].

to develop an n-valued logic, with $2 \le n \le \infty$. Such that it is isomorphic to the fuzzy set theory in the some way as the two-valued logic is isomorphic to classical set theory.

Zadeh modified the lukasiewicz logic and established an infinite-valued logic. By defining the following primary logic operations:

$$\sim a = 1 - a$$
$$a \wedge b = \min\{a, b\}$$
$$a \vee b = \max\{a, b\}$$
$$a \Rightarrow b = \min\{1, 1 + b - a\}$$
$$a \Leftrightarrow b = 1 - |a - b|$$

It has been shown, in logic theory that all these logical operations become the same as those for the two -valued logic when n=2. And also when n=3. More importantly, when n=∞, this logic dose not restrict the truth values to be rational. They can be any real numbers in [0, 1]. It has also been shown that this infinite -valued logic is isomorphic to the fuzzy set theory that employs the min, max and (1-a) operations for fuzzy set intersection, union, and complement, respectively.

Thus the fuzzy logic is allowing truth values to be any number in the interval [0, 1]. If p is atomic proposition, then will now let t (P) denoted the truth of p. so t (p) ∈ [0, 1]. For any proposition in fuzzy logic t (p) =1 means that p is absolutely true, t (P) =0 is that p is absolutely false and t (P) =0.65 just means that the truth of p is 0.65. Thus in real world propositions are often only partly true. It is hard to characterize the truth of "Ali is old" as unambiguously true or false. If Ali is 60 years old. In some respect he is old; being eligible for senior citizen benefits at many establishments, but in other respects is not old since he is not eligible for social security. So in fuzzy logic we would allow t (Ali is old) to take values in $[0,1]$.

To describe fuzzy logic mathematically. We introduce the following concepts and notation. Let X be universe set and F be fuzzy set associated with a membership function, $\mu_F(x), x \in X$,if $y = \mu_F(x_0)$ is a point in $[0,1]$ representing the truth value of the proposition "x_0 is a ", or simply "a", then the truth value of " not a" is given by

$\sim y = \mu_F(x_0 \text{ is not a}) = 1 - \mu_F(x_0 \text{ is a}) = 1 - \mu_F(x_0) = 1 - y$

Consequently, for n members $x_1,\ldots\ldots\ldots,x_n$ is in X with n corresponding truth values $y_i = \mu_F(x_i)$ in $[0,1]$, $i = 1,\ldots\ldots\ldots,n$ by applying the extension principle. The truth values of "not a " is defined as $\sim y_i = 1 - y$, $i = 1,\ldots\ldots\ldots,n$.

Here, we note that when n=∞ is allowed to define the logical operations and, or, not, implication, and equivalence as follow:

for any $a,b \in X$. $\mu_F(a \wedge b) = \mu_F(a) \wedge \mu_F(b) = \min\{\mu_F(a), \mu_F(b)\}$.

$\mu_F(a \vee b) = \mu_F(a) \vee \mu_F(b) = \max\{\mu_F(a), \mu_F(b)\}$

$\mu_F(\sim a) = 1 - \mu_F(a)$

$\mu_F(a \Rightarrow b) = \mu_F(a) \Rightarrow \mu_F(b) = \min\{1, 1 + \mu_F(b) - \mu_F(a)\}$

$\mu_F(a \Leftrightarrow b) = \mu_F(a) \Leftrightarrow \mu_F(b) = 1 - |\mu_F(a) - \mu_F(b)|.$

For multi-point cases, e.g. $a_i, b_j \in X$ with $\mu_F(a_i), \mu_F(b_j) \in [0,1]$ $i = 1,........., n$,

$j = 1,........., m$ where $1 \le n$, $m \le \infty$, we can define

$$\mu_F(a_1,.........., a_n) \wedge \mu_F(b_1,..........b_m) = \max_{1 \le i \le n, 1 \le j \le m}\{\min\{\mu_F(a_i), \mu_F(b_j)\}\}.$$

Remark 3.1.1:-

- Fuzzy propositional calculus generalizes classical propositional calculus by using the truth set $[0,1]$ instead of $\{0,1\}$
- The set of building blocks is a set V of symbols representing atomic or elementary propositions.
- The set of formulas F is built up from V using the logical connectives $\wedge, \vee, \sim$.
- As in the two -valued and three-valued propositional calculus, a truth evaluation is gotten by taking any function $t: V \to [0,1]$

and extending it to a function $t: V \to [0,1]$ by replacing each element $a \in V$ which appears in the formula by its value $t(a)$ which is an element in $[0,1]$.

- this give an expression in elements of $[0,1]$ and the connectives $\wedge, \vee, \sim$.
- this expression is evaluated by letting $x \vee y = \max\{x, y\}$ $x \wedge y = \min\{x, y\}$, $\sim x = 1 - x$.

- For elements x and y in $[0,1]$. We get an equivalence relation on F by letting two formulas be equivalent if they have the same truth evaluation for all t.
- A formula is a tautology if it always has truth value 1.
- Two formulas p and q are logically equivalent when $t(p) = t(q)$ For all truth valuations t.
- As in three-valued logic the law of the excluded middle and the law of contradiction fail.

3.1.1 Fuzzy Expression:-

In fuzzy expression (formula), a fuzzy proposition can have its truth value in the interval $[0,1]$ the fuzzy expression function is a mapping function from $[0,1]$ to $[0,1]$, $F:[0,1] \to [0,1]$

If we generalize the domain in n-dimension, the function becomes as Follows:

$F:[0,1]^n \to [0,1]$

Therefore we can interpret the fuzzy expression as n-ray relation from n fuzzy sets to $[0,1]$. In the fuzzy logic, the operations such as negation (~), conjunction ($\wedge$) and disjunction ($\vee$) are used as in the classical logic.

Definition 3.1.1.1:-

The fuzzy logic is a logic represented by the fuzzy expression (formula) which satisfies the followings.

1. Truth values, 0 and 1, and variable x_i ($\in [0,1], i = 1,2,\ldots\ldots,n$) are fuzzy expressions.
2. If F is a fuzzy expression, ~ F is also fuzzy expression.

3. If F and H are fuzzy expressions, $F \wedge H$ and $F \vee H$ are also fuzzy expressions.

3.1.1.1 Operation on Fuzzy Expression:-

There are some operators in the fuzzy expression such as (~) negation, ($\wedge$) conjunction, ($\vee$) disjunction, and ($\Rightarrow$) implication. However the meaning of operators may be different according to the literature. If we follow lukasiewicz's definition, the operators are defined as follows for $a,b \in [0,1]$.

1. negation $\sim a = 1 - a$
2. Conjunction $a \wedge b = \min(a,b)$.
3. disjunction $a \vee b = \max(a,b)$
4. implication $a \Rightarrow b = \min(1, 1+b-a)$

The properties of fuzzy operators are following:-

(1) involution	~(~a)=a
(2) commutativity	a ∧b=b ∧a , a ∨b=b ∨a
(3) associativity	(a ∧b) ∧c=a ∧ (b∧ c) , (a ∨b) ∨c=a ∨ (b ∨c)
(4) distributivity	a ∨ (b ∧c)=(a ∨b) ∧ (a ∨c) a ∧ (b ∨c)=(a ∧b) ∨ (a ∧c)
(5) idempotency	a ∧a=a, a ∨a=a
(6) Absorption	a ∨ (a ∧b)=a, a ∧ (a ∨b)=a
(7) Absorption by 0 and 1	a ∧0=0, a ∨1=1
(8) identity	a ∧1=a , a ∨0=a
(9) de Morgan's law	~(a ∧b)=~a ∨~b, ~(a ∨b)=~a ∧~b

Table 1

Example 3.1.1.1.1:-

We can see that the two properties are not satisfied in the following examples:-

1. law of contradiction

Assume (a) is in $[0,1]$. $a \wedge \sim a = \text{Min}[a, \sim a] =$

$$\text{Min}\,[a, 1-a] = \begin{pmatrix} a & \text{if} \quad 0 \le a \le 0.5 \\ 1-a & \text{if} \quad 0.5 \le a < 1 \end{pmatrix}$$

Thus $0 < a \wedge \sim a \le 0.5$ then $a \wedge \sim a \ne 0$

2. law of excluded middle

Suppose (a) is in $[0,1]$.

$$a \vee \sim a = \max[0,1] = \max[a, 1-a] = \begin{Bmatrix} a & \text{if} \quad 0.5 \le a < 1 \\ 1-a & \text{if}\ 0 < a \le 0.5 \end{Bmatrix}$$

Thus $0.5 \le a \vee \sim a < 1$

Then $a \vee \sim a = 1$ if $a = 0$ or 1
$a \vee \sim a < 1$ otherwise

Example 3.1.1.1.2:-

Let a=1, b=0

1. $\sim a = 0$
2. $a \wedge b = Min(1,0) = 0$
3. $a \vee b = \max(1,0) = 1$
4. $a \Rightarrow b = Min(1, 1-1+0) = 0$

3.1.2 Basic Connectives:-

Consider a piece of information of the form" if (x is A and y is not B), then (z is c or z is d)". An approach to the translation of this type of knowledge is model it as fuzzy sets. To translate completely the sentence above, we need to model the connectives "and ", "or", and "not", as well as the conditional "if ...then ...".This combining of evidence, or "data fusion", is essential in building expert systems, or in synthesizing controllers. But the connectives experts use are domain dependent they very from field to field. The connectives used in data fusion in medical science are different from those in geophysics ,so there are many ways to model these connectives. The search for appropriate models for "and" has led to a class of connectives called "T-norms". Similarly, for modeling "or" there is a class called "t-conorms". We will investigate ways for modeling basic connectives used in combining knowledge that comes in the form of fuzzy sets. These models may be viewed as extensions of the analogous connectives in classical two-valued logic.A model is obtained for each choice of such extensions, and one concern is with isomorphisms between the algebraic systems that arise.

3.1.2.1 T-Norms:-

Consider first the connective "and". When A and B are crisp subsets of a set U, then the table

A / B	0	1
0	0	0
1	0	1

Table 2

Gives the truth evolution of "A and B" in terms of the possible truth values 0 and 1 of A and B. The table just species a map $\wedge:\{0,1\}\times\{0,1\}\to\{0,1\}$.when A and B are fuzzy subsets of U, truth values are the members of the interval [0, 1]. And we need to extend this map to a map $\wedge:[0,1]\times[0,1]\to[0,1]$ one such extension is given by $x\wedge y=\min\{x,y\}$ This mapping does agree with the table above when x and y belong to {0, 1}. We make the following observations about $x\wedge y=\min\{x,y\}$:

1. 1 acts as an identity. That is , $1\wedge x=x$.
2. $\wedge$ is commutative. That is , $x\wedge y=y\wedge x$.
3. $\wedge$ is associative. That is , $x\wedge(y\wedge z)=(x\wedge y)\wedge z$.
4. $\wedge$ is increasing in each argument. that is if $v\le w$ and $x\le y$ then $v\wedge x\le w\wedge y$
5. note that $\wedge$ is idempotent that is $x\wedge x=x$

Any binary operation

$T:[0,1]\times[0,1]\to[0,1]$

satisfying these properties is a candidate for modeling the connective "and" in the fuzzy setting. They were termed "triangular norms", or "T-norms" for short.

Remark 3.1.2.1.1:-

We will use these T-norms as a family of possible connectives for fuzzy intersection. Now T-norms are binary operations on [0, 1] and a common practice is to denote them by $T(x,y)$.

Definition 3.1.2.1.1:-

A binary operation $T:[0,1]\times[0,1]\to[0,1]$ is a T-norm if it satisfies the following:-

$\forall\, x,y,z,w\in[0,1]$

1. $T(x,0)=0,\ T(x,1)=x$ boundary condition
2. $T(x,y)=T(y,x)$ commutativity

3. $T(T(x,y),z)=T(x,T(y,z))$ associativity

4. if $w\le x$ and $y\le z$ then $T(w,y)\le T(x,z)$

Monotonicity (is increasing in each argument)

The first, second, and fourth conditions give $T(0,x)\le T(0,1)=0$

We have the following examples.

Example 3.1.2.1.1:-

1. $T_0(x,y)=\begin{cases} x\wedge y & \text{if } x\vee y=1 \\ 0 & \text{otherwise} \end{cases}$

2. $T_1(x,y)=0\vee(x+y-1)$

3. $T_2(x,y)=\dfrac{xy}{2-(x+y-xy)}$

4. $T_3(x,y)=xy$

5. $T_4(x,y)=\dfrac{xy}{x+y-xy}$

6. $T_5(x,y)=x\wedge y$

Proposition 3.1.2.1.1:-

If T is a T-norm, then for $x,y\in[0,1]$, then $T_0(x,y)\le T(x,y)\le T_5(x,y)$

Proof:-

We now from above that $T_0(x,y)=\begin{cases} x\wedge y & \text{if } x\vee y=1 \\ 0 & \text{otherwise} \end{cases}$

And $T_0(x,y)=0$ unless $x\vee y=1$. Now $T(x,y)=x\wedge y$ if $x\vee y=1$, so $T_0\le T$.

Since $T(x,y)\le x\wedge y$, so $T(x,y)\le T_5(x,y)$.

Remark 3.1.2.1.2:-

The T-norm $\wedge$ is the only idempotent one, that is the only T-norm such that $T(x,x)=x$ for all x.

And for any T-norm $T(x,x)=x$ is never great then x.

Thus we find that it's not necessary that $T=\wedge$

There will be an element x such that $T(x,x)<x$..

Definition 3.1.2.1.2:-

A T-norm T is convex if when every $T(x,y) \le c \le T(x_1,y_1)$ then there is an (r) between x and x_1 and an (s) between y and y_1 such that $c = T(r,s)$.

Remark 3.1.2.1.3:-

For T-norms, the condition of convexity is equivalent to continuity. We refer to the condition as convex.

Corollary 3.1.2.1.1:-

If a T-norm T is convex and $a < b$ then there is $c \in [a,b]$ such that $a = T(b,c)$.

Proof:-

$T(a,b) \le T(1,a) = a < T(1,b)$, so by convexity, there is such c.

Definition 3.1.2.1.3:-

A T-norm T is Archimedean if it is convex, and for each $a,b \in (0,1)$, there is apositive integer n such that $a^{[n]} = \underbrace{T(a,a,\ldots\ldots,a)}_{\text{n times}} < b$

In general we will write $T(a,a) = a^{[2]}$ and $T(a,a,a) = a^{[3]}$, so on.
We use $a^{[n]}$ instead of a^n for this T-norm power to distinguish it from a multiplied by itself n times.
The examples T_1, T_2, T_3 and T_4 are all Archimedean. For convex T-norms the condition for Archimedean simplifies, as the corollary to the following proposition.

Proposition 3.1.2.1.2:-

If a T-norm T Archimedean, then for $a,b \in (0,1)$, $T(a,b) < b$.

Proof:-

If T Archimedean, then for $a \in (0,1)$, clearly that $T(a,a) < a$ lest $a^{[n]} = a$ for all n.

if $a<b$, then $T(a,b)\le T(b,b)<b$.

1. If $a>b$, then $T(a,b)\le T(1,b)=b$, then $T(a^{[n]},b)=b$. For all n, but for sufficiently large n, $a^{[n]}\le b$. and $b=T(a^{[n]},b)\le T(b,b)$, an impassivity.

Corollary 3.1.2.1.2:-

The following are equivalent for a convex T-norm T.

1. T is Archimedean.
2. $T(a,a)<a$ for all $a\in(0,1)$.

Proof:-

Archimedean clearly implies the second condition. Assume that $T(a,a)<a$,for all $a\in(0,1)$,and let $b\in(0,1)$.

Then $\wedge_n a^{[n+1]}=\wedge_n a^{[n]},a=T(a,\wedge_n a^{[n]})$, whence $\wedge_n a^{[n]}=0$, and the corollary follows.

Definition 3.1.2.1.4:-

1. A T-norm T is nilpotent if for $a\ne 1, a^{[n]}=0$ for some positive integer n, the n depending on a .
2. A T-norm is strict if for $a\ne 0, a^{[n]}>0$ for every positive integer n.
3. An element $a\in(0,1)$ is called a zero divisor of T if there exit some $b\in(0,1)$ such that $T(a,b)=0$.

Example 3.1.2.1.2:-

1. for T_1 each $a\in(0,1)$ is both nilpotent element and zero divisor of T_1 As well as of the T_0
2. T_2 has neither nilpotent elements nor zero divisor .
3. T_3 has no nilpotent elements and no zero divisor also T_3 is strict.

Remark 3.1.2.1.4:-

1. If $a\in[0,1]$ is an idempotent element of a T-norm T then ,by induction we also have $a^{[n]}=a$ for all $n\in N$. In particular, this

means that no element of $(0,1)$ can be both idempotent and nilpotent.

2. Each nilpotent element a of a T-norm T is also a zero divisor of T (if $n>1$) is the smallest integer such that $a^{[n]}=0$ then $T(a,a^{[n-1]})=0$ with $a^{[n-1]}>0$, but not conversely.
3. if a T-norm T has a nilpotent element a then there is always an element $b\in(0,1)$ such that $b^{[2]}=0$, indeed, if $n>1$ is the smallest integer that $b^{[n]}=0$ then $b=a^{[n-1]}$ satisfies $b^{[2]}=0$.
4. If $a\in(0,1)$ is a nilpotent element (a zero divisor) of a T-norm T then each number $b\in(0,a)$ is also a nilpotent element (a zero divisor) of T.

Proposition 3.1.2.1.3:-

For each T-norm T the following are equivalent

1. T has zero divisor
2. T has nilpotent elements

Proof:-

If T has a zero divisor, i.e. if $T(a,b)=0$ for some $a>0$ and $b>0$, then for $c=\min(a,b)>0$ we obtain $T(c,c)=0$ showing that c is a nilpotent element of T. Conversely each nilpotent element a of T-norm T is also a zero divisor of T (if $n>1$ is the smallest integer such that $a^{[n]}=0$ then $T(a,a^{[n-1]})=0$ with $a^{[n-1]}>0$).

3.1.2.2 Negations:-

The complement of a fuzzy set F has been defined by $F'(x)=1-F(x)$. This the same as following: by the function $\Gamma:[0,1]\to[0,1]$ By $x\to 1-x$. This latter function is an involution of the lattice L=$([0,1],\leq)$.

That it is ordered reversing and appling it twice gives the identity map. In fuzzy set theory, such a map $\Gamma:[0,1]\to[0,1]$ is called a strong negation .A strong negation Γ satisfies:

1. $\Gamma(0)=1,\ \Gamma(1)=0$
2. Γ is non increasing.
3. $\Gamma(\Gamma(x))=x$

A map satisfying only the first two conditions is a negation. It is clear that there are many of them, any non increasing map that starts at 1 and goes to 0. Such simple maps as $\Gamma(x)=1$ if $x=1$ and =0 otherwise, are negations.

3.1.2.3 T-conorms:-

The notion of T-norm plays the role of intersection, or in logical terms," and" .the dual of that notion is that of union, "or". In the case of sets, union and intersection are related via complements. The well-know de Morgan formulas do that, they are $A\cup B=(A^c\cap B^c)^c$ $A\cap B=(A^c\cup B^c)^c$. But in fuzzy setting we have many "complements". A negation plays such a role. For a binary operation T on $[0,1]$,we can define its dual with respect to any negation Γ namely $C(x,y)=\Gamma(T(\Gamma(x),\Gamma(y))$. Thus the last equation holds if and only if $T(x,y)=\Gamma(C(\Gamma(x),\Gamma(y))$ so if these equations hold , then we say that T And C are dual with respect to Γ .in the case T is T-norm then C is calledT-conorm is the dual of some T-norm with respect to some negation.

Definition 3.1.2.3.1:-

A binary operation C on $[0,1]$ i.e. a function $C:[0,1]^2\to[0,1]$ which for all $x,y,z,w\in[0,1]$. is aT-conorm if and only if satisfies the following condition.

1. $C(x,y)=C(y,x)$ (commutativity)
2. $C(x,C(y,z))=C(C(x,y),z)$ (associativity)
3. if $w\le x$ and $y\le z$ then $C(w,y)\le C(x,z)$ (Monotonicity)
4. $C(x,0)=x$, $C(x,1)=x$ (boundary condition)

Example 3.1.2.3.1:-

The following are the basic t-conorms

1. $C_5=\max(x,y)$ (maximum)
2. $C_3=x+y-x.y$ (probabilistic sum)
3. $C_1=\min(x+y,1)$ (Lukasiewicz t-conorm,bounded sum)
4. $C_0(x,y)=\begin{cases}1 & \text{if} \quad (x,y)\in(0,1]^2 \\ \max(x,y) & \text{otherwise}\end{cases}$ (drastic sum)

The following proposition gives properties characterizing t-conorms.

Proposition 3.1.2.3.1:-

A function $C:[0,1]^2\to[0,1]$ is aT-conorm if and only if there exists a T-norm T such that for all $(x,y)\in[0,1]^2$ $C(x,y)=\Gamma(T(\Gamma(x),\Gamma(y))$
or $C(x,y)=1-T(1-x,1-y)$ ... (*)

Proof:-

If T is a T-norm then obviously the operation C defined by (*) satisfies condition (1),(2),(3),(4) of T-norm and the condition (4) of T-conorm.And therefore a T-conorm.

On the other hand if C is aT-conorm then define the function $T:[0,1]^2\to[0,1]$ by $T(x,y)=1-C(1-x,1-y)$ ... (**)

Again, it is trivial to check that T is a T-norm and that (**) holds.

Remark 3.1.2.3.1:-

1. TheT-conorm given by (*) is called the dualT-conorm of$T$,and the T-norm given by (**) is said to be the dual T-norm of$C$.

2. The proof of proposition (3.1.2.3.1) makes it clear that also each T-norm is the dual operation of some T-conorm note that (T_5, C_5) (T_3, C_3), (T_1, C_1) and (T_0, C_0) are pairs of T-norm and T-conorms which are mutually dual to each other.

3. all T-conorms coincide on the boundary of $[0,1]^2$, as consequence of these additional boundary conditions which hold for all $x \in [0,1]$ $C(1,x) = C(x,1) = 1$ $C(0,x) = x$.

4. The duality changes the order if. For some T-norms T_1 and T_2 we have $T_1 \le T_2$ and if C_1 and C_2 are the dual T-conorms of T_1 and T_2 respectively. then we get $C_1 \ge C_2$,consequently, for each T-conorm C we have $C_5 \le C \le C_0$. i.e. the maximum C_5 is the Weakest and drastic sum C_0 is the strongest T-conorm. i.e.

$$C_5 < C_3 < C_1 < C$$

Definition 3.1.2.3.2:-

A T-conorm is Archimedean if it is dual to a T-norm that is Archimedean, is nilpotent if it is dual to a nilpotent T-norm, and is strict if it is dual to a strict T-norm. Thus for nilpotent T-conorm C if and only if $x \in (0,1]$, $x^{[n]} = 1$ for some n, where $x^{[n]}$ means x conformed to itself n times. And for archimedean property, for each $(x, y) \in (0,1)^2$ there such $n \in N$, $x^{[n]} > y$.

Definition 3.1.2.3.3:-

Let T be a T-norm and C be T-conorm, then we say that T is distributive over C if for all $x, y, z \in [0,1]$ $T(x, C(y,z)) = C(T(x,y), T(x,z))$ .And that C is distributive over T if for all $x, y, z \in [0,1]$. $C(x, T(y,z)) = T(C(x,y), C(x,z))$.

Remark 3.1.2.3.2:-

If T is distributive over C and C is distributive over T then (T, C) is called a distributive pair (of T-norm and T-conorms).

Proposition 3.1.2.3.2:-

Let T be a T-norm and C aT-conorm then we have

1. C is distributive over T if and only if $T = T_s$
2. T is distributive over C if and only if $C = C_s$
3. (T, C) is a distributive pair iff $T = T_s$ and $C = C_s$

Proof:-

Let $T = T_s$ obviously, eachT-conorm is distributive over T_s because

1. of Monotonicity condition (3) of the T-conorm. Conversely, if C is distributive over T then for all $x \in [0,1]$ we have

   $x = C(x, T(0,0)) = T(C(x,0), C(x,0)) = T(x,x)$,then for all $(x,y) \in [0,1]^2$

with $y \leq x$ the Monotonicity condition (4) of T-norm implies $y = T(y,y) \leq T(x,y) \leq T_s(x,y) = y$ which, together with condition (2), means $T = T_s$. Proof of (2) and (3) is just the combination of (1) and (2).

Remark 3.1.2.3.3:-

If T is aT-norm, C the dualT-conorm and if T is distributive over C (or C is distributive over T) then we necessarily have $T = T_s$ and $C = C_s$.

3.1.2.4 Fuzzy implications:-

In classical two –valued logic, the implication may be expressed on {0, 1} by the formula $(a \Rightarrow b)$ is binary operation on the truth values {0, 1}.In fuzzy logic, our set of truth values is [0, 1] and so material implication($\Rightarrow$) should be binary on [0, 1] .such operations should agree with the classical case for {0, 1}.

A fuzzy implication is a map $\Rightarrow:[0,1]\times[0,1]\rightarrow[0,1]$. Satisfying

$\Rightarrow$	0	1
0	1	1
1	0	1

Table 3

Example 3.1.2.4.1:-

Here some examples of a fuzzy implication.

1. $(x\Rightarrow y)=\begin{Bmatrix}1 & \text{if} & x\le y\\ 0 & \text{if} & x>y\end{Bmatrix}$

2. $(x\Rightarrow y)=((1-x+y)\wedge 1)$

3. $(x\Rightarrow y)=((1-x)\vee y)$

The class of all possible fuzzy implications consists of all functions $(\Rightarrow)$ defined on the unit square with the given values on the four corners. There are three basic constructions of fuzzy implications, They arise from three ways to express implication in the classical case. The following are equivalent for that case.

- $(x\Rightarrow y)=\vee\{z:x\wedge z\le y\}$
- $(x\Rightarrow y)=x^c\vee y$
- $(x\Rightarrow y)=x^c\vee(x\wedge y)$

These three conditions make sense on $[0,1]$ when a T-norm is used for $\wedge$, aT-conorm for $\vee$, and a negation for (complement) .We can give the following casa:-

Definition 3.1.2.4.1:-

An R-implication is a map $\Rightarrow:[0,1]\times[0,1]\rightarrow[0,1]$ of the form

$(x\Rightarrow y)=\vee\{z\in[0,1]:T(x,z)\le y\}$ Where T is T-norm .

Thus (R-implication : $x\Rightarrow y=\sup\{z\in[0,1]:T(x,z)\le y\}$) , typical examples of R-implication are the (Gödel and Gaines) implications

Definition 3.1.2.4.2:-

A T-norm implication is a map $\Rightarrow:[0,1]\times[0,1]\to[0,1]$ $(x\Rightarrow y)=T(x,y)$. Typical example of T-norm implication are the (Mamdani) ($x\Rightarrow y=\min\{x,y\}$) ,and (Larsen) (($x\Rightarrow y)=xy$) implications.

Definition 3.1.2.4.3:-

C-implications is a map $\Rightarrow:[0,1]\times[0,1]\to[0,1]$ of the form $x\Rightarrow y=C(\Gamma(x),y)$, Where C isT-conorm and Γ is a negation on $[0,1]$. Typical examples of C-implication are the (Lukasiewicz and Kleen-Dienes Implications.).

The most often used fuzzy implication operations are listed in the following table.

Name	Definition
Early zadeh	$(x\Rightarrow y)=\max\{1-x,\min(x,y)\}$
Lukasiewicz	$(x\Rightarrow y)=\min\{1,1-x+y\}$
Mamdani	$(x\Rightarrow y)=\min\{x,y\}$
Larsen	$(x\Rightarrow y)=xy$
Standard strict	$(x\Rightarrow y)=\begin{cases}1 & \text{if } x\le y\\ 0 & \text{otherwise}\end{cases}$
Gödel	$(x\Rightarrow y)=\begin{cases}1 & \text{if } x\le y\\ y & \text{otherwise}\end{cases}$
Gaines	$(x\Rightarrow y)=\begin{cases}1 & \text{if } x\le y\\ y/x & \text{otherwise}\end{cases}$
Kleen-Dienes	$(x\Rightarrow y)=\max\{1-x,y\}$
Kleen-Dienes-lukasiewicz	$(x\Rightarrow y)=1-x+xy$
Yager	$(x\Rightarrow y)=y^x$

Table 4

Remark 3.1.2.4.1:-

If T is any T-norm then the following calculations show that an R-implication is an implication.

Let T be any T-norm.

- $(1 \Rightarrow 0) = \vee\{z \in [0,1] : T(1,z) \le 0\} = 0$ since $T(1,z) = z$
- $(0 \Rightarrow 0) = \vee\{z \in [0,1] : T(0,z) \le 0\} = 1$ since $T(0,z) = 0$
- $(0 \Rightarrow 1) = \vee\{z \in [0,1] : T(0,z) \le 1\} = 1$ since $T(0,z) \le 1$
- $(1 \Rightarrow 1) = \vee\{z \in [0,1] : T(1,z) \le 1\} = 1$ since $T(1,z) \le 1$

For R-implications $\Rightarrow$,it is always the case that $(x \Rightarrow y) = 1$ for $x \le y$ since $T(x,1) = x \le y$.

3.2 Fuzzy Relations:-

Relations, or associations among objects, are of fundamental importance in the analysis of real-world systems. Mathematically ,the concept is a very general. There are many kinds of relations: order relations, equivalence relations and other relation with various important properties .relations are a ubiquitous in mathematics and their generalizations of fuzzy theory are important .in this section we present some of these generalizations with an emphasis on binary relations.especially fuzzy equivalence relations ,which generalize ordinary equivalence relations .To generalizing relations to fuzzy relations is easy .An n-ary relation is a subset R of the Cartesian product $X_1 \times X_2 \times \dots\dots \times X_n$ of (n) sets. The generalizing to fuzzy case is the natural one.

Definition 3.2.1:-

A crisp relation R represents that of from sets X to Y for $x \in X$ and $y \in Y$, its membership function $\chi_R(x,y)$ this membership function maps $X \times Y$ to set $\{0,1\}$, $\chi_R : X \times Y \to \{0,1\}$.

$$\chi_R(x,y) = \begin{cases} 1 & \text{iff } (x,y) \in R \\ 0 & \text{iff } (x,y) \notin R \end{cases}$$

We can extend the above definition to define the fuzzy relation.

Definition 3.2.2:-

Let X and Y be two universes. A binary fuzzy relation $\mu_R(x,y)$ or for short $R(X,Y)$ on $X \times Y$ is defined as

$R(X,Y) = \{\{(x,y), \mu_R(x,y)\} : (x,y) \in X \times Y\}$, where $\mu_R : X \times Y \to [0,1]$ is a grade of membership function. If $X = Y$. Then $R(X,Y)$ is called binary fuzzy relation on X .

Although an n-ary fuzzy relation on a product space $X = X_1 \times X_2 \times ....\times X_n$ may be defined by an n-variant membership function $\mu_R(x_1,......,x_n)$ in the similar way.

Remark 3.2.1:-

In the definition 3.2.1 if the universes X and Y are uncountable (continuous) we may have $R(X,Y) = \int_{x \times y} \mu_R(x,y)/(x,y)$ ..(*) is a binary fuzzy relation on $X \times Y$ .if X and Y are countable (discrete) universes, then $R(X,Y) = \sum_{x \times y} \mu_R(x,y)/(x,y)$. Also the integral symbol denotes the set of all tuples $\mu_R(x,y)/(x,y)$ on $X \times Y$. It is possible to express (*) with $R(X,Y) = \int_x \int_y \mu_R(x,y)/(x,y)$.

Example 3.2.1:-

Let $X = \{1,2,3\}$ then "approximately equal" is the binary fuzzy relation

$$1/(1,1) + 1/(2,2) + 1/(3,3) + 0.8/(1,2) + 0.8/(2,3) + 0.8/(2,1) + 0.8/(3,2) + 0.3/(1,3) + 0.3/(3,1)$$

The membership function $\mu_R(x,y)$ of this relation can be described by

$$\mu_R(x,y) = \begin{Bmatrix} 1 & \text{when } x = y \\ 0.8 & \text{when } |x-y| = 1 \\ 0.3 & \text{when } |x-y| = 2 \end{Bmatrix}$$

Example 3.2.2:-

Let $X = [0,250]$ be the interval of height of persons, and suppose that $\mu_R(x,y)$ represents the "much taller man". Where

$$\mu_R(x,y) = \begin{Bmatrix} 0 & \text{for } x-y \le 20 \\ \frac{x-y}{20} & \text{for } 0 < x-y < 20 \\ 1 & \text{for } x-y \ge 20 \end{Bmatrix}$$

Then $R(X,Y) = \int_{x \times y} \mu_R(x,y)/(x,y)$

Remark 3.2.2:-

A crisp relation R represents the relation from crisp set X to Y its domain and range can be defined as $dom(R) = \{x / x \in X, y \in Y, \mu_R(x,y) = 1\}$

$ran(R) = \{y / x \in X, y \in Y, \mu_R(x,y) = 1\}$

Definition 3.2.3:-

A fuzzy relation R defined on crisp set X and Y ,the domain and range of this relation are defined as $\mu_{dom(R)}(x) = \max_{y \in Y} \mu_R(x,y)$ and $\mu_{ran(R)}(y) = \max_{x \in X} \mu_R(x,y)$.

Definition 3.2.4:-

The inverse of a binary fuzzy relation $R(X,Y)$ on $X \times Y$ denoted by $R^{-1}(Y,X)$ is a relation on $Y \times X$ given by $\mu_{R^{-1}}(y,x) = \mu_R(x,y)$ for all $x \in X, y \in Y$.

Remark 3.2.3:-

- In definition 3.2.3 a set X becomes the support of $dom(R)$ and $dom(R) \subseteq X$. Set Y is the support of $ran(R)$ and $ran(R) \subseteq Y$
- Because fuzzy relations in general are fuzzy sets we can define the Cartesian product to be a relation between two or more fuzzy set.

Definition 3.2.5:-

Let F_1 be a fuzzy set on universe X and F_2 be fuzzy set on Y then the Cartesian product between fuzzy sets F_1 and F_2 will result in a fuzzy relation R which is contained with in the full Cartesian product space or $F_1 \times F_2 = R \subset X \times Y$. Where the fuzzy relation R has membership function

$\mu_R(x,y) = \mu_{F_1 \times F_2}(x,y) = \min(\mu_{F_1}(x), \mu_{F_2}(y))$, $(x,y) \in F_1 \times F_2$

Remark 3.2.4:-

- In the above definition $\mu_R(x,y) \le \mu_{F_1}(x)$ and $\mu_R(x,y) \le \mu_{F_2}(x)$ $\forall (x,y) \in X \times Y$.
- F_1, F_2 and $\mu_{F_1 \times F_2}$ together define a fuzzy relation. For two fuzzy relations R_1, R_2 defined on the same product set $F_1 \times F_2$. Then we can define operation on two fuzzy relations as follows.

Definition 3.2.6:-

The intersection of R_1, R_2 is defined by

$(R_1 \cap R_2)(x,y) = \mu_{R_1 \cap R_2}(x,y) = \min\{\mu_{R_1}(x,y), \mu_{R_2}(x,y)\}$. $(x,y) \in F_1 \times F_2$

For n relations, we extend it to the following

$\mu_{R_1 \cap R_2 \cap \ldots \cap R_i}(x,y) = \underset{R_i}{\wedge} \mu_{R_i}(x,y)$

Definition 3.2.7:-

The union of R_1, R_2 is defined by

$(R_1 \cup R_2)(x,y) = \mu_{R_1 \cup R_2}(x,y) = \max\{\mu_{R_1}(x,y), \mu_{R_2}(x,y)\}$, $(x,y) \in F_1 \times F_2$.

$\mu_{R_1 \cup R_2 \cup \ldots \cup R_j}(x,y) = \vee_{R_j} \mu_{R_j}(x,y)$

Definition 3.2.8:-

The complement relation R^c for fuzzy relation R shall be defined by the following membership function.

$\forall (x,y) \in X \times Y$, $\mu_{R^c}(x,y) = 1 - \mu_R(x,y)$.

Definition 3.2.9:-

We say that $R_1 \subset R_2$ if $\mu_{R_1} \le \mu_{R_2}$.

3.2.1 Matrix representation of fuzzy relation:-

Definition 3.2.1.1:-

Given a certain vector, if an element of this vector has its value between 0 and 1, we call this vector a fuzzy vector.

Definition 3.2.1.2:-

If a fuzzy relation R is given in the form of fuzzy matrix, its elements represent the membership values of this relation. This matrix is denoted by M_R, and membership values by $\mu_R(i,j)$. Then $M_R = (\mu_R(i,j))$.

Remark 3.2.1.1:-

Fuzzy matrix is a gathering of such vectors. Given a fuzzy matrix $M_{R_1} = (a_{ij})$ And $M_{R_2} = (b_{ij})$ we can perform operations on these fuzzy matrices as follow:-

- sum $M_{R_1} + M_{R_2} = \max[a_{ij}, b_{ij}]$
- max product $M_{R_1} \bullet M_{R_2} = \max_k[\min(a_{ik}, b_{kj})]$
- scalar product for M_R by $\lambda \bullet M_R$ where $0 \le \lambda \le 1$

For example a fuzzy set (vector) F_1 that has four element hence column vector of size 4×1, and for fuzzy set(vector) F_2 that has five elements, hence a row vector of size 1×5 the resulting fuzzy relation R will be represented by a matrix of size 4×5 i.e. R will have four rows and five columns.

Example 3.2.1.1:-

Suppose we have two fuzzy sets, F_1 defined on a universe of three discrete temperatures, $X=\{x_1,x_2,x_3\}$ and F_2 defined on a universe of two discrete pressures $Y=\{y_1,y_2\}$ and we want to find the fuzzy cartesian product between them. Fuzzy set F_1 could represent the "ambient" temperature and fuzzy set F_2 the" near optimum" pressure for a certain heat exchanger and the cartesian product might represent the conditions (temperature-pressure pairs) of the exchanger that are associated with " efficient" operations. For example, let $F_1=\frac{0.2}{x_1}+\frac{0.5}{x_2}+\frac{1}{x_3}$, $F_2=\frac{0.3}{y_1}+\frac{0.9}{y_2}$

note that F_1 can be represented as a column vector of size 3×1 and F_2 can represented by a row vector of 1×2. The fuzzy Cartesian product using $\mu_{F_1\times F_2}(x,y)=\min(\mu_{F_1}(x),\mu_{F_2}(y))$ the resets in a fuzzy relation R (of size 3×2) representing "efficient" conditions of R

Remark 3.2.1.2:-

- For fuzzy matrices $M_{R_1}=(a_{ij})$ and $M_{R_2}=(b_{ij})$ if $a_{ij}\le b_{ij}$ holds for all i,j , then we say that matrix M_{R2} is bigger than M_{R1} thus $a_{ij}\le b_{ij}\Leftrightarrow M_{R_1}\le M_{R_2}$.

- Also we note that for all elements of any fuzzy matrix $0 \le a_{ij} \le 1$ and it's denote as the grade of membership of cartesian product of $(F_1 \times F_2)$.

- If expressing the fuzzy relation by matrices i.e. the union of M_{R_1} and M_{R_2} given by $M_{R_1 \cup R_2} = M_{R_1} + M_{R_2}$.

- We define $0=\begin{bmatrix} 0 \ldots\ldots 0 \\ . \\ . \\ 0 \ldots\ldots 0 \end{bmatrix}$, $E=\begin{bmatrix} 1 \ldots\ldots 1 \\ . \\ . \\ 1 \ldots\ldots 1 \end{bmatrix}$ and for a fuzzy matrix M_R .then $0 < M_R < E$.

- A fuzzy matrix M_R is called constant if all its rows are equal.

- Let $M_R = (a_{ij})$ is fuzzy matrix .we can define the complement of a fuzzy matrix as $(M_R)^c = (1 - a_{ij})$.

- For $M_{R_1} = (a_{ij})$ and $M_{R_2} = (b_{ij})$ then $M_{R_1} = M_{R_2}$ iff $(a_{ij}) = (b_{ij})$ for all i and j.

- Let 0 and $I = [M_{ij}]$ where $M_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$ denote the zero and identity matrices respectively.

Then $M_R^{m+1} = M_R^m \cdot M$, $M^0 = I$ where $M_R \cdot M_R \cdot \ldots\ldots \cdot M_R$. (m-times). and also $M_R \cdot 0 = 0 \cdot M_R = 0$, $M_R \cdot I = I \cdot M_R = M_R$,

- Let M_{R_1}, M_{R_2} *and* M_{R_3} be any fuzzy matrices then $M_{R_1} \cdot (M_{R_2} \cdot M_{R_3}) = (M_{R_1} \cdot M_{R_2}) \cdot M_{R_3}$.

3.2.2 Operation of fuzzy relation on (different product sets):-

For two fuzzy relation R_1 and R_2 we cannot perform their union and intersection by the same rule because they are defined on different product sets $X \times Y$ and $Y \times Z$, respectively .however, we can perform their compositions, since they have a common set Y in between X and Z. Among some other important compositions the (max-min and max-product).

In classical set theory the composition of two relations $f : X \to Y$ and $g : Y \to Z$ in $X \times Y$ and $Y \times Z$ respectively

$g \circ f = \{(x,z) \in X \times Z : g(f(x)) = z\} =$

$\{(x,z) \in X \times Z : (x,y) \in f$ and $(y,z) \in g$, for some $y \in Y\}$.

To generalizing this idea to fuzzy composition of fuzzy relation. we may define $R_1 \circ R_2$ by the following definition.

Definition 3.2.2.1:-

Let R_1 be fuzzy relation on $X \times Y$ and R_2 be fuzzy relation on $Y \times Z$. The composition of R_1 and R_2 is defined by $R_3 = R_1 \circ R_2$, where

$\mu_{R_1 \circ R_2}(x,z) = \max_{y \in Y}\{\min\{\mu_{R_1}(x,y), \mu_{R_2}(y,z)\}\}$, ...(*)

Where $(x,z) \in X \times Z$, and $(x,y) \in X \times Y$, $(y,z) \in Y \times Z$. And note that $R_1 \subseteq X \times Y$, $R_2 \subseteq Y \times Z$ .also we can write (*) as follows.

$\mu_{R_1 \circ R_2}(x,z) = \vee_{y \in Y}\{\mu_{R_1}(x,y) \wedge \mu_{R_2}(y,z)\}$.

Remark 3.2.2.1:-

The above definition is called (max-min composition)

Example 3.2.2.1:-

Let X, Y, Z be three universes .such that $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and $Z = \{z_1, z_2, z_3\}$. let R_1 be fuzzy relation for $X \times Y$ and let R_2 be relation on $Y \times Z$,such that

$$M_{R_1} = \begin{matrix} & y_1 & y_2 \\ x_1 & 0.7 & 0.5 \\ x_2 & 0.8 & 0.4 \end{matrix}, \quad M_{R_2} = \begin{matrix} & z_1 & z_2 & z_3 \\ y_1 & 0.9 & 0.6 & 0.2 \\ y_2 & 0.1 & 0.7 & 0.5 \end{matrix}$$

Then max-min composition R_3 of R_1, R_2 can be defined on cartesian space $X \times Z$.

$$M_{R_3} = \begin{matrix} & z_1 & z_2 & z_3 \\ x_1 & 0.7 & 0.6 & 0.5 \\ x_2 & 0.8 & 0.6 & 0.4 \end{matrix}$$

Where $\mu_{R_3}(x_1, z_1) = \max[\min(0.7, 0.9), \min(0.5, 0.1)] = 0.7$

Definition 3.2.2.2:-

The fuzzy max-product composition of two fuzzy relations R_1, R_2 ,such that $R_3 = R_1 \circ R_2$, can be defined by $\mu_{R_3} = \vee_{y \in Y}\{\mu_{R_1}(x,y) \cdot \mu_{R_2}(y,z)\}$.

Remark 3.2.2.2:-

- It should be noted that neither crisp nor fuzzy compositions are commutative in general, that is $R_1 \circ R_2 \neq R_2 \circ R_1$.
- $(R_1(x,y)) \circ R_2(y,z))^{-1} = R_2^{-1}(z,y) \circ R_1^{-1}(y,x)$
- $(R_1(x,y) \circ R_2(y,z)) \circ R(z,w) = R_1(x,y) \circ (R_2(y,z) \circ R_3(z,w))$

3.2.3 Alpha-cuts of fuzzy relation:-

We know about α-cut for fuzzy sets, and we know a fuzzy relation is one kind of fuzzy sets. Therefore, we can apply the α-cut to the fuzzy relation.

Definition 3.2.3.1:-

We can obtain α-cut relation from a fuzzy relation by taking the pairs which have membership degrees on less than α. Assume $R \subseteq X \times Y$, and R_α is a α-cut relation, then $R_\alpha = \{(x,y) / \mu_R(x,y) \geq \alpha, x \in X, y \in Y\}$.

Note that R_α is a crisp relation.

Definition 3.2.3.2:-

Fuzzy relation can be said to be composed of several R_α's as following: $R = \bigcup_\alpha \alpha R_\alpha$ Here α is a value in the level set R_α is a α-cut relation, αR_α is a fuzzy relation. The membership function of αR_α is defined as $\mu_{\alpha R_\alpha}(x,y) = \alpha \cdot \mu_{R_\alpha}(x,y)$, for $(x,y) \in X \times Y$

Then we can to compose a fuzzy relation R into several αR_α.

Example 3.2.3.1:-

Assume we have a fuzzy relation R such that, $M_R = \begin{bmatrix} 0.9 & 0.4 & 0.0 \\ 0.2 & 1.0 & 0.4 \\ 0.0 & 0.7 & 1.0 \\ 0.4 & 0.2 & 0.0 \end{bmatrix}$

Now the level set with degree of membership function is $A = \{0, 0.2, 0.4, 0.7, 0.9, 1.0\}$,then we can have some $\alpha - cut$ relations in the following Manner:

$$R_{0.4} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad R_{0.7} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_{0.9} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_{1.0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So we can obtain the relation R can be decomposed as following.

$$M_R = 0.4\times\begin{bmatrix}1&1&0\\0&1&1\\0&1&1\\1&0&0\end{bmatrix}\cup 0.7\times\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\\0&0&0\end{bmatrix}\cup 0.9\times\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}\cup 1.0\times\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

3.2.4 Projection and cylindrical extension:-

Definition 3.2.4.1:-

We can project a fuzzy relation $R\subseteq X\times Y$ with respect to X or Y as in the following manner. For all $x\in X, y\in Y$

$\mu_{R_X}(x)=\max_y \mu_R(x,y)$: *projection* to X

$\mu_{R_Y}(y)=\max_x \mu_R(x,y)$: *projection* to Y

Here the projected relation of R to X is denoted by R_X and to Y is by R_Y.

Example 3.2.4.1:-

There is a relation $R\subseteq X\times Y$. The projection with respect to X or Y shall be

$$M_R = \begin{array}{c} \begin{matrix} b_1 & b_2 & b_3 \end{matrix} \\ \begin{bmatrix}0.1&0.2&1.0\\0.6&0.8&0.0\\0.0&1.0&0.3\end{bmatrix}\begin{matrix}a_1\\a_2\\a_3\end{matrix}\end{array}$$

$$M_{R_X} = \begin{matrix}a_1\\a_2\\a_3\end{matrix}\begin{bmatrix}1.0\\0.8\\1.0\end{bmatrix} \qquad M_{R_Y} = \begin{array}{c}\begin{matrix} b_1 & b_2 & b_3 \end{matrix}\\ \begin{bmatrix}0.6&1.0&1.0\end{bmatrix}\end{array}$$

Definition 3.2.4.2:-

The extending of the projection in 2-dimensions relation to n-dimensional fuzzy set, assume relation R is defined in the space of $X_1\times X_2\times\dots\times X_n$. Projecting this relation to subspace of $X_{i1}\times X_{i2}\times\dots\times X_{ik}$, gives a projected relation: $R_{X_{i1}\times X_{i2}\times\dots\times X_{ik}}$

$$\mu_{R x_{i1} \times x_{i2} \times \ldots \times x_{ik}}(x_{i1}, x_{i2}, \ldots, x_{ik}) = \max_{x_{j1}, x_{j2} \ldots x_{jm}} \mu_R(x_1, x_2, \ldots, x_n).$$

Here $x_{j1}, x_{j2}, \ldots, x_{jk}$ represent the omitted dimensions, and $x_{i1}, x_{i2}, \ldots, x_{ik}$ the remained dimensions, and thus $\{x_1, x_2, \ldots, x_n\} = \{x_{i1}, x_{i2}, \ldots, x_{ik}\} \cup \{x_{j1}, x_{j2}, \ldots, x_{jm}\}$

Definition 3.2.4.3:-

As the opposite concept of projection, cylindrical extension is possible. If a fuzzy set or fuzzy relation R is defined in space $X \times Y$,this relation can be extended to $X \times Y \times Z$ and we can obtain a new fuzzy set, this fuzzy set is written as $CY(R)$, $\mu_{CY(R)}(a,b,c) = \mu_R(a,b)$, $a \in X, b \in Y, c \in Z$.

Example 3.2.4.2:-

In the previous example, relation R_X is the projection of R to direction X . If we extend it again to direction Y, we can have an extended relation $CY(R_X)$. For example:

$$\mu_{CY(R_X)}(a_1, b_1) = \mu_{R_X}(a_1) = 1.0$$
$$\mu_{CY(R_X)}(a_1, b_2) = \mu_{R_X}(a_1) = 1.0$$
$$\mu_{CY(R_X)}(a_2, b_1) = \mu_{R_X}(a_2) = 0.8$$

$$M_{CY}(R_X) = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.8 & 0.8 & 0.8 \\ 1.0 & 1.0 & 1.0 \end{bmatrix} \end{array}$$

The new relation $CY(R_X)$ is now in $X \times Y$.

Remark 3.2.4.1:-

Let two fuzzy relations be defined as follows: $R_1 \subseteq X_1 \times X_2$, $R_2 \subseteq X_2 \times X_3$. Even though we want to apply the intersection operation between R_1 and R_2, it is not possible because the dominos of R_1 and R_2 are different from each Other, if we obtain cylindrical extensions $CY(R_1)$ and $CY(R_2)$ to space

$X_1 \times X_2 \times X_3$, and then $CY(R_1)$ and $CY(R_2)$ have the same domain . We can now apply operations on the two extended $CY(R_1)$ and $CY(R_2)$. Therefore

$join(R_1, R_2) = CY(R_1) \cap CY(R_2)$.

Then projection and cylindrical extension are often used to make domains same for more than one.

3.2.5 Classification of fuzzy relation:-

We assume that fuzzy relation R is defined on $X \times X$.

Definition 3.2.5.1:-

We call R is reflexive fuzzy relation if for all $x \in X$, $\mu_R(x,x) = 1$.

Definition 3.2.5.2:-

If $\exists x \in X, \mu_R(x,y) \neq 1$, then the relation is (irreflexive).

Definition 3.2.5.3:-

If $\forall x \in X, \mu_R(x,y) \neq 1$, then it is called (antireflexive).

Definition 3.2.5.4:-

fuzzy relation R is called symmetric if it satisfies the following condition.

$\forall (x,y) \in X \times X$, $\mu_R(x,y) = \mu_R(y,x)$.

Definition 3.2.5.5:-

A relation R is called (antisymmetric) if for $x \neq y$ either

$\mu_R(x,y) \neq \mu_R(y,x)$ or, $\mu_R(x,y) = \mu_R(y,x) = 0$ $\forall x, y \in X$, or

If $\mu_R(x,y) > 0$ and $\mu_R(y,x) > 0$ imply $x = y$.

Definition 3.2.5.6:-

A relation R is called (perfectly antisymmetric) if $x \neq y$ whenever $\mu_R(x,y) > 0$ then $\mu_R(y,x) = 0$, $\forall x, y \in X$.

Example 3.2.5.1:-

For a set $X = \{2,3,4,5\}$ there is a relation such that for $x, y \in X$, "x is close to y". Concerning this relation when x=y, the relation is perfectly satisfied and thus $\mu_R(x, y) = 1$. Let's denote this reflexive one in a matrix as in the following:

$$\begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccc} 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1.0 & 0.9 & 0.8 & 0.7 \\ 0.9 & 1.0 & 0.9 & 0.8 \\ 0.8 & 0.9 & 1.0 & 0.9 \\ 0.7 & 0.8 & 0.9 & 1.0 \end{bmatrix} \end{array}$$

Also the relation "x is close to y" is a symmetric relation.

Definition 3.2.5.7:-

A relation R is called transitive relation if $\forall\ x, y, z \in X$

$\mu_R(x, z) \geq \max_y\{\min(\mu_R(x, y), \mu_R(y, z)\}$.

And if we replace min ($\wedge$) by any T-norm, we get a generalization of transitive relation.

Definition 3.2.5.8:-

If a fuzzy relation R ; Satisfies the following condition, we call it a "fuzzy equivalence relation" or " similarity relation.":

1. reflexive relation

   $x \in X$, $\mu_R(x, x) = 1$

2. symmetric relation

   $\forall (x, y) \in X \times X$, $\mu_R(x, y) = \mu_R(y, x)$

3. transitive relation

   $\forall\ x, y, z \in X$, $\mu_R(x, z) \geq \max_y\{\min(\mu_R(x, y), \mu_R(y, z))$.

A fuzzy relation R on X is T-fuzzy equivalence relation for the T-norm T, If R is reflexive, symmetric, and T-transitive. If $T = \wedge$ then we say that R is a fuzzy equivalence relation.

Definition 3.2.5.9:-

If fuzzy relation R satisfies the following conditions, we call it "fuzzy compatibility relation" or " resemblance relation":-

1. reflexive relation .
2. symmetric relation .

Definition 3.2.5.10:-

Given fuzzy relation R, if the following are well kept for all $x, y, z \in X$, this relation is called "pre-order relation."

1. reflexive relation .
2. transitive relation .

Remark 3.2.5.1:-

- Also if certain relation is transitive but not reflexive, this relation is called "semi-pre-order" or "nonreflexive fuzzy pre-order".
- If the membership function follows the relation $\mu_R(x,x) = 0$

for all x we use the term "antireflexive fuzzy pre-order".

Definition 3.2.5.11:-

If relation R satisfies the following for all $x, y, z \in X$.

It's called " fuzzy partial order relation":

1. reflexive relation .
2. antisymmetric relation .
3. transitive relation .

Remark 3.2.5.2:-

We can get a corresponding crisp relation R_1 from given fuzzy order relation R by arranging the value of membership function as follows.

1. if $\mu_R(x,y) \geq \mu_R(y,x)$ then $\mu_{R_1}(x,y) = 1$

$$\mu_{R_1}(y,x) = 0$$

2. If $\mu_R(x,y) = \mu_R(y,x)$ then $\mu_{R_1}(x,y) = \mu_{R_1}(y,x) = 0$

Remark 3.2.5.3:-

- If the corresponding order relation of a fuzzy order relation is total order or linear order, this fuzzy relation is named as "fuzzy total order" and if not it's called "partial order".
- When the second condition of the fuzzy order relation is transformed to" perfect antisymmetric", the fuzzy order relation becomes a perfect fuzzy order.

(2) Perfect antisymmetric if $\forall(x,y) \in X \times X$, $x \neq y$, $\mu_R(x,y) > 0$, then $\mu_R(y,x) = 0$.

- When the first condition (reflexivity) does not exist, the fuzzy order relation is called "fuzzy strict order."

Definition 3.2.5.12:-

Let R_1 and R_2 be fuzzy relations in $X \times Y$, $Y \times Z$ respectively and let T be a T-norm the composition $R_3 = R_1 \circ R_2$ of R_1 and R_2 with respect to T is the fuzzy relation on $X \times Z$,with membership function

$$\mu_{R_1 \circ R_2}(x,z) = \vee_{y \in Y}[T(\mu_{R_1}(x,y), \mu_{R_2}(y,z))] .$$

1. When $T(x,y) = x \wedge y$, $R_1 \circ R_2$ is referred to as a max-min composition.
2. When $T(x,y) = x \cdot y$, $R_1 \circ R_2$ is a max-product composition.

3.3 Fuzzy partition:-

Definition 3.3.1:-

Let F be a fuzzy set on X satisfying $F \neq \phi$ and $F \neq X$. The pair (F . F^c) is defined as fuzzy partition. usually. If m subsets are defined in X, m - tuple$(F_1, \ldots, F_m)$ holding the following conditions is called a fuzzy partition.

1. $\forall i, F_i \neq \phi$
2. $F_i \cap F_j = \phi$ for $i \neq j$

3. $\forall x \in X, \ \sum_{i=1}^{m} \mu_{F_i}(x) = 1$

3.3.1 Fuzzy partition by an equivalence relation:-

An equivalence relation on a set gives a partition of that set and vice versa. The analogy for fuzzy equivalence relations suggests properties for the notion of fuzzy partition. If $R : X \times X \to [0,1]$, if fuzzy relation on a set X, there is associated the family $\{R_x : X \to [0,1] : y \to R(x,y)\}$ of fuzzy subsets of X. If R were an equivalence relation, then R_x would be the equivalence containing x, so this is an exact analog to the crisp case. When R is a fuzzy equivalence relation, we have

1. $R_x(x) = 1$ for each $x \in X$.
2. $R_x(y) = R_y(x)$ for $x, y \in X$.
3. $R_x(y) \geq R_z(x) \wedge R_z(y)$ for $x, y.z \in X$.

This suggests that a fuzzy partition of X could be defined as a family $P = \{R_x : x \in X\}$ of fuzzy subsets of X satisfying these three properties.

3.3.2 Fuzzy partition by α-cuts:-

Theorem 3.3.2.1:-

if R is a fuzzy relation then R is a fuzzy equivalence relation if and only if each α-cut, is an equivalence relation on X.

Proof:-

Let R be a fuzzy equivalence relation by definition of

$R_\alpha = \{(x,y) : R(x,y) \geq \alpha, \forall \alpha \in [0,1]\}$ then $(x,x) \in R_\alpha$ since $R(x,x) = 1 \geq \alpha$.

Thus R_α is reflexive.

If $(x,y) \in R_\alpha$, then $R(x,y) \geq \alpha$ so $R(y,x) \geq \alpha$ from symmetry, whence $(y,x) \in R_\alpha$.

If (x,y) and $(y,z) \in R_\alpha$, then $R(x,z) \geq R(x,y) \wedge R(y,z) \geq \alpha$, so $(x,z) \in R_\alpha$, thus R_α is an equivalence relation.

Conversely:-

Let for each α-cut, R_α is an equivalence relation on X ,thus

Let $(x,x) \in R_\alpha$ since $R(x,x) \geq \alpha$ let $R(x,x)=1$ thus R is reflexive

Let $(x,y) \in R_\alpha$ and $(y,x) \in R_\alpha$ then $R(x,y) \geq \alpha$ and $R(y,x) \geq \alpha$.

Thus $R(x,y) = R(y,x)$.

Let (x,y) and $(y,z) \in R_\alpha$ and let $R(x,y)$ and $R(y,z) = \alpha$ thus

$R(x,y) \wedge R(y,z) = \alpha$ and since R_α is an equivalence relation and $(x,z) \in R_\alpha$

Thus $R(x,z) \geq \alpha = R(x,y) \wedge R(y,z)$.

- So with each fuzzy equivalence relation R on a set X there is associated a family of equivalence relations of X, namely the α-cut R_α .one for each $\alpha \in [0,1]$.
- Each of these equivalence relations induces a partition P_α of X, so we also have the associated family $\{P_\alpha : \alpha \in [0,1]\}$ of partitions of X. Since the α-cut of a fuzzy set determine that fuzzy set.

Theorem 3.3.2.2:-

Let $E(X)$ be the set of all equivalence relations on the set X. Then $E(X)$ is complete lattice, and each R_α is an element of $E(X)$.

Proof

See [3]

Remark 3.3.2.1:-

In $E(X)$ complete lattice, we have for any subset $B \subseteq [0,1]$

$$\bigwedge_{\alpha\in B} R_\alpha = \bigcap_{\alpha\in B} R_\alpha = \{(x,y) : R(x,y) \geq \alpha \text{ for all } \alpha \in B\} = \{(x,y) : R(x,y) \geq \bigvee_{\alpha\in B} \alpha\} = R_{\bigvee_{\alpha\in B} \alpha}.$$

Thus $\bigvee_{\alpha\in B} R_\alpha = \bigcup_{\alpha\in B} R_\alpha = \{(x,y) : R(x,y) \geq \alpha \text{ for some } \alpha \in B\} \geq R_{\bigwedge_{\alpha\in B} \alpha}$.

This last inequality is equality if B is finite.

Definition 3.3.2.1:-

A partition tree on a set X is a family $\{P_\alpha : \alpha \in [0,1]\}$ of partitions of X such that $p_0 = X$, for any subset B of$[0,1]$, $P_{\vee_{\alpha \in B} \alpha} = \wedge_{\alpha \in B} P_\alpha$.

On should note that when $B = \phi$, then $\vee_{\alpha \in B} \alpha = 0$ and $\wedge_{\alpha \in B} P_\alpha = X$.

Theorem 3.3.2.3:-

Let R be a fuzzy relation on a set X, and let $\{R_\alpha : \alpha \in [0,1]\}$ be its set of α-cuts. Then R is a fuzzy partial order on X if and only if each R_α is a partial order on X.

Proof:-

Suppose R is a fuzzy partial order on X, we will prove R_α is partial order on X. For reflexive, since $R(x,x) = 1 \geq \alpha \Rightarrow (x,x) \in R_\alpha$, for antisymmetric $R(x,y) \geq \alpha$.and $R(y,x) \geq \alpha$., and since $R(x,y) = R(y,x)$, $x = y$.

Thus $(x,y) \in R_\alpha$ and $(y,x) \in R_\alpha$, $\Rightarrow x = y$.

For transitivity suppose that (x,y) and (y,z) are in R_α then $R(x,y) \geq \alpha$ and $R(y,z) \geq \alpha$, whence $R(x,y) \wedge R(y,z) \geq \alpha$ and so $(x,z) \in R_\alpha$.

Conversely:-

Suppose that each R_α is partial order we proof that R is partial order on X.

We just prove transitive, which means that $R(x,z) \geq R(x,y) \wedge R(y,z)$.

If $R(x,y) \wedge R(y,z) = \alpha$ then (x,y) and $(y,z) \in R_\alpha$, since R_α is transitive $(y,z) \in R_\alpha$, so $R(x,z) \geq \alpha = R(x,y) \wedge R(y,z)$

thus $R(x,z) \geq R(x,y) \wedge R(y,z)$.

Remark 3.3.2.2:-

So with each fuzzy partial order on a set X.

There is associated a family of partial ordering namely the $\alpha - cut$ of this relation .Those α – cuts determine the fuzzy partial order.

CHAPTER 4

4.1 Approximate Reasoning:-

One of basic tools for fuzzy logic and approximate reasoning is the notion of a linguistic variable that in [1973 by zadeh] was called variable of higher order rather than a fuzzy variable , Where the linguistic variables, that is, variables whose values are not numbers but words or sentences rather than numbers .All material in this chapter taken from [31],[16],[20],[5],[25],[33],[27],[11],[28],[34],[3],[15],[1],[23],[19],[14],[21],[7],[2],[6],[29],[25],[1].

we can defined linguistic variable as follows :-

Definition 4.1.1:-

The linguistic variable is defined by the following quintuple.

Linguistic variable $=(x,T(x),X,G,M)$, where

x : Name of variable ,

X : set of universe of discourse which defined the characteristics of the variable.

$T(x)$: set of linguistic terms which can be a value of the variable.

G : Syntactic grammar which produces terms in $T(x)$.

M : Semantic rules which map terms in $T(x)$ to fuzzy sets in X.

Example 4.1.1.:-

Let's consider a linguistic variable "A" whose name is "Age".

$A=(age,T(\text{age}),X,G,M)$

Age: name of the variable A

$T(age)$: {young, very young, very very young, ...}

Set of terms used in the discussion of age

X : [0,100] universe of discourse

$G(Age):T^{i+1}=\{\text{young}\}\cup\{\text{very } T^i\}$

$M(young)=\{(x,\mu_{young}(x))\setminus x\in[0,100]\}$

$$\mu_{young}(x) = \begin{cases} 1 & \text{if } x \in [0,25] \\ (1 + \frac{x-25}{5})^{-2} & \text{if } x \in [25,100] \end{cases}$$

In the above example, the term "young" is used as a basis in the $T(age)$, and thus this kind of term is called a "primary term". When we add modifiers to the primary terms, we can define new terms (fuzzy terms). In many cases, when the modifier "very" is added, the membership is obtained by square operation. For example, the membership function of term "very young" is obtained from that of "young".

$\mu_{very\ young}(x) = (\mu_{young}(x))^2$.

The fuzzy linguistic terms often consist of two parts:

1. Fuzzy predicate (primary term): expensive, old, rare, dangerous, good, etc.
2. Fuzzy modifier: very, likely, almost impossible, extremely unlikely, etc.

The modifier is used to change the meaning of predicate and it can be grouped into the following two classes:

1. Fuzzy truth qualifier or fuzzy truth value: quite true, very true, more or less true, mostly false, etc.
2. Fuzzy quantifier: many, few, almost, all, usually, etc.

In the following sections, we will introduce the fuzzy predicate, fuzzy modifier, and fuzzy quantifier.

4.1.1 Fuzzy Predicate:-

As we know now, a predicate proposition in the classical logic has the following form.

"x is a man"

"y is p"

X and y are variables, and "man" and "p" are crisp sets. The sets of individuals satisfying the predicates are written by "man(x)" and "p(y)".

Definition 4.1.1.1:-

If the set defining the predicates of individual is a fuzzy set, the predicate is called a fuzzy predicate.

Example 4.1.1.1:-

Let consider the following statement

"z is expensive".

"w is young".

The terms "expensive" and "young" are fuzzy terms. Therefore the sets "expensive (z)" and "young (w)"are fuzzy sets.

When a fuzzy predicate "x is p" is given, we can interpret it in two ways.

1. P(x) is a fuzzy set. The membership degree of x in the set P is defined by the membership function $\mu_{P(x)}$
2. $\mu_{P(x)}$ is the satisfactory degree of x for the property P. therefore, the truth value of the fuzzy predicate is defined by the membership function, truth value $= \mu_{P(x)}$.

4.1.2 Fuzzy Modifier (hedges):-

As we know, a new term can be obtained when we add the modifier "very" to a primary term. In this section we will see how semantic of the new term and membership function can be defined.

Example 4.1.2.1:-

Let's consider a linguistic variable "age", linguistic terms "young" and "very young" are defined in the universal set X, $X = \{x / x \in [0,100]\}$ The variable age takes a value in the set $T(age)$

$T(age) =$ {Young, very young, very very young...}

The term "young" is represented by a membership function $\mu_{young}(x)$ when we represent the term "very young", we can use the square of $\mu_{young}(x)$ as follows, $\mu_{very\ young}(x)=(\mu_{young}(x))^2$.

There is other modifier like very, low, slight, more or less, fairly, slightly, almost, barely, mostly, roughly, approximately, these modifier have the effect of modifying the membership function $\mu_F(x)$ such as

"Very" $\mu_F(x)= (\mu_F(x))^2$ "more or less" $\mu_F(x)=(\mu_F(x))^{1/2}$

"Very very" $\mu_F(x)=(\mu_F(x))^4$ "somewhat" $\mu_F(x)=(\mu_F(x))^{1/2}$

"Plus" $\mu_F(x)=(\mu_F(x))^{1.25}$

"indeed" $\mu_F(x)=\begin{cases} 2[\mu_F(x)]^2 & \text{if } 0\le \mu_F \le 0.5 \\ 1-2[1-\mu_F(x)]^2 & \text{if } 0.5\le \mu_F \le 1 \end{cases}$

"Slightly" $\mu_F(x)=(\mu_F(x))^{1/3}$

"A little" $\mu_F(x)=(\mu_F(x))^{1.3}$

"Minus" $\mu_F(x)=(\mu_F(x))^{0.75}$

"Extremely" $\mu_F(x)=(\mu_F(x))^3$

4.1.3 Fuzzy Qualifier:-

4.1.3.1 Fuzzy truth values:-

Baldwin defined fuzzy truth qualifier in the universal set $X=\{x/x\in[0,1]\}$ as follows. T= {true, very true, fairly true, absolutely true,..., absolutely false, fairly false ,false}. The qualifiers in T define "fuzzy truth values ": and they can be defined by the membership functions. If we take Baldwin's membership function $\mu_{true}(x)$, the truth qualifiers are represented in the following membership functions

$\mu_{true}(x)=x \quad , x\in[0,1]$

$\mu_{very\ true}(x)=(\mu_{true}(x))^2 \ , \quad x\in[0,1]$

$\mu_{fairly\ true}(x)=(\mu_{true}(x))^{1/2} \ , \quad x\in[0,1]$

$\mu_{false}(x)=1-\mu_{true}(x) \quad , \ x\in[0,1]$

$$\mu_{very\ false}(x) = (\mu_{false}(x))^2 \quad , \quad x \in [0,1]$$

$$\mu_{fairly\ false}(x) = (\mu_{false}(x))^{1/2} , \quad x \in [0,1]$$

$$\mu_{absolutely\ true}(x) = \begin{Bmatrix} 1 & \text{for } x = 1 \\ 0 & \text{otherwise} \end{Bmatrix}$$

$$\mu_{absolutely\ false}(x) = \begin{Bmatrix} 1 & \text{for } x = 0 \\ 0 & \text{otherwise} \end{Bmatrix}$$

Example 4.1.3.1:-

Let's consider a predicate using the primary term "young" and fuzzy truth qualifier "very false"

P="Ali is young is very false".

Suppose the term "young" is defined by the function μ_{young}.

$$\mu_{young} = \begin{Bmatrix} 1 & x \in [0,25] \\ (1 + \frac{x - 25}{5})^{-2} & x \in [25,100] \end{Bmatrix}$$

The term "very false" can be defined by the following.

$$\mu_{very\ false} = (1 - \mu_{true}(x))^2 = (1 - \mu_{young}(x))^2 = \begin{Bmatrix} 0 & x \in [0,25] \\ (1 - (1 + \frac{x - 25}{5})^{-2})^2 & x \in [25,100] \end{Bmatrix}$$

Therefore, if Ali has age less than 25, the truth value of the predicate p is 0. If he is in [25,100], the truth value is calculated from $\mu_{very\ false}$.

Example 4.1.3.2:-

Let's consider a predicate p in the following. P="20 is young."Assume the terms "young" and "very young" are defined as shown in figurer 16 we see the membership degree of 20 in "young" is 0.9. Therefore, the truth value of the predicate p is 0.9. Now we can modify the predicate p by using fuzzy truth qualifiers as follows.

p_1 = "20 is young is truth".

p_2 = "20 is young is fairly true."

p_3 = "20 is young is very true."

p_4 = "20 is young is false."

The truth values are changed according to the qualifiers as in fig 17.

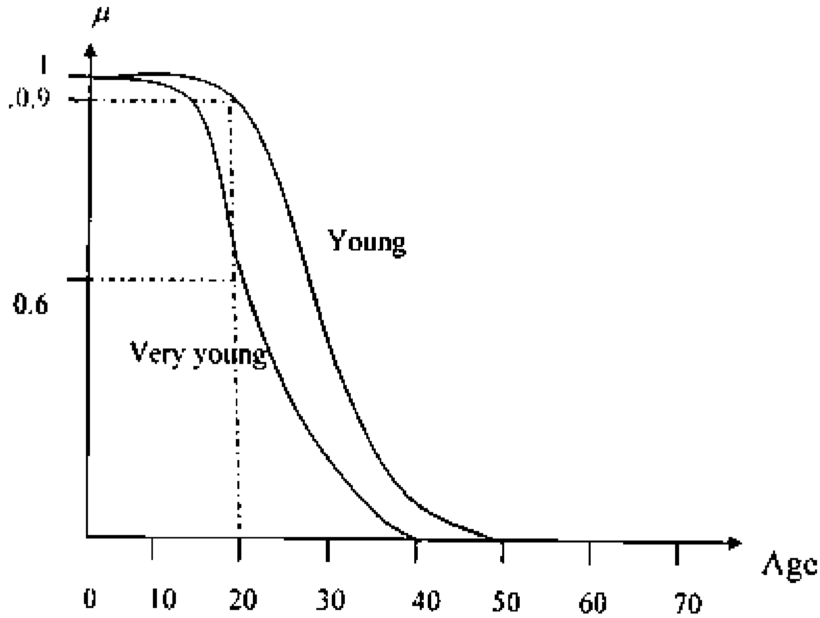

Figure 16

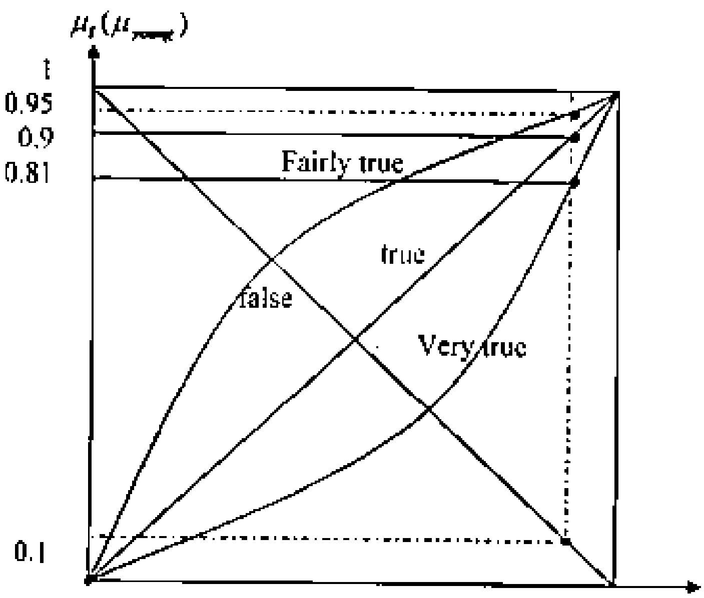

Figure 17

We know already $\mu_{young}(20)$ is 0.9. That is, the truth value of p is 0.9. For the predicate p_1,we use the membership function "true" in the figure and obtain the truth value 0.9. For p_2 the membership function "fairly true" is used and 0.95 is obtained. In the same way, we can calculate for p_3 and p_4, summarize the truth values in the following.

For p_1:0.9

For p_2:0.95

For p_3:0.81

For p_4:0.1.

4.2 Application of fuzzy set in (Artificial Neuron Network):-

Neural network theory grew out of artificial intelligence research, or the research in designing machines with cognitive ability. A neural network is a computer program or hardwired machine that is designed to learn in manner similar to the human brain.

The basic building of a brain and the neural network is the neuron. The basic human neuron adapted from Beale and Jackson (1990) is shown below in figure 18 – (biological neuron). As described by Jackson . (1990), all inputs to the cell body of the neuron arrive along (dendrites). Dendrites can also act as outputs interconnecting interneurons. Mathematically, the dendrite's function can be approximated as a summation. Axons, on the other hand, are found only on output cells. The axon has an electrical potential. If excited past a threshold it will transmit an electrical Signal. Axons terminate at (synapses) that connect it to the dendrite of another neuron. When the electrical input to a synapse reaches a threshold, it will pass the signal through to the dendrite to which it is connected. The human brain contains approximately 10^{10} interconnected neurons creating its massively parallel computational capability.

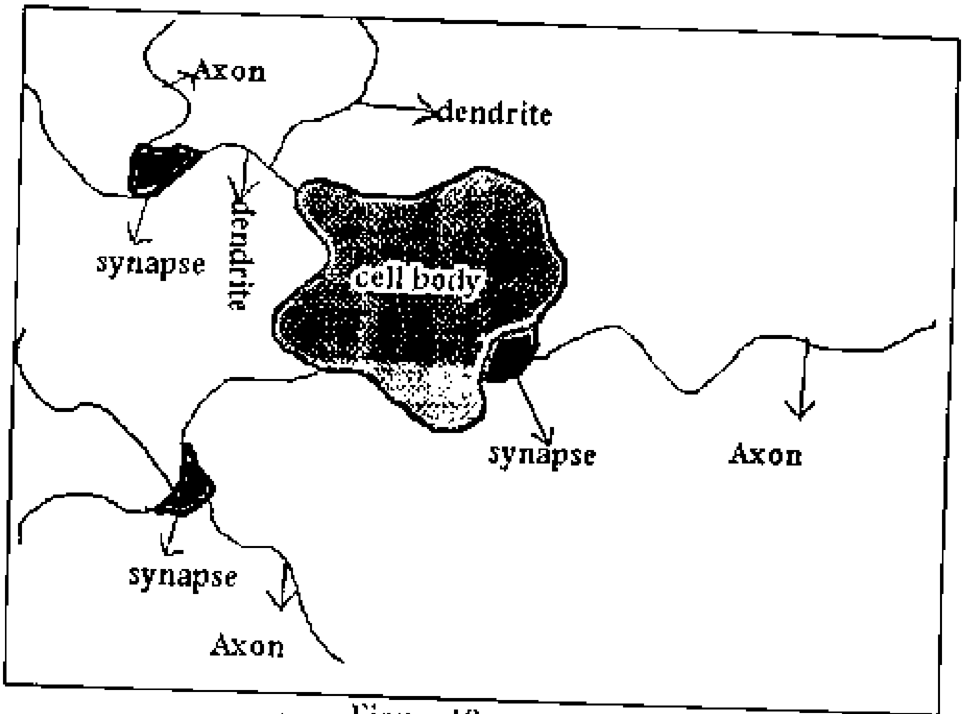

Figure 18

The artificial neuron was developed in an effort to model the human neuron. The artificial neuron was adapted from Karalopoulos (1996) and Haykin (1994). Inputs enter the neuron and are multiplied by their respective synaptic.

4.2.1 Basic Network Components:-

A neural network is general mathematical computing paradigm that models the operations of biological neural systems. In 1943 McCulloch, neurobiologist, and Pitts, Astatistician published a seminal paper title "a logical calculus of ideas imminent in neurons activity" in bulletin of mathematical biophysics. This paper inspired the development of the modern digital computer or the electronic brain, as john vonNeumann called it. At approximately the same time, Frank Rosenblatt was also motivated by this paper to investigate the computation of the eye, which eventually led to the first generation of neural network, known as the perceptron.

4.2.1.1 McCulloch and Pitts' Neuron Model:-

Among numerous neural network models that have been proposed over the years, all share a common building block known as a neuron and networked interconnection stricture. The most widely used neuron model is based on McCulloch and Pitts' work as in Figure 19

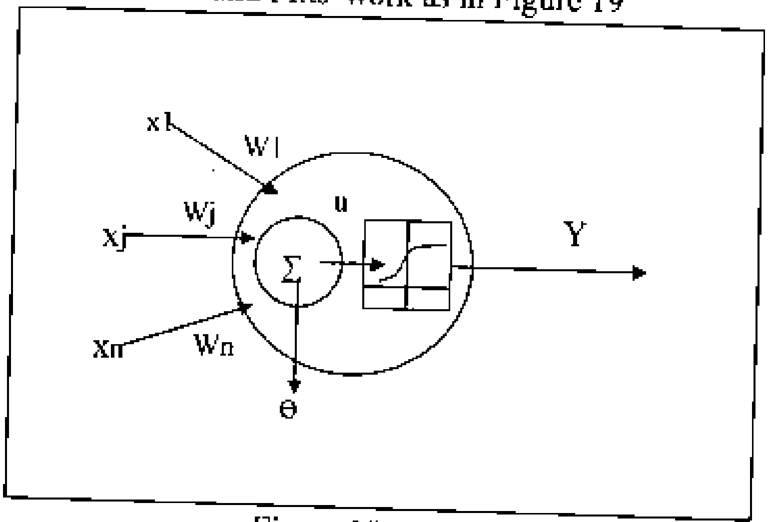

Figure 19

Each neuron consists of two parts: the net function and activation function. Determines how the network input $\{x_j : 1 \le j \le n\}$ are combined inside the neuron. In above figure, a weighted linear combination is adopted: $u = \sum_{j=1}^{n} w_j x_j + \theta$ $\{w_j : 1 \le j \le n\}$ are parameters known as synaptic weights. The quantity θ is called the (bias or threshold) and is used to model the threshold. In the literature, other types of network input combination methods have been proposed as well. They are summarized in the following table

Net functions	Formula
Linear	$u = \sum_{j=1}^{n} w_j x_j + \theta$
Higher order (2nd order formula exhibited)	$u = \sum_{j=1}^{n} \sum_{k=1}^{n} w_{jk} x_j x_k + \theta$
Delta(Σ-Π)	$u = \prod_{j=1}^{n} w_j x_j$

Table 5

The output of neuron, denoted by Y, is related to the network input u, via a linear or nonlinear transformation called the activation function:

$$Y = f(u)$$

In various neural network models, different activation functions have been proposed the most commonly used activation functions are summarized in the following table.

Activation function	Formula
Sigmoid	$f(u) = \dfrac{1}{1+e^{\frac{-u}{T}}}$ where T =temperature parameter
Hyperbolic tangent	$f(u) = \tanh(\frac{u}{T})$
Inverse tangent	$f(u) = \frac{2}{\pi}\tan^{-1}(\frac{u}{T})$
Threshold	$f(u) = \begin{Bmatrix} 1 & u > 0 \\ -1 & u < 0 \end{Bmatrix}$
Gaussian radial basis	$f(u) = \exp\left\{-\lVert u-m\rVert^2/\sigma^2\right\}$
Linear	$f(u) = au + b$

Table 6

4.2.1.2 Perceptron Model:-

The original perceptron model proposed by Rosenblatt in the 1950.

In the perceptron model, a single neuron with a linear weighted net function and threshold activation function is employed the input to this neuron $x = (x_1, x_2, \ldots, x_n)$ is a feature vector in R^n.

The net function $u(x) = w_0 + \sum_{i=1}^{n} w_i x_i$ and the output $y(x)$ are obtained from $u(x)$ via a threshold activation function:- $y(x) = \begin{Bmatrix} 1 & u(x) \geq 0 \\ 0 & u(x) < 0 \end{Bmatrix}$

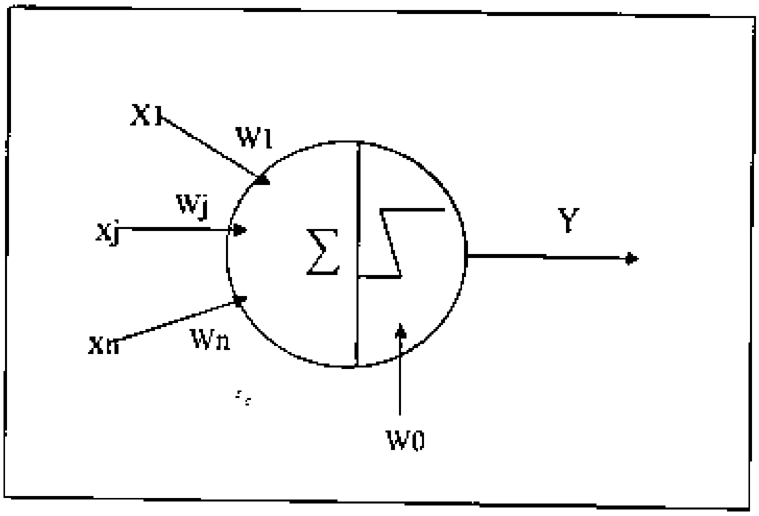

Figure 20

Remark 4.2.1.2.1:-

We can write $u(x) = w_0 + f(<w,x>) = w_0 + f(\sum_{i=1}^{n} w_i x_i)$ and

$<w,x> = w^T x = w_1 x_1 + ... + w_n x_n$

4.2.2 Characteristics of ANN:-

1. computational model of the brain
2. learning capability
3. learning by updating weights

4.2.3 Structure of ANN:-

Neural networks can be categorized into

1. feed forward
2. feedback

The feed forward neural networks have only feed forward links, i.e neural networks which do not have feed back cycle. The output of anode will not directly or indirectly be used as input of that node. To the contrary, the feedback neural networks have feedback cycle. The output of a node may be used as an input of a node. Figure 21(a) and (b) show the feed forward and feedback neural network, there is no guarantee that the networks become stable because of point; some may have limit-cycle, or become chaotic or divergent. These are common characteristics of non-linear systems which have feedback.

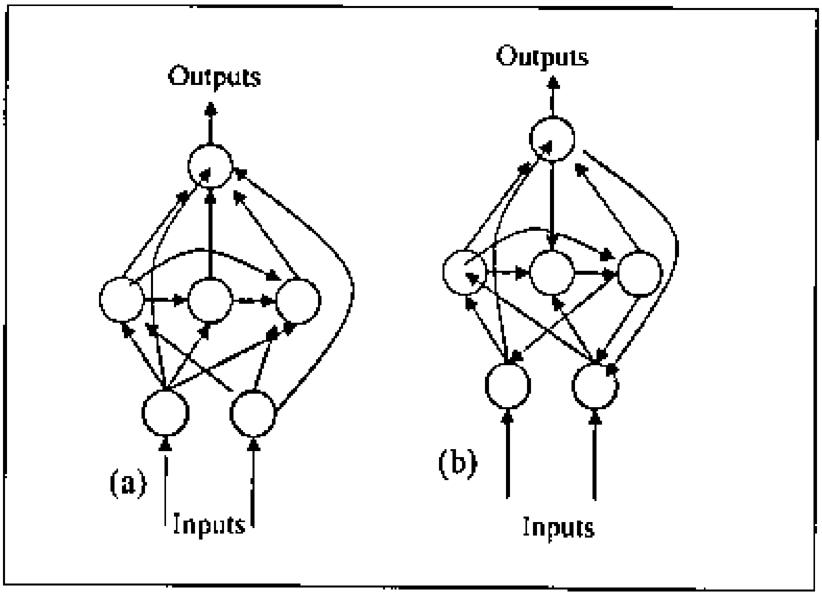

Figure 21

4.2.4 Learning Algorithms:-

There are two types of learning algorithms in the neural networks. The first type is (supervised learning) it uses asset of training data which consist of pairs of input and output. During learning, the weights of a neural network are changed so that input-output mapping becomes more and more close to the training data.

The second type is (unsupervised learning). While the supervised learning presents target answers for each input to a neural network, the unsupervised learning has the target answers. It adjusts the weights of a neural network in response to only input patterns without the target answers. In the unsupervised learning. The network usually classifies the input patterns into similarity categories.

4.3 Fuzzy Neural Networks (FNN):-

In the nearly three decades since it publication the pioneering work of McCulloch and Pitts has had a profound influence on the development of the theory of neural nets, although the McClloch-pitts model of a neuron has contributed a great deal to the understanding of the behavior of neural -like systems, it fails to reflect the fact that the behavior of even the simplest type of nerve cell exhibits not only randomness but, more importantly, a type of imprecision which is associated with the lock of sharp transition from the occurrence of an event to its non-occurrence it is possible that a better model for the behavior of an nerve cell may be provided by what might be called a fuzzy neuron, which is generalization of the McCulloch-Pitts model. The concept of a fuzzy neuron employs some of the concepts and techniques of the theory of fuzzy sets.

4.3.1 Classification of Fuzzy Neural Networks:-

We may classify all FNN models as three main types as follow:-

1. type that based on fuzzy operators:

(i) Feed forward neural networks (1980)

(ii) Feed back neural networks (1990)

2. Fuzzified neural networks .

Where inputs and connecting weight are fuzzy set, and internal operations are based on extension principle and fuzzy arithmetic.

3. fuzzy inference networks

(i) Mamdani type (1990)

(ii) Takagi-Sugeno type (1990)

(iii) Generalized type (1990)

We will be only concerned with the type (1) that based on fuzzy operators and her we don't discuss a learn problem but we just concerned the strictures of fuzzy neural network.

4.3.2 Simple Fuzzy Neurons:-

Consider a simple neural net in figure 22 all singles and weights are real numbers. The two input neurons do not change the input signals so their output is the same as their input. The signal x_i interacts with the weight w_i to product the product $p_i = w_i x_i$, i = 1,2

The input information p_i is aggregated, by addition, to produce the input net $= p_1 + p_2 = w_1 x_1 + w_2 x_2$, to the neuron .the neuron uses it's transfer function f ,which could be a sigmoid function, $f(x) = (1+e^{-x})^{-1}$,to compute the output $y = f(net) = f(w_1 x_1 + w_2 x_2)$.

This simple neural net, which employs multiplication, addition, and sigmodal f ,will be called (standard) neural net.

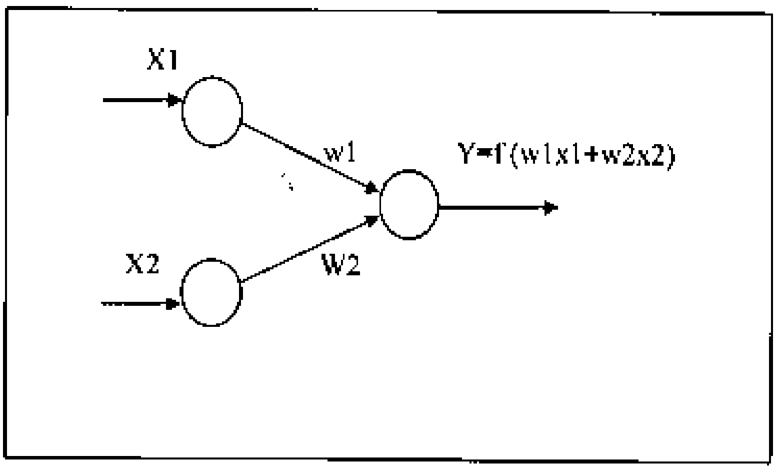

Figure 22

If we employ other operations like a t-norm, or a t-conorm to combine the incoming data to a neuron we obtain what we call hybrid neural net.

These modifications lead to a fuzzy neural architecture based on fuzzy arithmetic operations. Let us express the inputs (which are usually membership degree of a fuzzy concept) x1, x2 and the weights w1, w2 over the unit interval [0, 1].

A hybrid neural net may not use multiplication, addition, or a sigmoidal function (because the results of these operation are not necessarily are in the unit interval).

Definition 4.3.2.1:-

A hybrid neural net is a neural net with crisp signals and weights and crisp transfer function. However,

1. we can combine xi and wi using a t-norm, t-conorm
2. we can aggregate p1 and p2 with a t-norm, t-conorm, or any other continuous function.
3. f can be any continuous function from input to output.

In the following we present some fuzzy neurons.

Definition 4.3.2.2:-

(AND fuzzy neuron) the signal xi and wi are combined by a triangular conorm C to produce $pi = C(w_i, x_i), i = 1,2$ the input information p_i is aggregated by a triangular norm T to produce the output

$y = AND(p_1, p_2) = T(p_1, p_2) = T(C(w_1, x_1), C(w_2, x_2))$

Of the neuron. So, if $T = \min$ and $C = \max$ then the (AND) neuron realizes the min-max composition $y = \min\{w_1 \vee x_1, w_2 \vee x_2\}$

Definition 4.3.2.3:-

(OR fuzzy neuron) the signal xi and wi are combined by a triangular norm T to produce $pi = T(w_i, x_i), i = 1,2$, the input information pi is aggregated by a triangular conorm C to produce the output

$y = OR(p_1, p_2) = C(p_1, p_2) = C(T(w_1, x_1), T(w_2, x_2))$ Of the neuron. So if $T = \min$ and $C = \max$ then the (OR) neuron realizes the max-min composition $y = \max\{w_1 \wedge x_1, w_2 \wedge x_2\}$.

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المجموعات الضبابية وتطبيقاتها في الشبكات العصبية الاصطناعية

-مقدمة:

بسم الله والحمد لله، والصلاة والسلام على رسول الله خاتم الأنبياء والمرسلين .

سوف نتناول في هذا البحث دراسة للمجموعات الضبابية, والتي ظهرت لأول مرة على يد العالم ((لطفي زاده)) في العام (1965) ,حيث نشر زاده أستاذ ورئيس قسم الهندسة الكهربائية بجامعة كاليفورنيا – بركلي ورقته العلمية الشهيرة ((المجموعات الضبابية)) , وكذلك قام زاده بعمل على نظرية(الإمكانية) في المنطق الرياضي , وقدم أيضا مفهوم المصطلحات اللغوية , وأستحدث ما يعرف اليوم بالمنطق الضبابي. وتعتبر المجموعات الضبابية إحدى الطرق الرياضية الجديدة في تحليل النظم المعقدة, والتي تحتاج نوعا من الذكاء لمعالجتها. ولقد كانت هنالك حاجة ملحة لمعرفة الطرق الصحيحة لمعالجة المشاكل التي تنتج عند معالجة النظم, ومن بين تلك المشاكل مشكلة عدم التأكد والغموض, الأمر الذي زاد من صعوبة استخلاص المعلومة الصحيحة من البيانات الخام. ولذلك فإن تصميم نظم ذكية أو قادرة على محاكاة الذكاء البشري . يتطلب قدرة تلك النظم على فهم المعلومات الناقصة أو غير المؤكدة, وأيضا قدرتها على الاستنتاج والبديهة. ولكن كل ذلك الأمر يعتمد على تطوير نظام ذو قاعدة رياضية ذكية, بحيث تستطيع استخلاص المعلومات الصحيحة من بيئة يغلب عليها التشويه في البيانات. ولذلك برزت الحاجة لإيجاد مثل هذا النظام .في هذا البحث سنناقش مفهوم المجموعات الضبابية التي تعتبر من أفضل الأدوات الرياضية لمعالجة مسألة عدم التأكد, وأيضا أبراز دور المجموعات الضبابية في مجال الذكاء الاصطناعي. حيث سنقوم بتطبيق المجموعات الضبابية مع مفهوم الشبكات العصبية لاستحداث نظام ذكي يعتمد على مقدرة الشبكات العصبية في التعلم والتكيف مع التغيرات في النظم وقدرة المنطق الضبابي للعمل مع بيانات غامضة و على أدراك اللغة الطبيعية أو الاصطناعية وكذلك على قدرتها على فهم التفكير التقريبي لدى البشر.

لقد تم تقسيم هذا البحث إلي أربعة فصول كالتالي:-

الفصل الأول:- هذا الفصل يحتوي بعض المفاهيم الأساسية لنظرية الضبابية , وأيضا أعطينا تعاريف وأمثلة لتوضيح فكرة الضبابية ولوضحنا كيف يمكن تمثيل المجموعات الضبابية , وأيضا بعض الطرق لتحديد دالة العضوية , ودرسنا بعض العمليات المعروفة مثل , الاتحاد والتقاطع , والمكملة , وبعض العمليات الأخرى, مع أهم النظريات عليها .

الفصل الثاني:- لقد قمنا بعرض ودراسة مفهوم جديد ألا وهو مبدأ التوسيع(لزاده) حيث يعتبر أداة هامة لأجراء العمليات علي المجموعات الضبابية , وأيضا علي الكميات الضبابية , وكذلك درسنا بعض الأفكار الجبرية المهمة منها (المجموعة المستوى أو قطاعات ألفا وصور قطاع ألفا , والمجموعة المحدبة الضبابية) ,وأيضا درسنا الكميات الضبابية , والعمليات عليها باستخدام مبدأ التوسيع وكذلك درسنا أنواعها مثل الأعداد الضبابية ,والفترات الضبابية , والعمليات عليها باستخدام قطاعات ألفا , وأيضا درسنا مفهوم الحساب الضبابي وكيف يمكن إجراء عمليات باستخدامه, مثل الجمع والضرب والقسمة والطرح , وأيضا مفهوم الأكبر – الأصغر . وأعطينا الأمثلة على هذا الفصل .

الفصل الثالث:- في هذا الفصل ناقشنا مفهوم المنطق الضبابي وعملياته , وروابطه مثل(تي - نورم, وتي- كونورم) والتضمين الضبابي , وكذلك درسنا مفهوم العلاقات الضبابية وكيف يمكن تمثيلها باستخدام المصفوفات, وأيضا درسنا عملياتها وتصنيفها ودرسنا مفهوم الإسقاط والتوسيع الاسطواني ,وأيضا درسنا مفهوم التجزيء الضبابي وتصنيفاته .

الفصل الرابع:-

ناقشنا في هذا الفصل مفهوم التفكير التقريبي في المنطق الضبابي ودرسنا تطبيقات المجموعات الضبابية في الشبكات العصبية حيث قمنا باستعراض النماذج الرياضية للعصبونات الاصطناعية وخصائصها وتركيباتها وطرق تعلمها ,ثم عرجنا على مفهوم الشبكات العصبية الضبابية وكيفية الحصول على خلية عصبية هجينة , ثم قمنا بتصنيف الخلايا العصبية الضبابية .

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قســـــم الرياضيات

عنـــوان البحـــث

((المجموعات الضبابية وتطبيقاتها في الشبكات العصبية الاصطناعية))

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بحث بعنوان :-

المجموعات الضبابية وتطبيقاتها في الشبكات العصبية الاصطناعية

استكمالاً لمتطلبات الإجازة العالية الماجستير في علوم الرياضيات

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العام الجامعي 2006ف

بسم الله الرحمن الرحيم

﴿وما أوتيتم من العلم إلا قليلا﴾

صدق الله العظيم

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