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Fredholm Integral Equations with Degenerate Kernel Method in Two Dimensional

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الملخص:

في هذا البحث ناقشنا المعادلة الخطية ثنائية الأبعاد للنوع الثاني لنوع فردهولم ذات النواة المتصلة، كما ناقشنا وجود ووحدانية الحل لهذه المعادلة وفق الشروط اللازم توفرها، وتكلمنا عن طريقة تقريب النواة، واستخدمنا طريقة النواة القابلة للفصل لتحويل المعادلة التكاملية إلى نظام جبري خطي، ومناقشة شروط وجود ووحدانية الحل لهذا النظام الجبري، وكذلك مناقشة التكافؤ بين الحل التحليلي والحل العددي، كما ذكرنا بعض الأمثلة العددية وقدّرنا الخطأ في حالة وجُوده.

الكلمات المفتاحية: معادلة فردهولم التكاملية، وجود ووحدانية الحل، طريقة النواة القابلة للفصل.

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Abstract

In this paper, The existence and uniqueness of solution is studied in $L_2[a,b]\times L_2[c,d]$. Moreover, we use a degenerate kernel method to transform the integral equation into a linear algebraic system. In addition, the existence and uniqueness of this linear algebraic is discussed. Finally, numerical examples are considered and the error, in each case is computed by Maple.

Keywords: - Fredholm Integral Equation, The existence and uniqueness solution, Degenerate kernel method.

1. Introduction

Integral equations are undoubtedly extremely significant in both practical and scientific domains. In reality, a variety of challenges and issues in disciplines including physics, chemical engineering, heat and mass, etc. led to these equations. The most of integral equations that are closely related to differential equations are known as Fredholm Integral Equations (FIEs). Hence, boundary value issues for differential equations are the source of (FIEs), which are then resolved using a variety of basic techniques. In the applied sciences, the use of integral equations and their various forms and kernels is a crucial topic where are used as mathematical models for many physical situations. When the kernel has either a continuous or discontinuous form, numerical methods play a significant role to solve one- and two-dimensional integral equations. The most known techniques are projection-iteration, collecation, Galerkin and degenerate kernel.

The degenerate kernel approach can be applied if the kernel is continuous and can be expressed as the product of two functions. The approximation kernel approach, also known as the iterated method, can be applied if the kernel can not be expressed as the product of two functions. For more information for the numerical methods can be found in (Atkinson, 1997, Hacia, 1993, Golberg, 1979, Golberg, 1990 and Delves & Mohamed, 1985).

Many problem in applied science for example in engineering can be translated into two-dimensional (2D-FIEs). Fallahzadeh in (Fallahzadeh, 2012) used the Gsussian radial basis function and triangular method to solve (2D-FIEs). Ziyaee and Tari (Ziyaee & Tari, 2015 and Abdelaziz, 2022) used the differential transform approach to solve the problem. Lin (Lin, 2014) employed wavelet-based techniques to solve two-dimensional integral equations numerically. The (2D-FIE) were numerically solved by Alipanah and Esmaeili (Alipanah & Esmaeili, 2011) using the Gaussian radial basis function. A computational technique for resolving (2D-FIEs) of the second kind was discovered by Tari and Shahmorad (Tari & Shahmorad, 2008). Two-dimensional triangular orthogonal functions were used by Mirzaee and Piroozfar (Mirzaee & Piroozfar, 2010) to numerically solve linear (2D-FIEs) of the second kind. This study covers the analytic and numerical approach of using the degenerate kernel method to solve the (2D-FIEs) (Abdou, 2000, Abdou & Mohamed & Ismail, 2002 and Abdou & Mohamed & Ismail, 2003). We will discuss the existence and uniqueness of the solution where the theorem is proved by two different ways: Picard and Banach fixed point

(Abdou & Hendi, 2005, Abdou & Elboraie & Elkojok, 2008 and chiavone & Costanda & Mioduchowski, 2002). Moreover, the approximate kernel method is discussed. The degenerate kernel method transform the integral equation into a linear algebraic system. Finally, some example are given to show how fast and efficient the computations are.

2. Fredholm Integral Equation (FIE) in two Dimensional

Fredholm integral equation in two dimensional take the form

$$\mu\,\varphi(x,y)-\lambda\int_a^b\int_c^d K(x,u;y,v)\varphi(u,v)\,dudv = f(x,y) \qquad (1)$$

Where μ constant defines the kind of the integral equation, for $\mu = 0$ and $\mu =$ constant $\neq 0$, we have respectively the **(FIE)** of the first and second kind, while λ is a constant has many physical meaning. The known functions $K(x,u;y,v)$ and $f(x,y)$ represent respectively, the continuous kernel of the integral equation and its free term. While $\varphi(x,y)$ represents the unknown function.

2.1 Existence and uniqueness solution

In order to prove the existence and uniqueness of solution of (1), we assume the following conditions:

(i) The kernel $K(x,u;y,v)$, in general, satisfies

$$\left\{\int_a^b\int_a^b\int_c^d\int_c^d |K(x,u;y,v)|^2\,dxdudydv\right\}^{\frac{1}{2}} \le C$$, C is a constant (very small)

(ii) The given function $f(x,y)$ and its partail derivatives with respect to x, y are continuous and its normality in $L_2[a,b]\times L_2[c,d]$ is given by

$$\|f(x,y)\|_{L_2[a,b]\times L_2[c,d]} = \left[\int_a^b\int_c^d |f(x,y)|^2\,dxdy\right]^{\frac{1}{2}} = D$$, D is a constant.

Theorem 1:- The solution of the two dimensional integral equation (1) is exist and unique, under the following condition

$$|\lambda| < \frac{|\mu|}{C} \,. \qquad (2)$$

We shall prove the above theorem by using two different methods Picard method and Banach fixed point theorem:

Proof: - To prove that the solution of equation (1) is exist, using Picard method, we pick up any real continuous function $\varphi_0(x,y)$ in $L_2[a,b]\times L_2[c,d]$, then we construct a sequence $\varphi_n(x,y)$ to have

$$\mu\varphi_n(x,y)= f(x,y)+\lambda\int_a^b\int_c^d K(x,u;y,v)\varphi_{n-1}(u,v)dudv \ , \mu\neq 0,\ n=1,2,... \qquad (3)$$

Where

$$\mu\phi_o(x,y)= f(x,y) \qquad (4)$$

It is convenient to introduce

$$\mu\psi_n(x,y)=\mu\left[\phi_n(x,y)-\phi_{n-1}(x,y)\right]$$

$$=\lambda\int_a^b\int_c^d K(x,u;y,v)\left[\phi_{n-1}(u,v)-\phi_{n-2}(u,v)\right]dudv \qquad (5)$$

The above formula, yields

$$\mu\psi_n(x,y)=\lambda\int_a^b\int_c^d K(x,u;y,v)\psi_{n-1}(u,v)dudv \qquad (6)$$

Also

$$\mu\psi_0(x,y)= f(x,y) \qquad (7)$$

Also, the first and second term of equation (5) leads us construct series

$$\varphi_n(x,y)=\sum_{i=0}^{n}\psi_i(x,y) \qquad (8)$$

Now, taking the norm of equation (6), we obtain

$$|\mu|\,\|\psi_n(x,y)\|=|\lambda|\left\|\int_a^b\int_c^d K(x,u;y,v)\psi_{n-1}(u,v)dudv\right\| \qquad (9)$$

Using Cauchy – Schwarz inequality and condition (i) we have

$$\| \psi_n(x,y) \| \le \frac{|\lambda|}{|\mu|} C \| \psi_{n-1}(x,y) \| \qquad (10)$$

Let n =1, in equation (10), then use condition (ii), we get

$$\| \psi_1(x,y) \| \le \frac{|\lambda|}{|\mu|} C\, D \qquad (11)$$

Also, using (10) at n =2, we get

$$\| \psi_2(x,y) \| \le \left(\frac{|\lambda|}{|\mu|} C \right)^2 D \qquad (12)$$

By induction, one has

$$\| \psi_n(x,y) \| \le \alpha^n D \quad , \quad \alpha = \left| \frac{\lambda}{\mu} \right| C \qquad (13)$$

This bound under condition $\alpha < 1$, makes the sequence $\psi_n(x,y)$ uniformly convergent.

Hence, we have

$$\varphi(x,y) = \sum_{i=0}^{\infty} \psi_i(x,y) \qquad (14)$$

Since each of $\psi_i(x,y)$ is continuous, hence $\varphi_i(x,y)$ is also continuous, convergent and represents the existence of the solution of (1).

To prove $\varphi(x,y)$ is the unique solution of (1) assume $\tilde{\varphi}(x,y)$ is another solution in $L_2[a,b] \times L_2[c,d]$ hence, we get

$$\phi(x,y) - \tilde{\phi}(x,y) = \frac{\lambda}{\mu} \int_a^b \int_c^d K(x,u;y,v) \left[\phi(u,v) - \tilde{\phi}(u,v) \right] du dv \quad (15)$$

Applying Cauchy-Schwarz inequality and using condition (i) we obtain

$$\| \varphi(x,y) - \tilde{\varphi}(x,y) \| \le \left| \frac{\lambda}{\mu} \right| C \| \varphi(x,y) - \tilde{\varphi}(x,y) \| \qquad (16)$$

But $\left|\frac{\lambda}{\mu}\right| C < 1$. Hence $\varphi(x,y) = \tilde{\varphi}(x,y)$

Therefore, $\varphi(x,y)$ represents a unique solution of equation (1)

Since Picard method fails to prove the existence and uniqueness of solution of (1) if $\mu = 0$ *i.e.* for the **IE** of the first kind. Also, if $f(x,y) = 0$ *i.e* for the homogeneous integral equation. Therefore, we go to prove the existence of a unique solution of (1) using Banach fixed point theorem. For this aim, write the integral equation (1) in the integral operator form

$$W\varphi = \frac{f}{\mu} + K\varphi . \qquad \text{(a)}$$

Where

$$K\varphi = \frac{\lambda}{\mu}\int_a^b\int_c^d K(x,u;y,v)\,\varphi(u,v)\,dudv . \qquad \text{(b)}$$

Hence, in view of the integral operator (a), (b), we can write equation (1) in the form

$$W\varphi = \varphi .$$

3. Degenerate kernel method to solve FIE in two dimensional

Suppose that the approximate kernel $K_{n,m}(x,u;y,v)$ takes the form

$$K_{n,m}(x,u;y,v) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i(x)b_i(u)c_j(y)d_j(v) \qquad (17)$$

where

$$\left|K_{n,m}(x,u;y,v) - K(x,u;y,v)\right| \to 0 \quad , \quad \text{as } n,m \to \infty \qquad (18)$$

Therefore, the integral equation (1) yields

$$\mu\varphi_{n,m}(x,y) - \lambda\int_a^b\int_c^d K_{n,m}(x,u;y,v)\varphi_{n,m}(u,v)dudv = f(x,y) + R_{n,m} \qquad (19)$$

Where $R_{n,m}$ is the error function of $O(n^{-r_1}m^{-r_2})$, r_1 and r_2 are constants.

Definition 1:- The degenerate kernel method is said to be convergent of order $r_1 + r_2$ in the domain $L_2[a,b]\times L_2[c,d]$ if and only if for large n, m there exist a constant $\gamma > 0$ independent of n, m such that

$$\left\|\varphi(x,y)-\varphi_{n,m}(x,y)\right\| \le \gamma\, n^{-r_1} m^{-r_2} \tag{20}$$

Using (17) in (19), we have

$$\mu\varphi_{n,m}(x,y)-\lambda\sum_{i,j=1}^{n,m} a_i(x)c_j(y)\int_a^b\int_c^d b_i(u)d_j(v)\varphi_{n,m}(u,v)dudv = f(x,y) \tag{21}$$

Assume the unknown constant

$$\mathrm{A}_{ij} = \int_a^b\int_c^d b_i(u)d_j(v)\varphi_{n,m}(u,v)dudv \tag{22}$$

Hence, the formula (21) becomes

$$\varphi_{n,m}(x,y) = \frac{\lambda}{\mu}\sum_{i,j=1}^{n,m} \mathrm{A}_{ij}\, a_i(x)c_j(y) + \frac{f(x,y)}{\mu} \quad , \quad (\mu \neq 0) \tag{23}$$

To determine $A_{ij}(i = 1,2,3,\ldots,n \; and \; j = 1,2,3,\ldots,m)$, by substituting (23) into (22), we have

$$\mathrm{A}_{ij} = \int_a^b\int_c^d b_i(u)d_j(v)\left[\frac{f(u,v)}{\mu} + \frac{\lambda}{\mu}\sum_{l,k=1}^{n,m} \mathrm{A}_{lk}a_l(u)c_k(v)\right]dudv$$

The previous formula can be adapted in the form

$$\mathrm{A}_{ij} = \frac{1}{\mu}f_{ij} + \frac{\lambda}{\mu}\sum_{l,k=1}^{n,m} D_{ijlk}\mathrm{A}_{lk} \quad , \quad \mu \neq 0, \tag{24}$$

where

$$f_{ij} = \int_a^b\int_c^d b_i(u)d_j(v)f(u,v)dudv, \tag{25}$$

and

$$D_{ijlk} = \int_a^b\int_c^d b_i(u)d_j(v)a_l(u)\, c_k(v)dudv \tag{26}$$

The formula (24) represents a system of linear algebraic equations of order $n \times m$.

To write the system in the vector form writes

$$\overline{\mathrm{A}} = \mathrm{A}_{ij} = \int_a^b \int_c^d b_i(u) d_j(v) \varphi_{n,m}(u,v)\, dudv \tag{27}$$

Moreover, define the operator

$$G(\overline{\mathrm{A}}) = \int_a^b \int_c^d b_i(u) d_j(v) \left[\frac{f(u,v)}{\mu} + \frac{\lambda}{\mu} \sum_{l,k=1}^{n,m} \mathrm{A}_{lk} a_l(u) c_k(v) \right] dudv \tag{28}$$

Hence, we have the vector form

$$\overline{\mathrm{A}} = G(\overline{\mathrm{A}})\ . \tag{29}$$

Where the elements of $\overline{A}$ and $G(\overline{A})$ are given by

$$\overline{\mathrm{A}} = \begin{pmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & \dots & \mathrm{A}_{1m} \\ \mathrm{A}_{21} & & \dots & \mathrm{A}_{2m} \\ . & & & \\ . & & & \\ . & & & \\ \mathrm{A}_{n1} & & \dots & \mathrm{A}_{nm} \end{pmatrix} ,\ \overline{G} = \begin{pmatrix} G_{11}(\overline{\mathrm{A}}) & G_{12}(\overline{\mathrm{A}}) & \dots & G_{1m}(\overline{\mathrm{A}}) \\ . & & & \\ . & & & \\ . & & & \\ G_{n1}(\overline{\mathrm{A}}) & & \dots & G_{nm}(\overline{\mathrm{A}}) \end{pmatrix} \tag{30}$$

Theorem 2:- Under the same assumptions of theorem 1, the sequence solution of equation (19) converges uniformally to the unique solution $\varphi(x,y)$ of equation (1) in the space $L_2[a,b] \times L_2[c,d]$.

Proof:- From equation (1) and equation (19), after neglecting the small error $R_{n,m}$, we have

$$\| \varphi(x,y) - \varphi_{n,m}(x,y) \| = \left| \frac{\lambda}{\mu} \right| \left\| \int_a^b \int_c^d [K(x,u;y,v)\varphi(u,v) - K_{n,m}(x,u;y,v)\varphi_{n,m}(u,v)] dudv \right\|$$

The above formula can be a adopted in the form

$$\|\phi(x,y) - \phi_{n,m}(x,y)\| \le \left| \frac{\lambda}{\mu} \right| \left\{ \left\| \int_a^b \int_c^d [K(x,u;y,v) - K_{n,m}(x,u;y,v)] \phi(u,v)\ dudv \right\| \right.$$

$$\left. + \left\| K_{n,m}(x,u;y,v) \left[\phi(u,v) - \phi_{n,m}(u,v) \right] dudv \right\| \right\} \tag{31}$$

Using the fact that $\left\| K(x,u;y,v) - K_{n,m}(x,u;y,v) \right\| \to 0$ as $n,m \to \infty$ and the condition

$$\left[\int_a^b \int_a^b \int_c^d \int_c^d \left| K_{n,m}(x,u;y,v) \right|^2 dxdudydv \right]^{\frac{1}{2}} \le C$$

we get

$$\left\| \varphi(x,y) - \varphi_{n,m}(x,y) \right\| \le \alpha \left\| \varphi(x,y) - \varphi_{n,m}(x,y) \right\| \; , \; \alpha = \left| \frac{\lambda}{\mu} \right| C$$

By Banach fixed point theorem, we have

$$\varphi(x,y) = \varphi_{n,m}(x,y), \text{ as } \quad n,m \to \infty \tag{32}$$

4. The existence and uniqueness solution of the linear algebraic system

The linear algebraic system of equation (24) or its equivalent linear vector (29), we define

$$\left\| \bar{A} \right\|_{l_2 \times l_2} = \left[\sum_{i=1}^{n} \sum_{j=1}^{m} \left(A_{ij} \right)^2 \right]^{1/2} \tag{33}$$

Theorem 3:- Under the following conditions

$$\left[\sum_{i,j=1}^{n,m} \int_a^b \int_c^d \left| b_i(u) d_j(v) \right|^2 dudv \right]^{1/2} \left[\sum_{i,j=1}^{n,m} \int_a^b \int_c^d \left| a_i(u) c_j(v) \right|^2 dudv \right]^{1/2} = \zeta \; , \zeta \text{ (is a small constant)} \tag{34}$$

$$\left[\sum_{i,j=1}^{n,m} \int_a^b \int_c^d \varphi^2 \left(u, v, A_{ij} \right) dudv \right]^{1/2} < \eta \quad , \quad (\eta \text{ is a constant }) \tag{35}$$

$$\left[\sum_{i,j=1}^{n,m} \int_a^b \int_c^d \left| \varphi(u,v,A_{ij}) - \varphi(u,v,B_{ij}) \right|^2 dudv \right]^{1/2} \le \varepsilon \text{, where } \varepsilon = \left| \bar{A} - \bar{B} \right| = \left[\sum_{i,j=1}^{n,m} \left| A_{ij} - B_{ij} \right|^2 \right]^{1/2} \tag{36}$$

Under the above conditions the linear algebraic system (24) or (29) has a unique solution in the space $l_2 \times l_2$.

To prove this theorem, we must consider the following lemmas

Lemma 1:- Under the conditions (34) and (35) the operator $\overline{G}$ of (29) is bounded.

Proof:- From equation (27) we have

$$\left|G_{ij}\left(\overline{A}\right)\right| \le \int_a^b\int_c^d \left| b_i(u) d_j(v)\right| \left| \frac{f(u,v)}{\mu} + \frac{\lambda}{\mu}\sum_{l,k=1}^{n,m} A_{lk} a_l(u) c_k(v) \right| dudv \qquad (37)$$

Summing over *i,j* then applying Cauchy-Minkwoski inequality, we get

$$\left[\sum_{i,j=1}^{n,m}\left|G_{ij}\right|^2\right]^{1/2} \le \left[\sum_{i,j=1}^{n,m}\int_a^b\int_c^d \left|b_i(u) d_j(v)\right|^2 dudv\right]^{1/2} \cdot \left[\sum_{i,j=1}^{n,m}\int_a^b\int_c^d \left[\frac{f(u,v)}{\mu} + \frac{\lambda}{\mu}\sum_{l,k=1}^{n,m} A_{lk} a_l(u) . c_k(v)\right]^2 dudv\right]^{1/2} \quad (38)$$

Applying the condition of the above theorem, we have

$$. \quad \left\| \overline{G}\left(\overline{A}\right) \right\| \le \left|\frac{\lambda}{\mu}\right| C^* \varepsilon \quad , \quad \mu \ne 0 \qquad (39)$$

Hence, $\overline{G}$ is bounded operator in Banach space $l_2 \times l_2$.

Lemma 2:- Under the condition (36), $\overline{G}$ is continuous in Banach space $l_2 \times l_2$.

Proof:- Let $\overline{A}$ and $\overline{B}$ be any two elements in $S_\beta{}^+$, therefore

$$\left|G_{ij}\left(\overline{A}\right) - G_{ij}\left(\overline{B}\right)\right| \le \int_a^b\int_c^d \left|b_i(u) d_j(v)\right| \left| \frac{f(u,v)}{\mu} + \frac{\lambda}{\mu}\sum_{l,k=1}^{n,m} a_l(u) c_k(v) \overline{A}_{lk} - \left(\frac{f(u,v)}{\mu} + \frac{\lambda}{\mu}\sum_{l,k=1}^{n,m} a_l(u) c_k(v) \overline{B}_{lk} \right) \right| dudv$$

Summing over *i,j*, and apply Cauchy-Minkwoski inequality, we get

$$\left[\sum_{i,j=1}^{n,m} \left|G_{ij}\left(\overline{A}\right) - G_{ij}\left(\overline{B}\right)\right|^2\right]^{1/2} \le \left[\sum_{i,j=1}^{n,m} \int_a^b\int_c^d \left|b_i(u) d_j(v)\right|^2 dudv\right]^{1/2}$$

$$+ \left|\frac{\lambda}{\mu}\right| \left[\sum_{i,j=1}^{n,m}\int_a^b\int_c^d \left\{\sum_{l,k=1}^{n,m} a_l(u) c_k(v) \left(\overline{A}_{lk} - \overline{B}_{lk}\right)\right\}^2 dudv\right]^{1/2}$$

5. Numerical Examples

In this section, a numerical example is solved to show the efficiency of the method. The program has been provided by the researcher in MAPLE.

Example 5.1- By using degenerate kernel method solve the integral equation:

$$\varphi(x,y)-2\int_0^1\int_0^1\left[6xuy^2v^2+9xuy^3v^4+10x^2u^3y^2v^2+15x^2u^3y^3v^4\right]\varphi(u,v)\,dudv$$

$$\varphi(x,y)=x\ \ y$$

$$=xy-x^2y^3-xy^3-x^2y^2-xy^2$$

Exact solution:

Solution

$$K(x,u;y,v)=6xuy^2v^2+9xuy^3v^4+10x^2u^3y^2v^2+15x^2u^3y^3v^4$$

$$=3xu\left(2y^2v^2+3y^3v^4\right)+5x^2u^3\left(2y^2v^2+3y^3v^4\right)$$

$$=\left(3xu+5x^2u^3\right)\left(2y^2v^2+3y^3v^4\right)$$

$$K(x,u;y,v)=\sum_{i=1}^{2}(2i+1)\,x^iu^{2i-1}\sum_{j=1}^{2}(j+1)\,y^{j+1}v^{2j}$$

We can write the kernel in the form:

$$K(x,u;y,v)=\sum_{\substack{i=1\\ j=1}}^{2,2}a_i(x)\,c_i(u)\,b_j(y)\,d_j(v)$$

$$a_i(x)=(2i+1)\,x^i,\quad b_i(u)=u^{2i-1},\quad i=1,2$$

$$c_j(y)=(j+1)y^{j+1},\quad d_j(v)=v^{2j},\ j=1,2$$

We find

$$f_{ij}=\int_a^b\int_c^d b_i(u)\,d_j(v)\,f(u,v)\,dudv$$

Such that $i = 1,2$, $j =1,2$

By using Maple13 we get

$$F = \begin{bmatrix} \frac{-47}{360} & \frac{-29}{288} & \frac{-19}{225} & \frac{-19}{225} \end{bmatrix}^T$$

And also

$$D_{ijlk} = \int_a^b \int_c^d b_i(u) d_j(v) a_l(u) c_k(v) dudv$$

By using Maple13 we get

$$D = \begin{bmatrix} \frac{2}{5} & \frac{1}{2} & \frac{1}{2} & \frac{5}{8} \\ \frac{2}{7} & \frac{3}{8} & \frac{5}{14} & \frac{15}{32} \\ \frac{6}{25} & \frac{3}{10} & \frac{1}{3} & \frac{5}{12} \\ \frac{6}{35} & \frac{9}{40} & \frac{5}{21} & \frac{5}{16} \end{bmatrix}$$

By using Maple13 we solved this linear system:

$$(I - \lambda\, D)\, A = F$$

And we get

$$A = \begin{bmatrix} \frac{1}{12} & \frac{1}{18} & \frac{1}{20} & \frac{1}{30} \end{bmatrix}^T$$

$$\varphi^D(x, y) = f(x, y) + \lambda \sum_{\substack{i=1\\ j=1}}^{2,2} A_{ij}\, a_i(x) c_j(y)$$

$$\varphi^D(x, y) = xy = Exact\ \ solution.$$

$$Error = 0$$

Example 5.2:- By using degenerate kernal method solve the integral equation:

$$\varphi(x, y) - \int_0^1 \int_0^1 \left[12u\, y^2 v^2 + 15u\, y^4 v^3 + 16\, x\, u^2 y^2 v^2 + 20\, x\, u^2 y^4 v^3 + 20\, x^2\, u^3 y^2 v^2 \right.$$
$$\left. + 25\, x^2 u^3 y^4 v^3 \right] \varphi(u, v)\, dudv = xy - x^2 y^4 - xy^4 - y^4 - x^2 y^2 - xy^2 - y^2$$

Exact solution: $\varphi(x, y) = xy$

Solution:

$$K(x,u;y,v)=12uy^2v^2+15uy^4v^3+16xu^2y^2v^2+20xu^2y^4v^3+20x^2u^3y^2v^2+25x^2u^3y^4v^3$$
$$=3u\left(4y^2v^2+5y^4v^3\right)+4xu^2\left(4y^2v^2+5y^4v^3\right)+5x^2u^3\left(4y^2v^2+5y^4v^3\right)$$
$$=\left(3u+4xu^2+5x^2u^3\right)\left(4y^2v^2+5y^2v^3\right)$$

$$K(x,u;y,v)=\sum_{i=1}^{3}(i+2)\,x^{i-1}u^i\sum_{j=1}^{2}(j+3)\,y^{2j}\,v^{j+1}$$

We can write the kernel in the form:

$$K(x,u;y,v)=\sum_{\substack{i=1\\j=1}}^{3,2}a_i(x)b_i(u)c_j(y)d_j(v),$$

$$a_i(x)=(i+2)\,x^{i-1},\ \ b_i(u)=u^i\quad,\quad i=1,2,3$$

$$c_j(y)=(j+3)\,y^{2j},\ \ d_j(v)=v^{j+1}\ \ ,\ \ j=1,2$$

Since

$$f_{ij}=\int_a^b\int_c^d b_i(u)d_j(v)f(u,v)\,dudv\quad,\quad i=1,2,3,\ j=1,2$$

By using Maple13 we get

$$F=\begin{bmatrix}\frac{-121}{240} & \frac{-359}{1440} & \frac{-577}{2800} & \frac{-257}{1440} & \frac{-113}{700} & \frac{-1007}{7200}\end{bmatrix}^T$$

And also

By using $D_{ijlk}=\int_a^b\int_c^d b_i(u)d_j(v)a_l(u)c_k(v)\,dudv\quad,\quad i=1,2,3,\ j=1,2,\ l=1,2,3,\ k=1,2$

Maple13 we get

$$D=\begin{bmatrix} \frac{6}{5} & \frac{15}{14} & \frac{16}{15} & \frac{20}{21} & 1 & \frac{25}{28} \\ 1 & \frac{15}{16} & \frac{8}{9} & \frac{5}{6} & \frac{5}{6} & \frac{25}{32} \\ \frac{4}{5} & \frac{5}{7} & \frac{4}{5} & \frac{5}{7} & \frac{4}{5} & \frac{5}{7} \\ \frac{2}{3} & \frac{5}{8} & \frac{2}{3} & \frac{5}{8} & \frac{2}{3} & \frac{5}{8} \\ \frac{3}{5} & \frac{15}{28} & \frac{16}{25} & \frac{4}{7} & \frac{2}{3} & \frac{25}{42} \\ \frac{1}{2} & \frac{15}{32} & \frac{8}{15} & \frac{1}{2} & \frac{5}{9} & \frac{25}{48} \end{bmatrix}$$

By using Maple13 we solved this linear system:

$(I-\lambda D)A=F$, and we get

$$A=\begin{bmatrix} \frac{1}{12} & \frac{1}{15} & \frac{1}{16} & \frac{1}{20} & \frac{1}{20} & \frac{1}{25} \end{bmatrix}^T$$

$$\varphi^D(x,y)=f(x,y)+\lambda\sum_{\substack{i=1\\j=1}}^{3,2} \mathrm{A}_{ij}a_i(x)c_j(y)$$

$$\varphi^D(x,y)=xy=Exact\ \ solution.$$

$Error = 0$

Example 5.3:- By using degenerate keral method, solve the integral equation.

$$\varphi(x,y)-\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} x^2\sin(xu)y^2\cos(yv)\varphi(u,v)\,dudv=f(x,y)$$

Such that

$$f(x,y)=xy+\frac{1}{4}\left(-2\sin\frac{\pi}{2}x+\pi x\cos\frac{\pi}{2}x\right)\left(-2+2\cos\frac{\pi}{2}x+\pi y\sin\frac{\pi}{2}y\right)$$

Exact solution: $\varphi(x,y)=xy$

$$K(x,u;y,v)=x^2\sin(xu).y^2\cos(yv)$$

$$= x^2\left(\frac{xu}{1!}-\frac{(xu)^3}{3!}+\frac{(xu)^5}{5!}-\ldots\right).y^2\left(1-\frac{(yv)^2}{2!}+\frac{(yv)^4}{4!}-\ldots\right)$$

$$=\sum_{i=1}^{\infty}\frac{(-1)^{i-1}x^{2i+1}u^{2i-1}}{(2i-1)!}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}y^{2j}v^{2j-2}}{(2j-2)!}$$

$$=\sum_{i=1}^{\infty} a_i(x)b_i(u)\sum_{j=1}^{\infty} c_j(y)d_j(v)$$

$$a_i(x)=(-1)^{i-1}x^{2i+1} \quad , \quad b_i(u)=\frac{u^{2i-1}}{(2i-1)!} \quad , \quad i=1,2,3,..$$

$$c_j(y)=(-1)^{j-1}y^{2j} \quad , \quad d_j(v)=\frac{v^{2j-2}}{(2j-2)!} \quad , \quad j=1,2,3,...$$

When $n=3$, we have:

$$K_3(x,u;y,v)=\sum_{i=1}^{3}\frac{(-1)^{i-1}x^{2i+1}u^{2i-1}}{(2i-1)!}\sum_{j=1}^{3}\frac{(-1)^{j-1}y^{2j}v^{2j-2}}{(2j-2)!}$$

since:

$$f_{ij}=\int_a^b\int_c^d b_i(u)d_j(v)f(u,v)\,dudv$$

By using Maple13 we get:

$$F=\begin{bmatrix}0.8889749\\0.6777050\\0.1064433\\0.1957566\\0.1569944\\0.0253158\\0.0162703\\0.0134099\\0.0021914\end{bmatrix}$$

$$D_{ijlk}=\int_a^b\int_c^d c_i(u)d_j(v)a_l(u)b_k(v)\,dudv$$

By using Maple13 solved the system.

$$(\mathrm{I}-\lambda D)A=F$$

And we get

$$A=\begin{bmatrix}1.686537\\1.053613\\0.145706\\0.419696\\0.262555\\0.036343\\0.037148\\0.023252\\0.003219\end{bmatrix}$$

$$\varphi^D(x,y)=f(x,y)+\lambda\sum_{i,j}^{3,3}A_{ij}\,a_i(x)c_j(y).$$

$$\varphi^D(x,y)=xy+\left(\frac{-1}{2}\sin\frac{\pi}{2}x+\frac{1}{4}\cos\frac{\pi}{2}x(\pi x)\right)\left(-2+2\cos\frac{\pi}{2}y+y\pi\sin\frac{\pi}{2}y\right)$$

$$+1.68653x^3y^2-1.053613x^3y^4+0.1457069x^3y^6-0.419696x^5y^2+0.26255x^5y^4$$

$$-0.036343x^5y^6+0.037148x^7y^2-0.02325x^7y^4+0.00321974x^7y^6$$

$$Error=\left|\varphi(x,y)-\varphi^D(x,y)\right|$$

The following Tables showa the exact (analytical) solution $\varphi(x)$ of the FIE against the numerical solution $S_n(x)$ for different values of x taken in the given interval [0, 1]. The error can be determined as the absolute value of the difference between the exact solution and numerical solution as shown in the fifth column of Table 1 for $n=3$ and Table 2 for $n=10$. We note that by increasing the number of points to n, we obtain more accurate solution of the (2D-FIE) as seen in the fifth column of Table 2.

x	***y***	***Exact solution***	***Approximate solution***	***Error***
0	$\frac{\pi}{7}$	0	0	0
$\frac{\pi}{14}$	$\frac{\pi}{7}$	0.1007102490	0.1008874949	1.7724×10^{-4}
$\frac{\pi}{7}$	$\frac{\pi}{7}$	0.2014204981	0.2027781163	1.3576×10^{-3}
$\frac{3\pi}{14}$	$\frac{\pi}{7}$	0.3021307471	0.3063925545	4.2618×10^{-3}
$\frac{2\pi}{7}$	$\frac{\pi}{7}$	0.4028409961	0.4120079423	9.1669×10^{-3}
$\frac{5\pi}{14}$	$\frac{\pi}{7}$	0.5035512451	0.5196751649	1.6123×10^{-2}
$\frac{3\pi}{7}$	$\frac{\pi}{7}$	0.6042614942	0.6303223960	2.6060×10^{-2}
$\frac{\pi}{2}$	$\frac{\pi}{7}$	0.7049717432	0.7485795466	4.3607×10^{-2}

Table 1: The exact and numerical solutions of the FIE

When $n = 10$, we have

$$K_{10}(x,u;y,v)=\sum_{i=1}^{10}\frac{(-1)^{i-1}x^{2i+1}u^{2i-1}}{(2i-1)!}\sum_{j=1}^{10}\frac{(-1)^{j-1}y^{2j}v^{2j-2}}{(2j-2)!}$$

x	***y***	***Exact solution***	***Approximate solution***	***Error***
0	$\frac{\pi}{7}$	0	0	0
$\frac{\pi}{14}$	$\frac{\pi}{7}$	0.1007102490	0.1007102483	7.0144×10^{-10}
$\frac{\pi}{7}$	$\frac{\pi}{7}$	0.2014204981	0.2014204925	5.6150×10^{-9}
$\frac{3\pi}{14}$	$\frac{\pi}{7}$	0.3021307471	0.3021307301	1.7101×10^{-8}
$\frac{2\pi}{7}$	$\frac{\pi}{7}$	0.4028409961	0.4028409597	3.6365×10^{-8}
$\frac{5\pi}{14}$	$\frac{\pi}{7}$	0.5035512451	0.5035511852	5.9922×10^{-8}
$\frac{3\pi}{7}$	$\frac{\pi}{7}$	0.6042614942	0.6042614087	8.4800×10^{-8}
$\frac{\pi}{2}$	$\frac{\pi}{7}$	0.7049717432	0.7049716252	1.1836×10^{-7}

Table 2: The exact and numerical solutions of the FIE

6. Conclusion

In this paper we discussed the existence and uniqueness of the solution of the (2D-FIEs). The degenerate kernel method were used to transform the integral equation into a linear algebraic system. We measure the error of the computations as the absolute value of the difference between the exact (analytical) solution $\varphi(x)$ and numerical solution. Hence, by comparing the error of the numerical computations using Maple. One can conclude that from the illustrated examples the method is efficient and the convergence is fast.

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